

HANDBOOK OF
COMPLEX ANALYSIS:
GEOMETRIC
FUNCTION
THEORY

VOLUME 2

Edited by
R. Kühnau

NORTH-HOLLAND

HANDBOOK OF COMPLEX ANALYSIS

GEOMETRIC FUNCTION THEORY

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GEOMETRIC FUNCTION THEORY

Volume 2

Edited by

R. KÜHNAU

*Martin-Luther-Universität Halle-Wittenberg
Halle (Saale), Germany*

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Preface

As is the case for Volume 1 of this Handbook, this is not a textbook but a collection of independent survey articles about new developments in Geometric Function Theory. Again, it would of course be impossible to cover an extensive field like this one completely. Nevertheless, I do not doubt that its publication will help to make this theory interesting to new people and will attract new researchers to solving its problems.

I am grateful to Professor Edgar Reich for his kind advice in connection with some of the articles. I most gratefully acknowledge the friendly cooperation and expert support of Dr. A. Sevenster and Mrs. Andy Deelen from Elsevier. Last but not least, I have to thank Zigmās Kryžius from VTEX Typesetting service for the excellent printing.

Reiner Kühnau

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CHAPTER 1

Quasiconformal Mappings in Euclidean Spaces

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1. Definitions

We denote by $\mathbb{R}^n$ Euclidean n -space and by D and D' domains in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, compactified Euclidean n -space equipped with the chordal metric and chordal diameter

$$q(x, y) = \frac{2|x - y|}{\sqrt{|x|^2 + 1}\sqrt{|y|^2 + 1}}, \quad q(E) = \sup_{x, y \in E} q(x, y),$$

where $x, y \in \overline{\mathbb{R}^n}$ and $E \subset \overline{\mathbb{R}^n}$. Next, for $x \in \mathbb{R}^n$ and $0 < r < \infty$, we let

$$\mathbb{B}^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}, \quad \mathbb{B}^n = \mathbb{B}^n(0, 1).$$

Suppose that $f : D \rightarrow D'$ is a homeomorphism and that $x \in D$. If $x \notin \{\infty, f^{-1}(\infty)\}$, then we call

$$H_f(x) = \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)} \quad (1.1)$$

the *linear dilatation* of f at x , where

$$L_f(x, r) = \max_{|x - y| = r} |f(y) - f(x)|, \quad l_f(x, r) = \min_{|x - y| = r} |f(y) - f(x)|. \quad (1.2)$$

We say that f is *conformal* at x if it preserves angles between curves at x . We use preliminary inversions to define the linear dilatation and conformality of f at x if $x = \infty$ and/or $f(x) = \infty$.

If f is a diffeomorphism, then $H_f = 1$ in D if and only if f is conformal in D . It is natural to ask what the situation is if f is an arbitrary homeomorphism. The answer for the case where $n = 2$ was given by Menchoff [120] in 1937.

THEOREM 1.1. *When $n = 2$, $H_f = 1$ in D if and only if f or $\bar{f}$ is a meromorphic function of a complex variable.*

The answer to the above question for the case where $n > 2$ was given some years later by Gehring [51] and Reshetnyak [137, 138].

THEOREM 1.2. *When $n > 2$, $H_f = 1$ in D if and only if $f = g|D$, where g is a Möbius transformation, that is, the composition of a finite number of reflections in $(n - 1)$ -spheres and planes.*

The above two results are the basis for the following definition of quasiconformality.

DEFINITION 1.3. A homeomorphism $f : D \rightarrow D'$ is *K -quasiconformal* if

$$H_f(x) \leq K \quad \text{for } x \in D.$$

2. Historical remarks

2.1. Mappings in the plane

Plane quasiconformal mappings have been studied for almost seventy five years. They appear first in the late 1920s in papers by Grötzsch [71] who considered the problem of determining the *most nearly conformal* homeomorphism between pairs of topologically equivalent plane configurations with one conformal invariant. They occur later under the name *quasiconformal* in a fundamental paper by Ahlfors [1] on covering surfaces.

In the late 1930s Teichmüller greatly extended the study of Grötzsch to mappings between closed Riemann surfaces and obtained a natural parameter space for surfaces of fixed genus g which is homeomorphic to $\mathbb{R}^{6g-6}$ [160]. At about the same time, Lavrentieff [98] and Morrey [124] generalized a classical result due to Gauss on the existence of isothermal coordinates by establishing versions of what is now known as the *measurable Riemann mapping theorem* for quasiconformal mappings. See also [8].

Later Ahlfors, Bers and their students extended the results of Teichmüller and applied plane quasiconformal mappings with success to a variety of areas in complex analysis including Kleinian groups and surface topology. See, for example, [2,5,19,20,43,95].

For more comprehensive accounts of this area see [6,47,80,118,127,147].

2.2. Mappings in higher dimensions

Higher-dimensional quasiconformal mappings had already been considered by Lavrentieff in the 1930s [99]. However, no systematic tool for studying this class was available until 1959 when Loewner [110] introduced the notion of *conformal capacity* and, in answer to a question raised by Pfluger, showed that $\mathbb{R}^n$ could not be mapped quasiconformally onto a proper subset of itself.

Subsequently Gehring [50,51] and Väisälä [169,170] applied Loewner's method and its equivalent extremal length formulation to develop the initial results for quasiconformal mappings in $\mathbb{R}^n$. Then in the late 1960s, Reshetnyak [136,137] and Martio, Rickman and Väisälä [114–116] initiated a series of papers which extended the higher-dimensional theory to noninjective quasiconformal, or *quasiregular*, mappings. This study culminated in Rickman's remarkable extension of the classical theorem of Picard for quasiregular mappings in $\mathbb{R}^n$ [142]. See [28] and [106] for alternative proofs.

See [10,84,140,143,153,171,175] for more complete accounts of this subject and related topics.

2.3. Mappings in arbitrary metric spaces

The methods developed to lift the study of quasiconformal mappings from the plane to higher dimensions and the resulting applications of this extension encouraged researchers to study and develop a surprisingly rich theory for quasiconformal mappings in more general spaces. See, for example, the work of Korányi and Reimann [93] and Pansu [131]

on the Heisenberg group, of Väisälä in infinite-dimensional Banach spaces [174] and of Heinonen and Koskela in metric measure spaces [74,77]. This is currently a very active area.

3. Role played by quasiconformal mappings

Plane quasiconformal mappings constitute an important tool in complex analysis. Bers' theorem on simultaneous uniformization [18] and Drasin's solution [42] of the inverse problem of Nevanlinna theory were important applications of the measurable Riemann mapping theorem. Sullivan's solution of the 60 year old Fatou–Julia problem on wandering domains [157] and the work of Douady and Hubbard [41] and Shishikura [148] showed that these mappings could be used very effectively in the study of complex dynamics. See also [36].

The geometric proofs usually required to establish quasiconformal analogues of theorems for conformal mappings often yield new insight into classical results and methods of complex function theory [103]. Quasiconformal mappings also arise in exciting and unexpected ways in other parts of mathematics, e.g., functional analysis, geometry and elasticity. See Sections 11–13.

Higher-dimensional quasiconformal mappings offer a new and nontrivial extension of complex analysis to $\mathbb{R}^n$. They constitute a closed class of mappings interpolating between homeomorphisms and diffeomorphisms for which many results of geometric topology hold regardless of dimension. Moreover methods developed to study these mappings have found important applications in other branches of mathematics, e.g., partial differential equations and topology. See Sections 14–16.

Finally the study of quasiconformal mappings in more general spaces has led to useful and illuminating extensions of Sobolev spaces and other tools of real analysis to metric measure spaces [38,72,73,78,94].

4. Tools to study quasiconformal mappings

A homeomorphism $f : D \rightarrow D'$ is K -quasiconformal if

$$H_f(x) \leq K$$

in D . One must integrate this local condition to get global properties for f . When $n = 2$, the Cauchy integral formula and its Pompeiu extension are available. When $n > 2$, the Ahlfors–Beurling method of extremal length has proved to be an effective tool for this purpose.

DEFINITION 4.1. Given a family Γ of curves $\gamma \subset \mathbb{R}^n$, we denote by $\text{adm}(\Gamma)$ the family of nonnegative Borel measurable functions ρ in $\mathbb{R}^n$ such that

$$\int_{\gamma} \rho \, ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma$. We call

$$\text{mod}(\Gamma) = \inf_{\rho} \int_{\mathbb{R}^n} \rho^n dm, \quad \rho \in \text{adm}(\Gamma),$$

the *conformal modulus* and $\lambda(\Gamma) = \text{mod}(\Gamma)^{1/(1-n)}$ the *extremal length* of Γ .

When Γ is a family of disjoint arcs in $\mathbb{R}^2$, we may think of $\text{mod}(\Gamma)$ as the electrical transconductance and $\lambda(\Gamma)$ as the resistance of a system of homogeneous electric wires $\gamma \in \Gamma$. Hence, $\text{mod}(\Gamma)$ is large if the wires are short or plentiful and small if they are long or scarce.

THEOREM 4.2. *$\text{mod}(\Gamma)$ is an outer measure on the curve families in $\overline{\mathbb{R}^n}$:*

- (i) $\text{mod}(\emptyset) = 0$,
- (ii) $\text{mod}(\Gamma_1) \leq \text{mod}(\Gamma_2)$ if $\Gamma_1 \subset \Gamma_2$,
- (iii) $\text{mod}(\bigcup \Gamma_j) \leq \sum_j \text{mod}(\Gamma_j)$.

We give two examples of curve families Γ which are useful in the study of quasiconformal mappings. See [50,126,146,171,175].

EXAMPLE 4.3. If Γ is a family of arcs in $\mathbb{R}^n$ which join concentric spheres of radii a and b , where $a < b$, then

$$\text{mod}(\Gamma) \leq \omega_{n-1} \left(\log \frac{b}{a} \right)^{1-n},$$

where $\omega_{n-1} = m_{n-1}(\partial \mathbb{B}^n)$.

EXAMPLE 4.4. Suppose that S_1 and S_2 are concentric spheres of radii a and b , where $a \leq b$, and that C_1 and C_2 are continua in $\overline{\mathbb{R}^n}$ which join 0 to S_1 and ∞ to S_2 , respectively. If Γ is the family of arcs joining C_1 to C_2 , then

$$\text{mod}(\Gamma) \geq \omega_{n-1} \left(\log \sigma_n \left(\frac{b}{a} + 1 \right) \right)^{1-n},$$

where σ_n depends only on n .

Geometric properties for a quasiconformal mapping f can be obtained by applying the following result to $\text{mod}(\Gamma)$ and $\text{mod}(f(\Gamma))$ for appropriate curve families Γ .

THEOREM 4.5. *If $f : D \rightarrow D'$ is K -quasiconformal, then*

$$K^{1-n} \text{mod}(\Gamma) \leq \text{mod}(f(\Gamma)) \leq K^{n-1} \text{mod}(\Gamma) \tag{4.1}$$

for all families Γ in D .

A homeomorphism $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is K -quasiconformal if

$$\limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)} \leq K \quad (4.2)$$

for all $x \in \mathbb{R}^n$. We illustrate the method mentioned above by showing how Examples 4.3 and 4.4 and Theorem 4.5 can be used to establish a global form of inequality (4.2).

THEOREM 4.6. *If $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is K -quasiconformal, then*

$$\frac{L_f(x, r)}{l_f(x, r)} \leq c \quad (4.3)$$

for all $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $c = c(K, n)$.

PROOF. By means of preliminary similarity transformations, we may assume that $x = 0$ and $f(0) = 0$. Fix $0 < r < \infty$, let

$$a = \min_{|x|=r} |f(x)| = l_f(0, r), \quad b = \max_{|x|=r} |f(x)| = L_f(0, r)$$

and suppose that $a < b$. Next set

$$C_1 = \{x \in \mathbb{R}^n: |f(x)| \leq a\}, \quad C_2 = \{x \in \mathbb{R}^n: |f(x)| \geq b\} \cup \{\infty\},$$

and let Γ be the family of arcs which join C_1 and C_2 in $\mathbb{R}^n \setminus (C_1 \cup C_2)$. Then inequality (4.1) and the estimates in Examples 4.3 and 4.4 imply that

$$\begin{aligned} \omega_{n-1}(\log 2\sigma_n)^{1-n} &\leq \text{mod}(\Gamma) \leq K^{n-1} \text{mod}(f(\Gamma)) \\ &\leq K^{n-1} \omega_{n-1} \left(\log \frac{b}{a} \right)^{1-n} \end{aligned}$$

and we obtain (4.3) with $c = (2\sigma_n)^K$. □

We use Theorem 4.5 to extend the notion of quasiconformality as follows.

DEFINITION 4.7. A homeomorphism $f: D \rightarrow D'$ is K -quasiconformal if it satisfies the inequalities in (4.1) for all families Γ in D . We denote by $K(f)$ the minimum K for which f is K -quasiconformal.

The quasiconformal mappings considered by Grötzsch and Teichmüller were continuously differentiable except in a finite set. Ahlfors and Bers introduced the larger class defined above so that the class of K -quasiconformal mappings would be closed under locally uniform convergence.

Some of the analytic properties of this extended class of mappings are listed below.

THEOREM 4.8. Suppose that f is K -quasiconformal in D . Then:

- (i) f is differentiable with Jacobian $J_f \neq 0$ a.e. in D ,
- (ii) f is in the Sobolev class $W_{1,\text{loc}}^n(D)$,
- (iii) $m(E) = 0$ implies $m(f(E)) = 0$.

If $K = 1$ and f is sense preserving, then f is conformal and hence, when $n > 2$, the restriction to D of a Möbius transformation.

One cannot say much more about the analytic properties of a quasiconformal mapping f . For example, for each $K > 1$, there exists a K -quasiconformal mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is differentiable with a vanishing Jacobian at each point of a set with Hausdorff dimension n . See [65].

The criterion analogous to Definition 1.3 which corresponds to the class of quasiconformal mappings defined above is as follows [51,171].

DEFINITION 4.9. A homeomorphism $f: D \rightarrow D'$ is K -quasiconformal if

$$H_f(x) < \infty \tag{4.4}$$

for $x \in D \setminus E_1$ where E_1 is of σ -finite $(n-1)$ -measure, and

$$H_f(x) \leq K \tag{4.5}$$

for $x \in D \setminus E_2$ where $m(E_2) = 0$.

It is surprising and important for certain applications of quasiconformal mappings, to observe that the function

$$H_f(x) = \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)}$$

may be replaced by

$$h_f(x) = \liminf_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)}$$

in inequalities (4.4) and (4.5), in Definition 4.9. See [76,91,92].

5. Mapping problems

Suppose that D and D' are domains in $\overline{\mathbb{R}^n}$. Two basic problems are as follows.

1. How can we decide if D and D' are quasiconformally equivalent, that is, if there exists a quasiconformal mapping f which maps D onto D' ?
2. If D and D' are quasiconformally equivalent, what can we say about the quasiconformal f which maps D onto D' with minimum $K(f)$?

Both problems are quite difficult and there exist only partial answers even for the case where $n = 2$. We give below some results concerning quasiconformal equivalence for the case where D' is the unit ball $\mathbb{B}^n$.

THEOREM 5.1. *When $n = 2$, D is quasiconformally equivalent to $\mathbb{B}^2$ if and only if ∂D is a nondegenerate continuum.*

The sufficiency in Theorem 5.1 is an immediate consequence of the Riemann mapping theorem. The necessity follows from an extremal length argument using the family of curves in Example 4.3.

The following example shows that when $n > 2$, unlike the situation in Theorem 5.1, one *cannot* characterize the domains quasiconformally equivalent to $\mathbb{B}^n$ in terms of their boundaries.

EXAMPLE 5.2. Suppose $n > 2$ and let $D = \mathbb{R}^n \setminus \bar{D}^*$ where

$$D^* = \{x = (x_1, \dots, x_n): x_1 > 0, |x_2| < 1\}.$$

Then D is quasiconformally equivalent to $\mathbb{B}^n$ but D^* is not.

The domain D in Example 5.2 can be folded 2-quasiconformally onto the half space

$$\mathbb{H} = \{x = (x_1, \dots, x_n): x_1 > 0\}$$

which, in turn, can be mapped onto $\mathbb{B}^n$ by means of a Möbius transformation. Hence D is quasiconformally equivalent to $\mathbb{B}^n$.

On the other hand, estimates for the moduli of the curve families in Examples 4.3 and 4.4 yield the following *necessary* condition for D to be quasiconformally equivalent to $\mathbb{B}^n$ when $n > 2$, a condition which is not satisfied by the domain D^* in Example 5.2. See [64].

THEOREM 5.3. *When $n > 2$, D is quasiconformally equivalent to $\mathbb{B}^n$ only if there exists a constant $c > 1$ such that, for $x \in \mathbb{R}^n$ and $r > 0$,*

- (i) $\bar{D}^* \cap \bar{\mathbb{B}}^n(x, r)$ lies in a component of $\bar{D}^* \cap \bar{\mathbb{B}}^n(x, cr)$,
 - (ii) $\bar{D}^* \setminus \bar{\mathbb{B}}^n(x, r)$ lies in a component of $\bar{D}^* \setminus \bar{\mathbb{B}}^n(x, r/c)$,
- where $D^* = \mathbb{R}^n \setminus \bar{D}$.

We saw above that when $n > 2$ one cannot characterize the domains D quasiconformally equivalent to $\mathbb{B}^n$ in terms of their boundaries ∂D . However, the following result shows one can characterize such domains D in terms of the part of D near ∂D [53].

THEOREM 5.4. *When $n > 2$, D is quasiconformally equivalent to $\mathbb{B}^n$ if there exists a neighborhood U of ∂D and a quasiconformal mapping g of $D \cap U$ into $\mathbb{B}^n$ such that $g(x) \rightarrow \partial \mathbb{B}^n$ as $x \rightarrow \partial D$ in $D \cap U$.*

The proof of Theorem 5.4 is based on methods used by Brown [32] and Mazur [119] to establish an n -dimensional version of the Schoenflies theorem when $n > 2$.

DEFINITION 5.5. $D \subset \overline{\mathbb{R}^n}$ is a *Jordan domain* if ∂D is homeomorphic to $\partial \mathbb{B}^n$.

A local version of Theorem 5.4 holds when D is a Jordan domain [49,53,176]. See also [32].

THEOREM 5.6. *When $n > 2$, a Jordan domain D is quasiconformally equivalent to $\mathbb{B}^n$ if, for each $y \in \partial D$, there exists a neighborhood U and a quasiconformal mapping g of $D \cap U$ into $\mathbb{B}^n$ such that $g(x) \rightarrow \partial \mathbb{B}^n$ as $x \rightarrow \partial D$ in $D \cap U$.*

The notion of internally chord-arc domains in $\mathbb{R}^2$ allows us to characterize the infinite cylinders D in $\mathbb{R}^3$ which are quasiconformally equivalent to $\mathbb{B}^3$. See [173].

DEFINITION 5.7. A Jordan domain $G \subset \mathbb{R}^2$ is *internally chord-arc* if there exists a constant $c \geq 1$ such that, for each open arc $\alpha \subset G$ with endpoints in ∂G , there exists an arc $\beta \subset \partial G$ with the same endpoints as α such that

$$l(\beta) \leq cl(\alpha).$$

THEOREM 5.8. *If $G \subset \mathbb{R}^2$ is a Jordan domain, then the cylinder $D = G \times \mathbb{R}^1$ is quasiconformally equivalent to $\mathbb{B}^3$ if and only if G is internally chord-arc.*

A domain $D \subset \mathbb{R}^3$ has a *flat* boundary if ∂D lies in a two-dimensional plane. A second class of plane domains, quasidisks, yields a characterization for the domains D in $\mathbb{R}^3$ with flat boundaries which are quasiconformally equivalent to B^3 . See [54].

DEFINITION 5.9. A Jordan domain $D \subset \overline{\mathbb{R}^2}$ is a *quasidisk* if it is the image of $\mathbb{B}^2$ under a quasiconformal self mapping of $\overline{\mathbb{R}^2}$.

THEOREM 5.10. *If D is a domain in $\mathbb{R}^3$ with flat boundary, then D is quasiconformally equivalent to $\mathbb{B}^3$ if and only if D is a half space or ∂D is a quasidisk.*

A little is known about extremal maps f which minimize $K(f)$ or related dilatations when $n > 2$. Indeed an example suggests they need *not* be C^2 . See [61].

6. Extensions of mappings

Suppose that $f: D \rightarrow \mathbb{B}^n$ is quasiconformal. We consider here the following two questions:

1. When does f have a homeomorphic extension to $\overline{D}$?
2. When does f have a quasiconformal extension to $\overline{\mathbb{R}^n}$?

The following theorem due to Väisälä [170] gives a complete answer to the first question. It is the analogue for quasiconformal mappings of a well-known theorem concerning the extension of conformal mappings due to Carathéodory [34]. See also [103].

THEOREM 6.1. *If $f : D \rightarrow \mathbb{B}^n$ is quasiconformal, then f has a homeomorphic extension $f^* : \overline{D} \rightarrow \overline{\mathbb{B}^n}$ if and only if D is a Jordan domain.*

Less is known concerning answers to the second question except when $n = 2$.

DEFINITION 6.2. A Jordan curve $C \subset \overline{\mathbb{R}^2}$ is a *quasicircle* if it is the image of $\partial \mathbb{B}^2$ under a quasiconformal self map of $\overline{\mathbb{R}^2}$.

Hence quasicircles are the boundaries of the quasidisks considered in Definition 5.9. The following elegant characterization for quasicircles is due to Ahlfors [3].

THEOREM 6.3. *A Jordan curve $C \subset \overline{\mathbb{R}^2}$ is a quasicircle if and only if there exists a constant $c \geq 1$ such that, for all $z_1, z_2 \in C \setminus \{\infty\}$,*

$$\min(\text{dia}(C_1), \text{dia}(C_2)) \leq c|z_1 - z_2|,$$

where C_1 and C_2 are the components of $C \setminus \{z_1, z_2\}$.

See [59] and [60] for many other characterizations of quasidisks. A number of these reflect the surprisingly many different ways quasiconformal mappings interact with other parts of mathematics.

We then have the following answers for the second problem concerning quasiconformal extensions to $\mathbb{R}^n$.

THEOREM 6.4. *When $n = 2$, a quasiconformal mapping $f : D \rightarrow \mathbb{B}^2$ has a quasiconformal extension f^* to $\overline{\mathbb{R}^2}$ if and only if ∂D is a quasicircle.*

THEOREM 6.5 [52,172]. *When $n > 2$, a quasiconformal mapping $f : D \rightarrow \mathbb{B}^n$ has a quasiconformal extension f^* to $\overline{\mathbb{R}^n}$ if and only if $D^* = \overline{\mathbb{R}^n} \setminus \overline{D}$ is a Jordan domain which is also quasiconformally equivalent to $\mathbb{B}^n$.*

Theorem 6.5 does not hold when $n = 2$. For in this case D^* is conformally equivalent to $\mathbb{B}^2$ whenever D is a Jordan domain while a quasiconformal mapping $f : D \rightarrow \mathbb{B}^2$ will have a quasiconformal extension to $\overline{\mathbb{R}^2}$ only if ∂D is a quasicircle and hence satisfies the geometric condition in Theorem 6.3.

The hypotheses in Theorem 6.5 are unfortunately quite implicit. It would be interesting to find a geometric characterization for quasispheres analogous to that given above in Theorem 6.3 for quasicircles.

7. Boundary correspondence and lifting

For $n \geq 2$ let $\mathbb{H}^n$ denote the upper half space

$$\mathbb{H}^n = \{x = (x_1, \dots, x_n) : x_n > 0\}.$$

If $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is K -quasiconformal, then f has a quasiconformal extension, also denoted by f , to $\overline{\mathbb{R}^n}$ and $g = f|_{\partial\mathbb{H}^n}$ is a self-homeomorphism of $\partial\mathbb{H}^n$. Then

$$l_f(x, r) \leq l_g(x, r) \leq L_g(x, r) \leq L_f(x, r)$$

for $x \in \partial\mathbb{H}^n \setminus \{\infty\}$ and hence

$$H_g(x) = \limsup_{r \rightarrow 0} \frac{L_g(x, r)}{l_g(x, r)} \leq \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)} = H_f(x) \leq c(K) < \infty.$$

Thus when $n > 2$, the boundary correspondence g induced by an n -dimensional self-quasiconformal mapping of $\mathbb{H}^n$ is an $(n - 1)$ -dimensional quasiconformal mapping [51,126]. When $n = 2$, the boundary correspondence g is a quasisymmetric or one-dimensional quasiconformal mapping [22].

The following result shows that the converse is true, i.e., quasiconformal self-mappings of $\partial\mathbb{H}^n$ are the boundary correspondences of n -dimensional quasiconformal mappings.

THEOREM 7.1. *Each quasiconformal mapping $g : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^n$ is the boundary correspondence for a quasiconformal mapping $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$.*

The proof for the case $n = 2$ was given by Beurling and Ahlfors in a fundamental paper [22] in 1956. Their argument was based on an integral representation which they then used to show that the boundary correspondence for a plane quasiconformal self-mapping of the unit disk $\mathbb{B}^2$ need not be absolutely continuous. An alternative *conformally natural* extension was later given by Douady and Earle in [40]. See also [100,132].

Ahlfors [4] established the result when $n = 3$ in 1963 using the decomposition theorem for plane quasiconformal mappings given in Corollary 8.2. The proof for $n = 4$ was given by Carleson [35] in 1974 using work of Moise in three-dimensional topology [122,123].

Finally Tukia and Väisälä [166] started from an idea of Carleson's and employed results of Sullivan [155] to prove the general result for $n > 2$ in 1980.

8. Measurable Riemann mapping theorem

If f is quasiconformal in D , then f has a nonsingular differential

$$df = df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

at a.e. $x \in D$. Because df is linear, there exists a unique ellipsoid $S = S(x) = S_f(x)$ about the origin 0 with semiaxes

$$1 = a_1 \leq a_2 \leq \cdots \leq a_n = H_f(x)$$

which df maps onto $\partial\mathbb{B}^n(0, r)$. Then the family $\{S_f(x)\}$ of such ellipsoids determines f up to composition by a conformal mapping.

THEOREM 8.1. *Suppose that $n = 2$ and that $S = S(x)$ is a family of such ellipsoids whose semiaxes and orientation are measurable functions of $x \in D$ with $a_n = a_n(x) \leq K$. Then there exists a mapping f which is K -quasiconformal in D with $S_f = S$ a.e. in D .*

When $n > 2$, there is no analogue for Theorem 8.1 because the corresponding system of partial differential equations is overdetermined. Can one get a weaker result where one specifies, for example, only the maximum semiaxis $a_n(x) = H_f(x)$ a.e. in D ?

The following decomposition theorem for plane quasiconformal mappings is an important consequence of Theorem 8.1.

COROLLARY 8.2. *When $n = 2$, given $\varepsilon > 0$ each f which is K -quasiconformal in D can be written in the form*

$$f = f_1 \circ f_2 \circ \cdots \circ f_m,$$

where $K(f_j) < 1 + \varepsilon$ and $m = m(\varepsilon, K)$.

When $n > 2$, examples suggest that Corollary 8.2 is not true without further restrictions on the domain D . Does it hold when $D = \mathbb{B}^n$ or $D = \mathbb{R}^n$?

9. Distortion and equicontinuity

The Hölder continuity and equicontinuity properties for quasiconformal mappings are consequences of the following distortion theorem.

THEOREM 9.1. *If $f : D \rightarrow D'$ is a K -quasiconformal mapping and if $\partial D \neq \emptyset$, then*

$$q(f(x), f(y))q(\overline{\mathbb{R}^n} \setminus D') \leq a \left(\frac{q(x, y)}{q(x, \partial D)} \right)^b$$

for $x, y \in D$, where $a = a(n)$, $b = K^{1/(1-n)}$ and q is the chordal metric.

Thus a K -quasiconformal mapping $f : D \rightarrow D'$ is $K^{1/(1-n)}$ Hölder continuous with respect to the chordal metric in D and with respect to the euclidean metric in $D \setminus \{\infty, f^{-1}(\infty)\}$. If $0 < b < 1$, then

$$f(x) = |x|^{b-1}x$$

is K -quasiconformal, where $K = b^{1-n}$. Hence the exponent $K^{1/(1-n)}$ is sharp.

The following equicontinuity criterion is an important consequence of Theorem 9.1. See [103].

THEOREM 9.2. *If $r > 0$ and if $\mathcal{F}$ is a family of mappings f which are K -quasiconformal in D and which omit two values a_f and b_f , where $q(a_f, b_f) \geq r$, then the mappings in $\mathcal{F}$ are equicontinuous in D .*

Theorem 9.2 implies the following compactness property for quasiconformal mappings.

THEOREM 9.3. *If $f_j : D \rightarrow D'_j$ are K -quasiconformal and if $f_j \rightarrow f$ pointwise in D , then one of following is true:*

- (1) *f is a homeomorphism and the convergence is locally uniform in D .*
- (2) *f assumes only two values, one only at one point.*
- (3) *f is constant.*

Theorem 9.3 yields the following compactness criterion for quasiconformality due to Beurling and Ahlfors. See [22] and [51].

Suppose that $\mathcal{F}$ is a family of homeomorphisms $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and that P_1, P_2 are a pair of fixed points in $\mathbb{R}^n$. We say that:

- (i) $f \in \mathcal{F}$ is *normalized* if $f(P_j) = P_j$ for $j = 1, 2$,
- (ii) $\mathcal{F}$ is *complete* if $S \circ f \circ T \in \mathcal{F}$ whenever $f \in \mathcal{F}$ and S, T are similarity mappings,
- (iii) $\mathcal{F}$ satisfies *condition (A)* if each infinite sequence of normalized mappings in $\mathcal{F}$ contains a subsequence which converges to a homeomorphism.

THEOREM 9.4. *A complete family $\mathcal{F}$ of homeomorphisms $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies condition (A) if and only if each $f \in \mathcal{F}$ is K -quasiconformal for some fixed K .*

10. Properties of the Jacobian

If $f : D \rightarrow D'$ is a homeomorphism, where $D, D' \subset \mathbb{R}^n$, and if f is differentiable a.e. in D , then

$$\int_E J_f dm \leq m(f(E)) < \infty \quad (10.1)$$

for each compact $E \subset D$. Inequality (10.1) implies that J_f is locally L^1 -integrable in D and globally integrable if $m(D') < \infty$. Stronger conclusions hold when f is quasiconformal.

THEOREM 10.1. *If $f : D \rightarrow D'$ is K -quasiconformal, where $D, D' \subset \mathbb{R}^n$, then J_f is locally L^p integrable in D for $1 < p < p(K, n)$, where*

$$p(K, n) \leq \frac{K}{K-1} \quad (10.2)$$

and $p(K, n) \rightarrow \infty$ as $K \rightarrow 1$.

Bojarski [23] established Theorem 10.1 in for the case $n = 2$ in 1955 by applying an inequality due to Caldéron and Zygmund to the Beurling transform and an infinite series representation for f . Astala showed that (10.2) holds with equality when $n = 2$ [12].

The proof for the case where $n > 2$ derives the higher integrability from the fact that J_f satisfies a reverse Hölder inequality on small cubes $Q \subset D$ [55]. See Theorem 14.1 in Section 14. Reshetnyak [140] showed in this case that $p(K, n) \rightarrow \infty$ as $K \rightarrow 1$.

If $m(D') < \infty$, then J_f is globally L^p integrable for some $p > 1$ when f is quasiconformal and D' satisfies an additional geometric condition. See [14, 117, 154].

For an entirely different view of the Jacobian of a quasiconformal mapping, see [29]. There the authors characterize up to a multiplicative constant the positive continuous functions defined on $\mathbb{B}^2$ that arise as averaged Jacobians of quasiconformal self-mappings f of $\mathbb{R}^2$.

11. Connections with functional analysis

Quasiconformal mappings are related to several different classes of functions which arise in functional analysis. We consider two of these here – functions of bounded mean oscillation [45, 48, 88, 134] and Royden algebras [144, 145].

We begin with functions of bounded mean oscillation.

DEFINITION 11.1. A function u locally integrable in $D \subset \mathbb{R}^n$ is said to be of *bounded mean oscillation* in D or in $BMO(D)$ if

$$\|u\|_D^* = \sup_{B \subset D} \frac{1}{m(B)} \int_B |u - u_B| dm < \infty, \quad u_B = \frac{1}{m(B)} \int_B u dm,$$

where the supremum is taken over all balls $B \subset D$.

If D is bounded, then

$$L^\infty(D) \subset BMO(D) \subset \text{loc} L^p(D) \quad (11.1)$$

for $1 \leq p < \infty$ [88]. The following results due to Reimann, Astala and Jones illustrate how quasiconformal mappings are related to the class BMO .

THEOREM 11.2 [133]. *If $f : D \rightarrow D'$ is K -quasiconformal, where $D, D' \subset \mathbb{R}^n$, then*

$$\|\log J_f\|_D^* \leq c = c(K).$$

In particular, we see that if J_f is the Jacobian of a quasiconformal mapping $f : D \rightarrow D'$, then $\log J_f$ is locally L^p integrable in D for $1 \leq p < \infty$.

THEOREM 11.3 [133]. *If $f : D \rightarrow D'$ is quasiconformal, where $D, D' \subset \mathbb{R}^n$, then there exist a constant c such that*

$$\frac{1}{c} \|u \circ f\|_D^* \leq \|u\|_{D'}^* \leq c \|u \circ f\|_D^* \quad (11.2)$$

for each function u continuous in D' .

It is possible that a homeomorphism $f : D \rightarrow D'$ is quasiconformal provided that inequality (11.2) holds for all u continuous in D' . This is true if f satisfies certain a priori analytic conditions [133] or if

$$\frac{1}{c} \|u \circ f\|_G^* \leq \|u\|_{G'}^* \leq c \|u \circ f\|_G^* \quad (11.3)$$

for each subdomain G of D and each function u continuous in $G' = f(G)$ [11].

Inequality (11.1) and the fact that

$$L^\infty(D) = L^\infty(\mathbb{R}^n)|_D$$

suggest the same extension property might hold for the class $BMO(D)$. The following shows that this is not always the case.

THEOREM 11.4 [89]. *If $D \subset \mathbb{R}^2$ is finitely connected, then*

$$BMO(D) = BMO(\mathbb{R}^2)|_D$$

if and only if each component of ∂D is a point or a quasicircle.

The same type of extension theorem also holds for the Sobolev class $W_1^2(D)$ of functions f with weak first-order derivatives that are L^2 integrable in D .

THEOREM 11.5 [69,90]. *If $D \subset \mathbb{R}^2$ is finitely connected, then*

$$W_1^2(D) = W_1^2(\mathbb{R}^2)|_D$$

if and only if each component of ∂D is a point or a quasicircle.

We turn next to the connection between quasiconformal mappings and Royden algebras. See [144] and [145].

DEFINITION 11.6. Given $D \subset \mathbb{R}^n$ let $A(D)$ denote the algebra under pointwise addition and multiplication of continuous functions u in the Sobolev class $W_1^n(D)$ with norm

$$\|u\| = \sup_D |u| + \left(\int_D |\nabla u|^n dm \right)^{1/n}.$$

Then $A(D)$ is the *Royden algebra* of D .

The following result due to Nakai [128] and Lewis [107] relates the quasiconformal equivalence of domains to the structure of their Royden algebras. See also [105].

THEOREM 11.7. *Suppose D and D' are domains in $\mathbb{R}^n$. Then there exists a quasiconformal mapping $f : D \rightarrow D'$ if and only if $A(D)$ and $A(D')$ are isomorphic as algebras.*

Thus geometric methods for determining the quasiconformal equivalence of domains in $\mathbb{R}^n$ can be used to study their Royden algebras while analytic methods applied to these algebras yield criteria for the quasiconformal equivalence of domains. See, for example, [150–152].

12. Connections with geometry and elasticity

We consider here how the notion of quasiconformality is related to two questions concerning bi-Lipschitz mappings between sets $E, E' \subset \mathbb{R}^n$.

DEFINITION 12.1. A mapping $f : E \rightarrow E'$ is *L-bi-Lipschitz* if

$$\frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y|$$

for $x, y \in E$; f is *locally L-bi-Lipschitz* if each $x \in E$ has a neighborhood U such that f is *L-bi-Lipschitz* in $E \cap U$.

If f is bi-Lipschitz in a domain D in $\mathbb{R}^n$ then f is quasiconformal in D . The converse is not true. For example,

$$f(x) = |x|^{b-1}x$$

is K -quasiconformal but not bi-Lipschitz in $\mathbb{R}^n$ when $b = K^{1-n}$. Nevertheless, quasiconformal mappings arise in questions concerning the extension of and injectivity of bi-Lipschitz mappings.

We begin with the extension problem. Suppose that $f : E \rightarrow E'$ is bi-Lipschitz, where $E, E' \subset \mathbb{R}^n$. When does f have a bi-Lipschitz extension to $\mathbb{R}^n$?

THEOREM 12.2 [167]. *When $n \geq 2$, a bi-Lipschitz mapping $f : E \rightarrow E'$ has a bi-Lipschitz extension to $\mathbb{R}^n$ if and only if f has a quasiconformal extension to $\mathbb{R}^n$.*

Theorem 12.2 gives a criterion for bi-Lipschitz extension in terms of the mapping f . There is also a criterion for extension in terms of the set E when E is a Jordan curve.

THEOREM 12.3 [58,162]. *If $E \subset \mathbb{R}^2$ is a Jordan curve, then every bi-Lipschitz mapping $f : E \rightarrow E'$ has a bi-Lipschitz extension to $\mathbb{R}^2$ if and only if E is a quasicircle.*

We turn next to the question of injectivity. Suppose that f is locally bi-Lipschitz in a domain $D \subset \mathbb{R}^2$. When is f injective in D ?

DEFINITION 12.4. For $D \subset \mathbb{R}^n$, let $L(D)$ denote the supremum of the numbers $L \geq 1$ such that each mapping f locally L -bi-Lipschitz in D is injective in D .

The constant $L(D)$ has a physical interpretation if we think of D as a homogeneous elastic body and f as the deformation experienced by D when subjected to a force field. Requiring that f be locally L -bi-Lipschitz bounds the strain in D under the force field. Then $L(D)$ measures the critical strain in D before D collapses onto itself.

Little is known about this constant except, for example, that

$$2^{1/4} \leq L(D) \leq 2^{1/2}$$

whenever D is a ball or half space [66,85–87]. However, the following yields a large class of *rigid* plane domains, domains for which $L(D) > 1$.

THEOREM 12.5 [57]. *A finitely connected proper subdomain D of $\mathbb{R}^2$ is rigid if and only if each component of ∂D is a point or a quasicircle.*

The following result shows that in the case of the plane, the questions of injectivity and bi-Lipschitz extension are closely related.

COROLLARY 12.6. *If D is a bounded simply connected domain in $\mathbb{R}^2$ and if f is locally L -bi-Lipschitz in D with $L < L(D)$, then f has an M -bi-Lipschitz extension to $\mathbb{R}^2$, where $M = M(L, L(D))$.*

Corollary 12.6 says that the shape of a deformed simply connected plane elastic body D is roughly the same as that of the original provided the strain in D does not attain the critical value $L(D)$. It would be interesting to obtain a higher-dimensional analogue of this result.

13. Connections with complex analysis

Quasiconformal mappings sometimes arise in function-theoretic problems which appear to be completely unrelated to this class. A good example is Teichmüller's theorem [160] which relates the extremal quasiconformal mappings between Riemann surfaces with the quadratic differentials on these surfaces.

For a more elementary example suppose f is analytic in a simply connected domain D of hyperbolic type in $\overline{\mathbb{R}^2}$ and let S_f denote the Schwarzian derivative of f

$$S_f = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2.$$

If $S_f = 0$ in D , then f is the restriction of a Möbius transformation to D and hence injective.

The following result of Lehto [101] extends the above relation between the injectivity of f and the size of S_f relative to the hyperbolic density

$$\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2},$$

where $g : D \rightarrow \mathbb{B}$ is conformal. See also [97,102].

THEOREM 13.1. *If f is analytic and injective in a simply connected domain D , then*

$$\|S_f(z)\|_D \leq 3, \quad (13.1)$$

where

$$\|S_f\|_D = \sup_{z \in D} |S_f(z)| \rho_D(z)^{-2}.$$

Thus an inequality of the form

$$\|S_f\|_D \leq a,$$

a an absolute constant, is a necessary condition for an analytic function to be injective. Nehari [129] showed that the same kind of inequality is also a sufficient condition when D is a disk.

THEOREM 13.2. *If f is analytic in a disk or half plane D , then f is injective whenever*

$$\|S_f(z)\|_D \leq \frac{1}{2}. \quad (13.2)$$

DEFINITION 13.3. For $D \subset \mathbb{R}^2$ let $\sigma(D)$ denote the supremum of the numbers $a \geq 0$ such that each f analytic in D is injective whenever $\|S_f\|_D \leq a$.

It is natural to ask for which domains $D \subset \mathbb{R}^2$ is $\sigma(D) > 0$ since these are the conformal analogues of the rigid domains D in Section 12 with $L(D) > 1$.

THEOREM 13.4 [3,56]. $\sigma(D) > 0$ if and only if ∂D is a quasicircle.

Quasiconformal mappings also play an unexpected role in the remarkable λ -lemma of Mañé, Sad and Sullivan concerning holomorphic motions [111] and in its subsequent extensions [21,149,159]. See also [15].

DEFINITION 13.5. Suppose that E is a subset of $\overline{\mathbb{R}^2}$. Then a map $f = f(z, w) : \mathbb{B}^2 \times E \rightarrow \overline{\mathbb{R}^2}$ is a *holomorphic motion* of E if

- (i) $f(z, w)$ is analytic in $\mathbb{B}^2$ for each fixed $w \in E$,
- (ii) $f(z, w)$ is injective in E for each fixed $z \in \mathbb{B}^2$,
- (iii) $f(0, w) = w$ for $w \in E$.

The surprising fact concerning holomorphic motions $f = f(z, w)$ of a set E is that even though no continuity in w is assumed, they extend to holomorphic motions of $\mathbb{R}^2$ and that for each fixed $z \in \mathbb{B}^2$ the corresponding injections are quasiconformal mappings in w of the extended plane.

THEOREM 13.6. *If $f : \mathbb{B}^2 \times E \rightarrow \overline{\mathbb{R}^2}$ is a holomorphic motion, then f has an extension $f^* : \mathbb{B}^2 \times \overline{\mathbb{R}^2} \rightarrow \overline{\mathbb{R}^2}$ such that:*

- (i) *f^* is a holomorphic motion of $\overline{\mathbb{R}^2}$,*
- (ii) *$f^*(z, w)$ is a quasiconformal self mapping of $\overline{\mathbb{R}^2}$ for each fixed $z \in \mathbb{B}^2$,*
- (iii) *$f^*(z, w)$ is a continuous function of (z, w) .*

14. Connections with differential equations

There are several examples of the interaction of quasiconformal mappings and partial differential equations [84].

The first proofs for the Liouville theorem in $\mathbb{R}^n$ with minimal hypotheses, e.g., Theorem 1.2 in Section 2, were based on important regularity theorems, due to De Giorgi [39], Morrey [124], Moser [125] and Ural'tseva [168], applied to weak solutions of the p -harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad (14.1)$$

with $p = n$.

Next the fact that the solutions of (14.1) with $p = n$ are counterparts in $\mathbb{R}^n$ of the usual harmonic functions in $\mathbb{R}^2$ stimulated subsequent research on solutions of (14.1) for general p . See [24,75,81,108,109].

The following theorem on reverse Hölder inequalities [55] was used to establish the higher integrability of the Jacobian of a quasiconformal mapping in Section 10.

THEOREM 14.1. *Suppose that $D \subset \mathbb{R}^n$ is a domain and that $1 < p < \infty$. If*

$$\left(\frac{1}{m(Q)} \int_Q |g|^p dm \right)^{1/p} \leq \frac{a}{m(Q)} \int_Q |g| dm$$

for each cube $Q \subset D$, then there exist constants b and $q > p$, which depend only on a , p and n , such that

$$\left(\frac{1}{m(Q)} \int_Q |g|^q dm \right)^{1/q} \leq \frac{b}{m(Q)} \int_Q |g| dm$$

for each cube $Q \subset D$.

This result has turned out to be a useful tool in the study of partial differential equations [68,82].

The bound below for the distortion of area under a quasiconformal mapping was conjectured in [63] and established by Astala [12] and then by Eremenko and Hamilton [44].

THEOREM 14.2. *If $f : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ is K -quasiconformal, then*

$$\frac{m(f(E))}{\pi} \leq c(K) \left(\frac{m(E)}{\pi} \right)^{1/K} \quad (14.2)$$

for each measurable set $E \subset \mathbb{B}^2$.

Inequality (14.2) was subsequently applied by Nesi [130], by Astala and Miettinen [16] and by Milton and Nesi [121] to obtain sharp bounds for the conductivity, e.g., heat conductivity, magnetic permeability or stiffness, of composite materials having two or more phases with different physical properties.

15. Connections with topology

The fact that each quasiconformal mapping $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ induces a quasiconformal boundary correspondence $g : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^n$ was a key step in the original proof of Mostow's important rigidity theorem.

THEOREM 15.1 [126]. *When $n > 2$, two compact Riemannian n -manifolds of constant negative curvature are diffeomorphic if and only if they are conformally equivalent.*

Sullivan considered quasiconformal versions of the following three basic theorems of point set topology:

1. *Schoenflies theorem*: A collared topological $(n - 1)$ -sphere bounds a topological n -ball.
2. *Annulus theorem*: Two disjoint collared topological $(n - 1)$ -spheres bound a topological n -annulus.
3. *Component problem*: An orientation preserving homeomorphism of a topological n -ball into $\mathbb{R}^n$ is connected to the identity by a path of homeomorphisms.

He proved in [155] that each of these results hold in the quasiconformal context in all dimensions. This is not true for the class of diffeomorphisms. Hence the quasiconformal category appears to be a natural replacement for the class of diffeomorphisms in this case.

16. Connections with discrete groups

The study of discrete groups of uniformly quasiconformal mappings has also had some unexpected applications in topology.

DEFINITION 16.1. A family $\mathcal{F}$ of self-homeomorphisms of $\overline{\mathbb{R}}^n$ has the *convergence property* if each infinite subfamily contains a sequence f_j such that either

- (i) there exists a homeomorphism f such that

$$f_j \rightarrow f \quad \text{and} \quad f_j^{-1} \rightarrow f^{-1}$$

as $j \rightarrow \infty$, uniformly in $\overline{\mathbb{R}^n}$, or

(ii) there exist points $x_0, y_0 \in \overline{\mathbb{R}^n}$, possibly equal, so that

$$f_j \rightarrow y_0 \quad \text{and} \quad f_j^{-1} \rightarrow x_0$$

as $j \rightarrow \infty$, uniformly in $\overline{\mathbb{R}^n} \setminus \{x_0\}$ and in $\overline{\mathbb{R}^n} \setminus \{y_0\}$, respectively.

Theorem 9.3 implies that each family of K -quasiconformal self-homeomorphisms of $\overline{\mathbb{R}^n}$ has the convergence property. In particular, each family of Möbius transformations has this property. See [62].

DEFINITION 16.2 [62]. A *convergence group* is a family G of self-homeomorphisms of $\overline{\mathbb{R}^n}$ which has the convergence property and forms a group under composition.

Thus every group G of Möbius transformations or K -quasiconformal mappings of $\overline{\mathbb{R}^n}$ is a convergence group. Moreover the elements of a discrete convergence group can be classified in the same way as in a Möbius group.

THEOREM 16.3. *If g is an element of a discrete convergence group, then*

- (i) *g has finite order, i.e., g is elliptic, or*
- (ii) *g has infinite order and one fixed point, i.e., g is parabolic, or*
- (iii) *g has infinite order and two fixed points, i.e., g is loxodromic.*

Surprisingly many properties of discrete Möbius groups also hold for discrete convergence groups.

EXAMPLE 16.4 [62]. Suppose that f and g are elements of a discrete convergence group. If f and g have a common fixed point and if g is loxodromic, then

$$fg^k = g^k f,$$

where $k \neq 0$. If f and g agree in an open set, then

$$f = g.$$

The study of convergence and quasiconformal groups appears to offer an interesting and fruitful way to extend the classical theory of Möbius groups. See, for example, [26,27,112,113,156,161,162,164,165].

Convergence groups have also led to recent important developments in topology and geometry. These include:

- (1) independent proofs of the Seifert fibered space conjecture by Gabai [46] and by Casson and Jungreis [37],
- (2) a second proof of the Nielsen realization problem by Gabai [46] and
- (3) a topological characterization of Gromov hyperbolic groups [67,70] by Bowditch [30,31].

This series of results is a striking example of how quasiconformal mappings have interacted and continue to interact with other parts of mathematics.

17. An application to medicine

We observed earlier in Section 14 how quasiconformal mappings have been used to study physical properties of composite materials. We conclude here with another recent and unusual application of these mappings to medicine, in particular, to the study of the brain. The background for this work is as follows.

The cortex of the human brain is a highly convoluted surface with folds and fissures which vary in size and position from one person to another. This fact has made it difficult for medical researchers to analyze and compare functional regions of the brain since regions of activation which appear close together may be quite far apart when measured on the cortical surface. However, the surface representing the cortical gray matter is topologically equivalent to a two-dimensional sheet S in $\mathbb{R}^3$ and hence neurologists would like to be able to unfold the brain's complicated geometry of bulges and onto a flat set in $\mathbb{R}^2$.

A number of computational tools have been developed to take advantage of this surface based approach. It is impossible to flatten the curved surface S without metric and areal distortion. On the other hand, the Riemann mapping theorem implies the existence of a canonical conformal mapping f of S onto a set in $\mathbb{R}^2$ which preserves angular information.

One group of researchers investigating this problem is using circle packing to find a canonical discrete approximation of the conformal mapping f which will map the cortical surface S quasiconformally onto a flat set in the Euclidean or hyperbolic plane with bounded angular distortion. See [79] and [96].

References

- [1] L.V. Ahlfors, *Zur Theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.
- [2] L.V. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. **3** (1953–1954), 1–58.
- [3] L.V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
- [4] L.V. Ahlfors, *Extension of quasiconformal mappings from two to three dimensions*, Proc. Natl. Acad. Sci. USA **51** (1964), 768–771.
- [5] L.V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413–429.
- [6] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, NJ (1966).
- [7] L.V. Ahlfors, *A somewhat new approach to quasiconformal mappings in $\mathbb{R}^n$* , Complex Analysis, Lecture Notes in Math., Vol. 599, Springer-Verlag, Heidelberg (1977), 1–6.
- [8] L.V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 385–404.
- [9] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Distortion functions for plane quasiconformal mappings*, Israel J. Math. **62** (1988), 1–16.
- [10] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York (1997).
- [11] K. Astala, *A remark on quasi-conformal mappings and BMO*, Michigan Math. J. **30** (1983), 209–212.
- [12] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [13] K. Astala and F.W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Anal. Math. **46** (1986), 16–57.

- [14] K. Astala and P. Koskela, *Quasiconformal mappings and global integrability of the derivative*, J. Anal. Math. **57** (1991), 203–220.
- [15] K. Astala and G.J. Martin, *Holomorphic motions*, Papers on Analysis, Report 83, Dept. of Mathematics and Statistics, Univ. Jyväskylä (2001), 27–40.
- [16] K. Astala and M. Miettinen, *On quasiconformal mappings and 2-dimensional G-closure problems*, Arch. Ration. Mech. Anal. **143** (1998), 207–240.
- [17] L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, Analytic Functions, Princeton Univ. Press, Princeton (1960), 89–119.
- [18] L. Bers, *Uniformization by Beltrami equations*, Comm. Pure Appl. Math. **14** (1961), 215–228.
- [19] L. Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [20] L. Bers, *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Math. **141** (1978), 73–98.
- [21] L. Bers and H.L. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.
- [22] A. Beurling and L.V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.
- [23] B. Bojarski, *Generalized solutions of a system of first order differential equations of elliptic type with discontinuous coefficients*, Mat. Sb. **43** (1957), 451–503 (in Russian).
- [24] B. Bojarski and T. Iwaniec, *Another approach to Liouville theorem*, Math. Nachr. **107** (1982), 253–262.
- [25] B. Bojarski and T. Iwaniec, *Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$* , Ann. Acad. Sci. Fenn. **8** (1983), 257–324.
- [26] P. Bonfert-Taylor, *Jørgensen's inequality for discrete convergence groups*, Ann. Acad. Sci. Fenn. **25** (2000), 131–150.
- [27] P. Bonfert-Taylor and G.J. Martin, *Quasiconformal groups with small dilatation I*, Proc. Amer. Math. Soc. **129** (2000), 2019–2029.
- [28] M. Bonk and J. Heinonen, *Quasiregular mappings and cohomology*, Acta Math. **186** (2001), 219–238.
- [29] M. Bonk, J. Heinonen and S. Rohde, *Doubling conformal densities*, J. Reine Angew. Math. **541** (2001), 117–141.
- [30] B.H. Bowditch, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. **11** (1998), 643–667.
- [31] B.H. Bowditch, *Convergence groups and configuration spaces*, Geometric Group Theory down under, de Gruyter, Berlin (1999), 23–54.
- [32] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.
- [33] M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. **75** (1962), 331–341.
- [34] C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der Abbildung des Innern einer Jordanschen Kurve auf einen Kreis*, Math. Ann. **73** (1913), 305–320.
- [35] L. Carleson, *The extension problem for quasiconformal mappings*, Contributions to Analysis, Academic Press, New York (1974), 39–47.
- [36] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York (1993).
- [37] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), 441–456.
- [38] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428–517.
- [39] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino **3** (1957), 25–43.
- [40] A. Douady and C.J. Earle, *Conformally natural extension of homeomorphisms of the unit circle*, Acta Math. **157** (1986), 23–48.
- [41] A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. **18** (1985), 287–343.
- [42] D. Drasin, *The inverse problem of the Nevanlinna theory*, Acta Math. **138** (1977), 83–151.
- [43] C.J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** (1969), 19–43.
- [44] A. Eremenko and D.H. Hamilton, *On the area distortion by quasiconformal mappings*, Proc. Amer. Math. Soc. **123** (1995), 2793–2797.
- [45] C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

- [46] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992), 447–510.
- [47] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
- [48] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York (1981).
- [49] D.B. Gauld and M.K. Vamanamurthy, *Quasiconformal extensions of mappings in n -space*, Ann. Acad. Sci. Fenn. **3** (1977), 229–246.
- [50] F.W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.
- [51] F.W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.
- [52] F.W. Gehring, *Extension of quasiconformal mappings in three space*, J. Anal. Math. **14** (1965), 171–182.
- [53] F.W. Gehring, *Extension theorems for quasiconformal mappings in n -space*, J. Anal. Math. **19** (1967), 149–169.
- [54] F.W. Gehring, *Quasiconformal mappings of slit domains in three space*, J. Math. Mech. **18** (1969), 689–703.
- [55] F.W. Gehring, *The L^p -integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.
- [56] F.W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 561–572.
- [57] F.W. Gehring, *Injectivity of local quasi-isometries*, Comment. Math. Helv. **57** (1982), 202–220.
- [58] F.W. Gehring, *Extension of quasiisometric embeddings of Jordan curves*, Complex Var. **5** (1986), 245–263.
- [59] F.W. Gehring, *Characterizations of quasidisks*, Banach Center Publ. **48** (1999), 11–41.
- [60] F.W. Gehring and K. Hag, *The ubiquitous quasidisk* (in preparation).
- [61] F.W. Gehring and F. Huckemann, *Quasiconformal mappings of a cylinder*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Brasov (1971), 171–186.
- [62] F.W. Gehring and G.J. Martin, *Discrete quasiconformal groups I*, Proc. London Math. Soc. **55** (1987), 331–358.
- [63] F.W. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. **388** (1966), 1–15.
- [64] F.W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.
- [65] F.W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. **6** (1973), 504–512.
- [66] J. Gevirtz, *Injectivity of quasi-isometric mappings of balls*, Proc. Amer. Math. Soc. **85** (1982), 345–349.
- [67] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Birkhäuser, Boston (1990).
- [68] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. of Math. Stud., Vol. 105, Princeton Univ. Press, Princeton (1983).
- [69] V.M. Gol'dstein and S.K. Vodop'janov, *Prolongement des fonctions de class L^1_p et applications quasi conformes*, C. R. Acad. Sci. Paris **290** (1980), 453–456.
- [70] M. Gromov, *Hyperbolic groups*, Essays in Group Theory, Math. Sci. Res. Inst. Publ., Vol. 8, Springer-Verlag, New York (1987), 75–263.
- [71] H. Grötzsch, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **84** (1932), 114–120.
- [72] P. Hajlasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403–415.
- [73] P. Hajlasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **688** (2000).
- [74] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York (2001).
- [75] J. Heinonen, T. Kipelaäinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Univ. Press, New York (1993).
- [76] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), 61–79.
- [77] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.
- [78] J. Heinonen, P. Koskela, N. Shanmugalingam and J.T. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. **85** (2001), 87–139.

- [79] M.K. Hurdal, P.L. Bowers, K. Stephenson, W.L. De Sumners, K. Rehm, K. Schaper and D.A. Rottenberg, *Quasi-conformally flat mapping the human cerebellum*, Lecture Notes in Comput. Sci., Vol. 1679, Springer-Verlag, Berlin (1999), 279–286.
- [80] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo (1992).
- [81] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Ann. of Math. **136** (1992), 589–624.
- [82] T. Iwaniec, *The Gehring lemma*, Quasiconformal Mappings and Analysis, Springer-Verlag, New York (1998), 181–204.
- [83] T. Iwaniec, *Nonlinear analysis and quasiconformal mappings from the perspective of pdes*, Banach Center Publ. **48** (1999), 119–140.
- [84] T. Iwaniec and G.J. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press, New York (2001).
- [85] F. John, *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.
- [86] F. John, *On quasi-isometric mappings I*, Comm. Pure Appl. Math. **21** (1968), 77–110.
- [87] F. John, *On quasi-isometric mappings II*, Comm. Pure Appl. Math. **22** (1969), 265–278.
- [88] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.
- [89] P.W. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. **29** (1980), 41–66.
- [90] P.W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.
- [91] S. Kallunki and P. Koskela, *Exceptional sets for the definition of quasiconformality*, Amer. J. Math. **122** (2000), 735–743.
- [92] S. Kallunki and P. Koskela, *Metric definition of μ -homeomorphisms*, Michigan Math. J. **51** (2003), 141–151.
- [93] A. Korányi and H.M. Reimann, *Quasiconformal mappings on the Heisenberg group*, Invent. Math. **80** (1985), 309–338.
- [94] P. Koskela and P. MacManus, *Quasiconformal mappings and Sobolev spaces*, Studia Math. **131** (1998), 1–17.
- [95] I. Kra, *On the Nielsen–Thurston–Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.
- [96] I. Krantz, *Conformal mappings*, Amer. Scientist **87** (1999), 436–445.
- [97] W. Kraus, *Über den Zusammenhang einiger Charakteristiken eines einfach zusammenhängenden Bereiches mit der Kreisabbildung*, Mitt. Math. Sem. Giessen **21** (1932), 1–28.
- [98] M.A. Lavrentieff, *Sur une classe de représentations continues*, Mat. Sb. **42** (1935), 407–423.
- [99] M.A. Lavrentieff, *Sur un critère différentiel des transformations homéomorphes des domaines à trois dimensions*, Dokl. Akad. Nauk SSSR **20** (1938), 241–242.
- [100] M. Lehtinen, *Remarks on the maximal dilatation of the Beurling–Ahlfors extension*, Ann. Acad. Sci. Fenn. **9** (1984), 133–139.
- [101] O. Lehto, *Domain constants associated with Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 603–610.
- [102] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987).
- [103] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, New York (1973).
- [104] O. Lehto, K.I. Virtanen and J. Väisälä, *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. **273** (1959), 3–13.
- [105] J. Lelong-Ferrand, *Étude d’une classe d’applications liées à des homomorphismes d’algèbres de fonctions, et généralisant les quasi-conformes*, Duke Math. J. **40** (1973), 163–186.
- [106] J.L. Lewis, *Picard’s theorem and Rickman’s theorem by way of Harnack’s inequality*, Proc. Amer. Math. Soc. **122** (1994), 199–206.
- [107] L.G. Lewis, *Quasiconformal mappings and Royden algebras in space*, Trans. Amer. Math. Soc. **158** (1971), 481–492.
- [108] P. Lindqvist, *On the definition and properties of p-superharmonic functions*, J. Reine Angew. Math. **365** (1986), 67–79.
- [109] P. Lindqvist and O. Martio, *Regularity and polar sets for supersolutions of certain degenerate elliptic equations*, J. Anal. Math. **50** (1988), 1–17.
- [110] C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414.

- [111] R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193–217.
- [112] G.J. Martin, *Discrete quasiconformal groups that are not the quasiconformal conjugates of Möbius groups*, Ann. Acad. Sci. Fenn. **11** (1986), 179–202.
- [113] G.J. Martin, *Quasiconformal and affine groups*, J. Differential Geom. **29** (1989), 427–448.
- [114] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. **448** (1969), 1–40.
- [115] O. Martio, S. Rickman and J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. **465** (1970), 1–13.
- [116] O. Martio, S. Rickman and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. **488** (1971), 1–31.
- [117] O. Martio and J. Väisälä, *Global L^p -integrability of the derivative of a quasiconformal mapping*, Complex Var. **9** (1988), 309–319.
- [118] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin (1987).
- [119] B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65.
- [120] D. Menchoff, *Sur une généralisation d'un théorème de M.H. Bohr*, Mat. Sb. **44** (1937), 339–354.
- [121] G.W. Milton and V. Nesi, *Optimal G -closure bounds via stability under lamination*, Arch. Ration. Mech. Anal. **150** (1999), 191–207.
- [122] E.E. Moise, *Affine structures in 3-manifolds, IV. Piecewise linear approximations of homeomorphisms*, Ann. of Math. **55** (1952), 215–222.
- [123] E.E. Moise, *Affine structures in 3-manifolds, V. The triangulation theorem and Hauptvermutung*, Ann. of Math. **56** (1952), 96–114.
- [124] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
- [125] J. Moser, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Commun. Pure Appl. Math. **13** (1960), 457–468.
- [126] G.D. Mostow, *Quasi-conformal mappings in n -space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.
- [127] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).
- [128] M. Nakai, *Royden algebras and quasi-isometries of Riemannian manifolds*, Pacific J. Math. **40** (1972), 397–414.
- [129] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
- [130] V. Nesi, *Quasiconformal mappings as a tool to study certain two-dimensional G -closure problems*, Arch. Ration. Mech. Anal. **134** (1996), 17–51.
- [131] P. Pansu, *Quasimétries des variétés à courbure négative*, Thesis, Université Paris VII (1987).
- [132] D. Partyka, *The maximal dilatation of Douady and Earle extension of a quasimetric automorphism of the unit circle*, Ann. Univ. Mariae Curie-Skłodowska **44** (1990), 45–57.
- [133] H.M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. **49** (1974), 260–276.
- [134] H.M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation*, Lecture Notes in Math., Vol. 487, Springer-Verlag, New York (1975).
- [135] Yu.G. Reshetnyak, *On conformal mappings in space*, Dokl. Akad. Nauk SSSR **130** (1960), 1196–1198 (in Russian).
- [136] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, Sibirsk. Mat. Zh. **8** (1967), 629–658 (in Russian).
- [137] Yu.G. Reshetnyak, *Liouville's theorem on conformal mappings for minimal regularity assumptions*, Sibirsk. Mat. Zh. **8** (1967), 835–840 (in Russian).
- [138] Yu.G. Reshetnyak, *On stability bounds in the Liouville theorem on conformal mappings of multidimensional spaces*, Sibirsk. Mat. Zh. **11** (1970), 1121–1139 (in Russian).
- [139] Yu.G. Reshetnyak, *Stability estimates in Liouville's theorem and the L^p -integrability of the derivatives of quasi-conformal mappings*, Sibirsk. Mat. Zh. **17** (1976), 868–896 (in Russian).
- [140] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, Transl. Math. Monographs, Vol. 73, Amer. Math. Soc., Providence, RI (1989).
- [141] Yu.G. Reshetnyak, *Stability Theorems in Geometry and Analysis*, Kluwer, Dordrecht (1994).

- [142] S. Rickman, *On the number of omitted values of entire quasiregular mappings*, J. Anal. Math. **37** (1980), 100–117.
- [143] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, Berlin (1993).
- [144] H.L. Royden, *Harmonic functions on open Riemann surfaces*, Trans. Amer. Math. Soc. **73** (1952), 40–94.
- [145] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, New York (1970).
- [146] J. Sarvas, *Symmetrization of condensers in n -space*, Ann. Acad. Sci. Fenn. **522** (1972), 1–44.
- [147] M. Seppälä and T. Sorvali, *Geometry of Riemann Surfaces and Teichmüller Spaces*, North-Holland, Amsterdam (1992).
- [148] M. Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. **20** (1987), 1–29.
- [149] Z. Ślodkowski, *Extensions of holomorphic motions*, Ann. Sc. Norm. Sup. Pisa **22** (1995), 185–210.
- [150] N. Soderborg, *Quasiregular mappings with finite multiplicity and Royden algebras*, Indiana Univ. Math. J. **40** (1991), 1143–1167.
- [151] N. Soderborg, *A characterization of domains quasiconformally equivalent to the unit ball*, Michigan Math. J. **41** (1994), 363–370.
- [152] N. Soderborg, *An ideal boundary for domains in n -space*, Ann. Acad. Sci. Fenn. **19** (1994), 147–165.
- [153] U. Srebro, *Extremal lengths and quasiconformal maps*, Israel Math. Conf. Proc. **14** (2000), 135–158.
- [154] S.G. Staples, *Global integrability of the Jacobian and quasiconformal maps*, Michigan Math. J. **40** (1993), 433–444.
- [155] D. Sullivan, *Hyperbolic geometry and homeomorphisms*, Geometric Topology, Academic Press, New York (1979), 543–555.
- [156] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. Math. of Stud., Vol. 97, Princeton Univ. Press, Princeton (1980), 465–496.
- [157] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou–Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418.
- [158] D. Sullivan, *Quasiconformal homeomorphisms and dynamics II. Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.
- [159] D. Sullivan and W.P. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.
- [160] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Mat.-Nat. Kl. **22** (1940), 1–197.
- [161] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. **5** (1980), 73–78.
- [162] P. Tukia, *Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane*, Ann. Acad. Sci. Fenn. **6** (1981), 89–94.
- [163] P. Tukia, *A quasiconformal group not isomorphic to a Möbius group*, Ann. Acad. Sci. Fenn. **6** (1981), 149–160.
- [164] P. Tukia, *On quasiconformal groups*, J. Anal. Math. **46** (1986), 318–346.
- [165] P. Tukia, *Convergence groups and Gromov’s metric hyperbolic spaces*, New Zealand J. Math. **23** (1994), 157–187.
- [166] P. Tukia and J. Väisälä, *Quasiconformal extension from dimension n to $n + 1$* , Ann. of Math. **115** (1982), 331–348.
- [167] P. Tukia and J. Väisälä, *Bilipschitz extensions of maps having quasiconformal extensions*, Math. Ann. **269** (1984), 561–572.
- [168] N.N. Ural’tseva, *Bounds for a modulus of continuity of the first derivatives for a class of elliptic differential equations*, Trudy Mat. Sem. LOMI **7** (1968), 184–222 (in Russian).
- [169] J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. **298** (1961), 1–36.
- [170] J. Väisälä, *On quasiconformal mappings of a ball*, Ann. Acad. Sci. Fenn. **304** (1961), 1–7.
- [171] J. Väisälä, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag, New York (1971).
- [172] J. Väisälä, *Quasimöbius maps*, J. Anal. Math. **44** (1984–1985), 218–234.
- [173] J. Väisälä, *Quasiconformal maps of cylindrical domains*, Acta Math. **162** (1989), 201–225.
- [174] J. Väisälä, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces*, Banach Center Publ. **48** (1999), 55–118.

- [175] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, New York (1988).
- [176] S. Yang, *Quasiconformally equivalent domains and quasiconformally extensions*, Complex Var. **30** (1996), 279–288.
- [177] V.A. Zorič, *A theorem of M.A. Lavrentieff on quasiconformal space maps*, Mat. Sb. **74** (1967), 417–433.

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CHAPTER 2

Variational Principles in the Theory of Quasiconformal Maps

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Abstract

This is the first part of a survey of the basic variational principles for quasiconformal maps concerning various classical and modern problems of the geometrical complex analysis. These topics turn out to be intrinsically connected with the Teichmüller space theory and the complex metric geometry of these spaces, as well as involving holomorphic motions, complex potential theory and harmonic maps.

1. Background: Quasiconformal maps and variations

First we touch very briefly on the foundations of the theory of quasiconformal maps and their deformations.

1.1. What does quasiconformality mean?

Take any characterizing property of conformality and, roughly speaking, allow it to be perturbed in bounded limits. The perturbation omits the strongest conformal rigidity; nevertheless, most of the basic properties will be preserved. Quasiconformal maps in the plane are nothing more than the results of such a perturbation. Quasiconformal maps admit various analytic and geometric approaches originated by Grötzsch, Ahlfors and Lavrentiev.

From the point of view of differential equations, quasiconformal maps are the homeomorphic generalized solutions of uniformly elliptic systems (not necessarily linear) of the first order

$$L_j(x, y, u, v, u_x, u_y, v_x, v_y) = 0, \quad j = 1, 2,$$

for two real functions $u(x, y), v(x, y)$ of two variables $(x, y) \in \mathbb{R}^2$.

Actually, we may restrict ourselves to a special case of the (linear) *Beltrami equation* and regard quasiconformal maps as homeomorphic solutions of the Beltrami equation

$$\partial_{\bar{z}} w = \mu \partial_z w, \tag{1.1}$$

where μ is a bounded measurable function in a domain $D \subset \mathbb{C}$ with $\|\mu\|_\infty < 1$, and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $z = x + iy$, are distributional derivatives belonging locally to L_2 , and w should satisfy this equation almost everywhere in D . By a homeomorphic solution we mean a solution which is a homeomorphism. Thus, quasiconformal maps preserve orientation.

We call μ the *Beltrami coefficient* of the map w . This coefficient defines a vector field of infinitesimal ellipses on the domain D , in other words, a conformal structure on D .

If $\mu(z) = 0$ almost everywhere in D , then the solutions of the equation (1.1) are holomorphic on D . In the general case, the coefficient μ determines a conformal metric $ds^2 = |dz + \mu(z)d\bar{z}|^2$ on the domain D , and homeomorphic solutions of (1.1) become conformal in this metric.

The value $K(w) = (1 + \|\mu\|)/(1 - \|\mu\|)$ is called the *maximal dilatation* of the map w . Another quantity naturally associated with this map is its *dilatation* $k(w) = \|\mu\|_\infty < 1$.

The maps with $k(w) \leq k_0 < 1$ (equivalently, $K(w) \leq K_0 < \infty$) are called *k_0 -quasiconformal* or *K_0 -quasiconformal*. This means

$$|\partial_z w|^2 + |\partial_{\bar{z}} w|^2 \leq \frac{1}{2} \left(K_0 + \frac{1}{K_0} \right) (|\partial_z w|^2 - |\partial_{\bar{z}} w|^2). \tag{1.2}$$

One of the geometric characterizations of quasiconformality is the boundedness of distortion of the circles in the spherical metric δ on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$H(z) := \limsup_{r \rightarrow 0} \frac{\max_{|\zeta|=r} \delta(w(z+\zeta), w(z))}{\min_{|\zeta|=r} \delta(w(z+\zeta), w(z))} \leq C(K), \quad (1.3)$$

where $K = K(w)$ and C depends only on K .

This definition can be applied to the maps defined on arbitrary, even discrete subsets E of $\widehat{\mathbb{C}}$, by setting the ratio in the right-hand side of (1.3) to be equal to one whenever $z + \zeta \notin E$. For the domains E , the notions of quasiconformality based on (1.1), (1.2) and (1.3) are equivalent.

Another geometric characterization involves the *extremal lengths* of curve families or the conformal modules of rectangles and ring domains, and requires that one of these conformal invariants increase or decrease at most by a factor of K .

For the theory of quasiconformal maps see, e.g., [Ah2, Kru5, LV].

1.2. Representation and variation formulas

Let us first introduce two basic integral operators. Define, for $\rho \in L_p(\mathbb{C})$, $p > 2$, two transforms

$$\begin{aligned} T_0 \rho(z) &= -\frac{1}{\pi} \iint_{\mathbb{C}} \rho(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{z} \right) d\xi d\eta, \\ \Pi \rho(z) &= -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\rho(\zeta) - \rho(z)}{(\zeta - z)^2} d\xi d\eta, \quad \zeta = \xi + i\eta. \end{aligned}$$

The first integral converges absolutely in $\mathbb{C}$, while the second one exists by the Calderón–Zygmund results as the principal Cauchy value. The operator Π is called the *Hilbert–Beurling transform*.

If ρ has a compact support, then

$$\Pi \rho(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{(\zeta - z)^2},$$

and instead of T_0 we shall consider the operator

$$T \rho(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{\zeta - z}.$$

Note that $\partial_{\bar{z}} T \rho = \rho$ and $\partial_z T \rho = \Pi \rho$.

When μ in (1.1) has a compact support, the solution w of this equation normalized by $w(z) = z + O(z^{-1})$ as $z \rightarrow \infty$ is of the form

$$w(z) = z - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{\zeta - z} = z + T \rho(z). \quad (1.4)$$

Substituting (1.4) into (1.1), one obtains

$$\rho = \mu + \mu \Pi \mu + \mu \Pi (\mu \Pi \mu) + \cdots. \quad (1.5)$$

Let $\|\mu\|_\infty \leq k < 1$, then the series (1.5) is convergent in $L_p(\mathbb{C})$ for $2 < p < p_0(k)$ (which was first discovered by Bojarski [Bo]). Thus the distributional derivatives $\partial_{\bar{z}} w = \rho$ and $\partial_z w = 1 + \Pi \rho$ belong to $L_p^{\text{loc}}(\mathbb{C})$, moreover, for any disk

$$\Delta_R = \{z \in \mathbb{C}: |z| < R\}, \quad 0 < R < \infty,$$

we have

$$\begin{aligned} \|\rho\|_{L_p(\Delta_R)}, \quad \|\Pi \rho\|_{L_p(\Delta_R)} &\leq M_1(k, R, p) \|\mu\|_{L_\infty(\mathbb{C})}; \\ \|h\|_{C(\bar{\Delta}_R)} &\leq M_1(k, R, p) \|\mu\|_\infty. \end{aligned}$$

Furthermore, if $\mu(z; t)$ is a C^1 smooth $L_\infty(\mathbb{C})$ function of a real (respectively, complex) parameter t , then $\partial_w h^{\mu(\cdot, t)}$ and $\partial_{\bar{w}} h^{\mu(\cdot, t)}$ are smoothly $\mathbb{R}$ -differentiable (respectively, $\mathbb{C}$ -differentiable) L_p functions of t , and, consequently, the function $t \mapsto h^{\mu(\cdot, t)}(z)$ is C^1 smooth as an element of $C(\bar{\Delta}_R)$ for any $R < \infty$ (see, e.g., [AB], [Kru5, Chapter 2]).

In particular, we have

$$w(z) = z + T\mu(z) + \omega(z) \quad \text{with } \|\omega\|_{C(\bar{\Delta}_R)} \leq M_2(k, R) \|\mu\|_\infty^2. \quad (1.6)$$

In the general case, omitting the assumption that μ has compact support, we must ensure the convergence of integral in (1.4). Thus we assume that $w(0) = 0$ and replace there $T\rho$ by $T_0\rho$.

Now fix $R \in (0, \infty)$ and put $\mu = \mu_1 + \mu_2$, where $\mu_1(z) = \mu(z)$ in Δ_R and $\mu_1(z) = 0$ in $\Delta_R^* = \{z \in \mathbb{C}: |z| > R\}$, while $\mu_2(z) = 0$ in Δ_R and $\mu_2(z) = \mu(z)$ in Δ_R^* (i.e., w^{μ_2} is conformal in Δ_R^*). Then

$$w^\mu = w^\sigma \circ w^{\mu_1} \quad (1.7)$$

with

$$\sigma = \left(\frac{\mu_2}{1 - \bar{\mu}_1 \mu} \frac{\partial_z w^{\mu_1}}{\partial_z w^{\mu_1}} \right) \circ (w^{\mu_1})^{-1}. \quad (1.8)$$

The representation (1.6) or

$$w^v(z) = T_0 v(z) + O(\|v\|_\infty^2)$$

can be now applied to w^{μ_1} and to

$$w^\lambda(z) = 1/w^\sigma(1/z) \quad \text{with } \lambda(z) = \sigma(1/z)z^2/\bar{z}^2.$$

Adding the third normalization condition, for example, that the point $z = 1$ also remains fixed (which determines a unique solution to the corresponding Beltrami equation (1.1)), we obtain as a result that the general variational formula

$$w^\mu(z) = z - \frac{z(z-1)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-2)} + O(\|\mu\|_\infty^2), \quad (1.9)$$

where the ratio $O(\|\mu\|^2)/\|\mu\|^2$ remains uniformly bounded on compact sets in $\mathbb{C}$ (moreover, in spherical metric on $\widehat{\mathbb{C}}$) as $\|\mu\| \rightarrow 0$. More generally, the maps w^μ with the arbitrary fixed points $z_1, z_2 \in \mathbb{C}$ and ∞ are represented by

$$w^\mu(z) = \frac{(z-z_1)(z-z_2)}{\pi} \iint_{\mathbb{C}} \frac{\mu(\zeta) d\xi d\eta}{(\zeta-z_1)(\zeta-z_2)(\zeta-\infty)} + O(\|\mu\|_\infty^2). \quad (1.10)$$

The results on dependence of w^μ on parameters mentioned above follow immediately from (1.9). Somewhat stronger results on dependence from parameters were established by Ahlfors and Bers in [AB] and became basic, e.g., for the Teichmüller space theory. For another application see, e.g., [Kru5, Kru16]. Note that if $w^\mu(z, t)$ depends holomorphically on a complex parameter $t \in \Delta$ for almost all $z \in \mathbb{C}$, then, by Schwarz's lemma, $\mu(\cdot, t)$ is holomorphic in t as an element of $L_\infty(\mathbb{C})$.

1.3. Other explicit variational formulas

Given a quasiconformal automorphism W^μ of the upper half-plane $U = \{z: \operatorname{Im} z > 0\}$ with the Beltrami coefficient μ and with fixed points $0, 1, \infty$, we extend μ to the lower half-plane $U^* = \{z: \operatorname{Im} z < 0\}$ by symmetry, setting

$$\mu(z) = \overline{\mu(\bar{z})}, \quad z \in U^*. \quad (1.11)$$

Then the corresponding quasiconformal automorphism w^μ of $\widehat{\mathbb{C}}$ satisfies $w^\mu(\bar{z}) = \overline{w^\mu(z)}$ for all $z \in \mathbb{C}$, and by the uniqueness theorem for the Beltrami equation, $w^\mu|_U = W^\mu$. Representing w^μ by (1.9), we get

$$\begin{aligned} W^\mu(z) &= z - \frac{z(z-1)}{\pi} \iint_U \left[\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-2)} + \frac{\overline{\mu(\bar{\zeta})}}{\bar{\zeta}(\bar{\zeta}-1)(\bar{\zeta}-2)} \right] d\xi d\eta \\ &\quad + O(\|\mu\|^2). \end{aligned} \quad (1.12)$$

Similarly, given a $\mu \in L_\infty(\Delta)$ with $\|\mu\|_\infty < 1$, we extend it to Δ^* by

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \quad z \in \Delta^*, \quad (1.13)$$

then the corresponding quasiconformal self-map of Δ extends to a quasiconformal automorphism w^μ of $\widehat{\mathbb{C}}$ satisfying the symmetry condition $w^\mu(1/\bar{z}) = 1/w^\mu(z)$ and is of the

form

$$w^\mu(z) = z - \frac{1}{\pi} \iint \frac{\mu(\zeta) d\xi d\eta}{\zeta - z} - \frac{z^3}{\pi} \iint \frac{\overline{\mu(\zeta)} d\xi d\eta}{1 - z\bar{\zeta}} + P(z) + O(\|\mu\|^2), \quad (1.14)$$

where $P(z)$ is a polynomial of the second order of the form

$$P(z) = a + ibz - \bar{a}z^2, \quad \text{Im } b = 0,$$

whose coefficients are uniquely determined from the normalization conditions. In particular, if these conditions are $w(0) = 0$, $w(1) = 1$, both formulas (1.9) and (1.14) are reduced to

$$w^\mu(z) = z - \frac{z(z-1)}{\pi} \iint_{\Delta} \left[\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} - \frac{\overline{\mu(\zeta)}}{\bar{\zeta}(\bar{\zeta}-1)(1-z\bar{\zeta})} \right] d\xi d\eta + O(\|\mu\|^2). \quad (1.15)$$

We provide also the following more specialized but useful formulas.

Let $S_k(\infty)$ be the class of univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the disk Δ admitting k -quasiconformal extensions $\tilde{f}^\mu$ to Δ^* such that $\tilde{f}^\mu(\infty) = \infty$, and let $S(\infty) = \bigcup_k S_k(\infty)$. Applying (1.10) with $z_1 = 0$ and $z_2 \rightarrow z_1$, we get for small $\|\mu\|_\infty$,

$$\tilde{f}^{\mu+v}(z) = \tilde{f}^\mu(z) - \frac{\tilde{f}^\mu(z)^2}{\pi} \iint_{\tilde{f}^\mu(\Delta^*)} \frac{\theta(\zeta) d\xi d\eta}{\zeta^2(\zeta - \tilde{f}^\mu(z))} + O(\|v\|_\infty^2), \quad |z| \leq R < \infty, \quad (1.16)$$

where by the chain rule (1.8),

$$\theta(w) = \left(\frac{v}{1 - |\mu|^2} \frac{\partial_z \tilde{f}^\mu}{\partial_z \tilde{f}^\mu} \right) \circ (\tilde{f}^\mu)^{-1}. \quad (1.17)$$

In particular, for $\mu = 0$,

$$f^v(z) = z - \frac{z^2}{\pi} \iint_{\Delta^*} \frac{v(\zeta) d\xi d\eta}{\zeta^2(\zeta - z)} + O(\|v\|^2), \quad z \in \Delta. \quad (1.18)$$

Respectively, for the classes $\Sigma_k(0)$ of univalent $\widehat{\mathbb{C}}$ -holomorphic functions $f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$ in Δ^* with k -quasiconformal extensions $\tilde{f}^\mu$ to Δ satisfying $\tilde{f}^\mu(0) = 0$ and $\Sigma(0) = \bigcup_k \Sigma_k(0)$, we have from (1.6)

$$\tilde{f}^{\mu+v}(z) = \tilde{f}^\mu(z) - \frac{1}{\pi} \iint_{\tilde{f}^\mu(\Delta)} \frac{\theta(\zeta) d\xi d\eta}{\zeta - \tilde{f}^\mu(z)} + O(\|v\|^2), \quad (1.19)$$

where θ is defined similar to (1.19) and the bound for the remainder is uniform in any disk Δ_R .

In fact, one can fix more points. For example, let f be an injective holomorphic map of some disk Δ_r , $0 < r < \infty$, with k -quasiconformal extensions $\tilde{f}^\mu$ to $\widehat{\mathbb{C}}$. Suppose that $f(0) = 0$, $f'(0) = 1$ and $f(1) = 1$. Then, for sufficiently small k and for $|z| \leq R < r_0(k, r)$, we have the representation

$$f^\mu(z) = z - \frac{z^2(z-1)}{\pi} \iint_{|\zeta|>r} \frac{\mu(\zeta) d\xi d\eta}{\zeta^2(\zeta-1)(\zeta-z)} + \omega_\mu(z), \quad (1.20)$$

where

$$\|\omega_\mu\|_{C(\overline{\Delta_R})} \leq M(k, r, R) \|\mu\|_\infty^2$$

and $r_0(k, r)$ is a well-defined function of k and r such that

$$\lim_{k \rightarrow 0} r_0(k, r) = \infty,$$

and the constant $M(k, r, R)$ depends only on k, r and R .

Moreover, one can construct special quasiconformal variations sharing arbitrary finite numbers of prescribed values; see in this connection [Kru5, Chapter 3], [Kru17, BiK, Ren2].

The next useful representation formula was provided by Lehto [Leh1].

Let $f \in S_k(\infty)$ and let n be the smallest integer $> K = (1+k)/(1-k)$, then

$$f(z) = P_{n-1}(z) - \frac{z^n}{\pi} \iint \frac{\partial_\xi f}{\zeta^n(\zeta-z)} d\xi d\eta, \quad (1.21)$$

where P_{n-1} is the Taylor polynomial of $f|_\Delta$ of order $n-1$. This representation is obtained by applying the Pompeiu formula

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{g(\zeta) d\zeta}{\zeta-z} - \frac{1}{\pi} \iint_{|\zeta|<R} \frac{\partial_{\bar{\zeta}} g}{\zeta-z} d\xi d\eta, \quad |z| < R,$$

to $\psi(z) = (f(z) - P_{n-1}(z))z^{-n}$. In particular, if $K < 2$, (1.21) implies (1.18).

Finally, let us mention that the fundamental Schiffer's variation in the theory of conformal maps also can be obtained on the basis of representation (1.6), cf. [Ga3].

1.4. A boundary quasiconformal variation

We shall make use of a special quasiconformal boundary variation constructed by Biluta in [Bi1]. Let us construct a map $\omega = \chi(w)$ of the plane $\mathbb{C}_w$, where $w = u + iv$, with a given cut

$$v = 0, \quad -a' \leq u \leq a'', \quad a' > 0, \quad a'' > 0, \quad (1.22)$$

which is the identity map outside the circular slice

$$S := \{(u, v): u^2 + (v + c \cot \sigma)^2 < c^2 / \sin^2 \sigma, v > 0\}, \quad (1.23)$$

while this slice is compressed to k in its original size in a given direction θ , where $0 < \theta < \pi$. Here, c is sufficiently small, $0 < c < a = \min(a', a'')$, $0 < \sigma < \pi/2$, and $0 < k < 1$. (In the plane $\mathbb{C}_z$, we obtain a region with exterior points.) This map $\omega = \chi(\omega) = \xi + i\eta$ has the form

$$\begin{aligned} \eta &= (k \sin^2 \theta + \cos^2 \theta)v - (1 - k) \sin \theta [u \cos \theta + c \cot \sigma \sin \theta \\ &\quad - \sqrt{c^2(1 + \cot^2 \theta)v - (v \cos \theta - u \sin \theta + c \cot \sigma \cos \theta)^2}], \\ \xi &= (\eta - v) \cot \theta + u. \end{aligned} \quad (1.24)$$

For small σ , one deduces by a direct calculation that the partial derivatives of ξ and η satisfy

$$\begin{aligned} \xi_u &= 1 + O(\sigma), & \xi_v &= -1(1 - k) \cot \theta + O(\sigma), \\ \eta_u &= O(\sigma), & \eta_v &= k + O(\sigma), \quad \sigma \rightarrow 0. \end{aligned} \quad (1.25)$$

Let us now apply this variation to a quasiconformal map $w = f(z)$ of a domain $B \subset \widehat{\mathbb{C}}$ onto a domain B' such that the boundary of B' contains the segment (1.22) as a part. Then by (1.8) the Beltrami coefficient of the composition $z = (\chi \circ f)^{-1}(\omega)$ satisfies

$$\begin{aligned} &|\mu_{(\chi \circ f)^{-1}}(\omega(w))|^2 \\ &= |\mu_{f^{-1}}(w)|^2 - 2(1 - |\mu_{f^{-1}}(w)|^2) \operatorname{Re}[\mu_\chi(w) \overline{\mu_{f^{-1}}(w)}] + O(\|\mu_\chi\|_\infty^2) \end{aligned}$$

with uniform estimate of the remainder term in B' . It follows that the inequality $|\mu_{(\chi \circ f)^{-1}}(\omega(w))| < |\mu_{f^{-1}}(w)|$ is equivalent to the condition $\operatorname{Re}(\mu_\chi \overline{\mu_{f^{-1}}}) > 0$, or

$$\operatorname{Re}[\mu_\chi(w) e^{-2i\theta_1(w)}] < 0, \quad (1.26)$$

where $\theta_1(w) = \frac{1}{2}[\arg \mu_{f^{-1}}(w) + \pi]$. Using equality

$$\mu_\chi(w) = \frac{1}{|\omega_w|^2} [\xi_u^2 + \eta_u^2 - \xi_v^2 - \eta_v^2 + 2i(\xi_u \xi_v + \eta_u \eta_v)],$$

one reduces the inequality (1.26) to

$$(\xi_u^2 + \eta_u^2 - \xi_v^2 - \eta_v^2) \cos 2\theta_1 + 2(\xi_u \xi_v + \eta_u \eta_v) \sin 2\theta_1 < 0. \quad (1.27)$$

Assuming again σ to be small and remembering that $\theta_1(w) = \theta$ inside the segment (1.22), one can write the condition (1.27) in the form

$$-(1 - k)[2 + (1 - k) \cos 2\theta / \sin^2 \theta] + O(\sigma) < 0. \quad (1.28)$$

For arbitrary fixed θ in $(0, \pi)$, we can always arrange for this condition to be satisfied by taking both $k' = 1 - k$ and σ sufficiently small.

1.5. An old problem of I.N. Vekua

The following problem was posed by I.N. Vekua already in 1961 in his graduate course on Generalized Analytic Functions at Novosibirsk State University and still remains open:

Let μ_n be the n th partial sum of the series (1.5). Set

$$f_n(z) = z - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu_n(\xi) d\xi d\eta}{\xi - z}.$$

Are f_n also homeomorphisms?

The only known result here is due to Belinskii [Bel4, Chapter 4], that if μ is sufficiently small and C^1 -smooth (up to jumps on a finite number of closed smooth curves), with small derivatives $\partial_z \mu, \partial_{\bar{z}} \mu$ in $\widehat{\mathbb{C}}$, then the first iteration (variation)

$$f_1(z) = z - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\mu(\xi) d\xi d\eta}{\xi - z}$$

provides a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$ whose Beltrami coefficient is $\tilde{\mu} = \mu + O(\|\mu\|^2)$ (the estimate of the remainder is locally uniform).

The next counterexample is due to Iwaniec and shows that the smoothness assumptions cannot be dropped completely.

Let $0 < a < b < 1$ be such that

$$a^2 + ab + b^2 > 2.$$

Set

$$\varepsilon = \frac{1}{a^2 + ab + b^2 - 1} \in (0, 1),$$

and define the functions

$$g(z) = \begin{cases} \varepsilon z(1 - |z|^2) & \text{if } |z| \leq 1, \\ 0 & \text{if } |z| \geq 1 \end{cases}$$

and

$$\mu(z) = \partial_{\bar{z}} g(z) = \begin{cases} -\varepsilon z^2, & |z| < 1, \\ 0, & |z| > 1. \end{cases}$$

Then

$$|\mu(z)| \leq \varepsilon < 1.$$

On the other hand, for these functions, we have $T\mu(z) = g(z)$, and it follows that the corresponding first iteration

$$f_1(z) = z + T\mu(z) = \begin{cases} (1 + \varepsilon)z - \varepsilon z|z|^2, & |z| \leq 1, \\ z, & |z| > 1, \end{cases}$$

satisfies

$$\begin{aligned} f_1(a) - f_1(b) &= (1 + \varepsilon)(a - b) - \varepsilon(a^3 - b^3) \\ &= (a - b)[1 + \varepsilon - \varepsilon(a^2 + ab + b^2)] = 0, \end{aligned}$$

which shows that f_1 is not injective in the unit disk Δ .

A simple modification of the above construction allows us to define $\varepsilon \in (0, 1)$ and a Beltrami coefficient μ , so that the second iteration

$$f_2(z) = z + T\mu(z) + T(\mu\Pi\mu)(z)$$

is not injective in Δ .

This shows that the answer for an arbitrary Beltrami coefficient is negative. Thus the question is reduced to establishing the sufficient conditions, which ensure the injectivity of iterations f_n .

2. General theory of extremal quasiconformal maps

First we touch very briefly on the foundations of deformation theory, which is given by the theory of quasiconformal maps.

2.1. Background: The Grötzsch problem

Let R and R' be two rectangles in the complex plane. Without loss of generality, we may assume that the vertices of these rectangles are, respectively, the points $0, a, a + ib, ib$ and $0, a', a' + ib', ib'$ ($a, b, a', b' > 0$).

In 1928, Grötzsch stated the following problem:

Consider C^1 diffeomorphisms of R into R' moving the vertices to vertices and preserving the ordering of sides, and find the map which is closest to being conformal.

In which case is there closeness? R and R' are conformally equivalent if and only if they are complex linearly equivalent, i.e.,

$$\frac{a'}{a} = \frac{b'}{b} = \lambda, \quad \lambda > 0.$$

Omitting this trivial case, it turns out that in general the extremal map is of a special form:

THEOREM (Grötzsch, 1928). For any C^1 diffeomorphism of R to R' that preserves the ordering of sides, we have

$$K(f) \geq K_0 = \frac{a'}{b'} : \frac{a}{b}$$

with equality if and only if f is the affine map

$$\begin{aligned} f_0(z) &= \frac{a'}{a}x + i\frac{b'}{b}y \\ &= \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right)z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right)\bar{z}, \quad z = x + iy. \end{aligned}$$

The proof of this theorem is rather simple. It involves estimating the lengths of the images of horizontal lines $\{0 \leq x \leq a, y = \text{const}\}$ by the area of R' , using inequality (1.2).

Much more important is that the Grötzsch theorem gave rise to the theory of maps whose deviation from conformality is smallest, and served as a background for the method of *extremal length* developed later by Ahlfors and Beurling, as well as their successors.

Such maps are now called *extremal quasiconformal* (or, less customarily, *möglichst conformal*) maps.

2.2. Teichmüller's theory of extremal quasiconformal maps

In 1939 Teichmüller gave an extremely fruitful extension of the Grötzsch problem to the maps of Riemann surfaces of finite analytic type.

Recall that a *Riemann surface* X is a connected one-dimensional complex manifold, i.e., a topological surface endowed with a conformal structure.

A Riemann surface X is called *analytically finite*, or, more precisely, of analytic type (g, n, m) , if it is conformally equivalent to a closed Riemann surface X_0 of genus g with n punctures and m holes bounded by analytic loops. Here

$$d := 6g - 6 + 2n + 3m > 0, \quad g \geq 0, n \geq 0, m \geq 0. \quad (2.1)$$

From the topological point of view, the fundamental group $\pi_1(X)$ of X is finitely generated.

If $m > 0$, the surface X has a border; for $m = 0$ it is (conformally equivalent to) a closed Riemann surface with or without punctures. The type $(g, n, 0)$ is often denoted by (g, n) .

It was a discovery of Fricke and Teichmüller to pass the so-called *marked* Riemann surfaces, i.e., to fix a basic surface X_0 and consider the (homeomorphic) Riemann surfaces X with distinguished homotopic classes $\alpha: X_0 \rightarrow X$, that is the pairs (X, α) .

There are various equivalent definitions of *marking*. For instance, in the simplest case of a closed Riemann surface X of genus $g \geq 1$, its marking means nothing more than fixing

(up to an inner automorphism) standard generators $\mathbf{a}_1, \dots, \mathbf{a}_{2g}$ of the fundamental group $\pi_L(X)$ for which the product of commutators satisfies

$$\prod_{j \text{ odd}} [a_j, a_{j+1}] = \mathbf{1}$$

and the geometric intersection numbers (of representing loops on X in corresponding homotopy classes) satisfy

$$\#(a_j, a_\ell) = \begin{cases} \delta_{j+1, \ell}, & j \text{ odd}, \\ 0, & j \text{ even}; j \leq \ell \end{cases}$$

(see, e.g., [Ab2, ZVC]).

One can show that every two homeomorphic Riemann surfaces with the same orientation and the same analytic type (g, n, m) are quasiconformally equivalent.

Now we may regard the fundamental *Teichmüller problem*:

Given two topologically equivalent marked Riemann surfaces X and X' of the type (g, n, m) with the same orientation. In the class of all homeomorphisms f of X onto X' find the map with smallest dilatation $K(f)$.

Teichmüller gave a solution to this problem in [Te1, Te2] although his proof cannot be considered as complete and sharp. In addition, it contains many gaps.

Teichmüller discovered an intimate connection between the solutions of this problem and holomorphic $(2, 0)$ -forms $\varphi = \varphi(z) dz^2$ on X , called *quadratic differentials*.

A crucial point is that the real dimension of the space $Q(X)$ of such forms equals d from (2.1); if $m = 0$, the complex dimension is $3g - 3 + n$.

The first sharp and complete proof was given by Ahlfors in his celebrated paper [Ah1]; later, new proofs were given by Bers, Krushkal, Hamilton, Strebel, Reich et al. All the proofs involve uniformization.

Ahlfors' proof was variational, but not direct. The first direct variational proof of existence of Teichmüller's extremal map in every homotopic class, which immediately provides the representation (2.2) is given in [Kru2].

The result is fundamental for the theory of extremal quasiconformal maps and the moduli problem. It states:

THEOREM 2.1 (Teichmüller theorem, part 1). *Let X and X' be quasiconformally equivalent Riemann surfaces of the same finite analytical type $(g, n, m) \neq (0, 1, 0), (0, 2, 0), (0, 3, 0), (1, 0, 0), (0, 0, 2), (0, 1, 1)$. Then, in each homotopy class α of homeomorphisms $X \rightarrow X'$, there exists a unique map f_0 with the Beltrami differential*

$$\mu_{f_0}(z) = k \frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad (2.2)$$

where k is a constant, $0 < k < 1$, and $\varphi \in Q(X) \setminus \{0\}$; the quadratic differential φ ($= \varphi(z) dz^2$) is determined up to a (constant) positive factor.

If X and X' are bordered surfaces ($m > 0$), then φdz^2 assumes positive values on ∂X .

Note that the Beltrami differentials of the form (2.2) are called also the *Teichmüller differentials*.

REMARKS. (1) For conformally equivalent X and X' , $k = 0$.

(2) For the torus ($g = 1, n = m = 0$), f_0 is only determined up to a conformal map.

(3) The inverse map

$$f_0^{-1} : X' \rightarrow X$$

is also extremal in its homotopy class, so, by analogy with (2.2),

$$\mu_{f_0^{-1}}(w) = k \frac{\overline{\psi(w)}}{|\psi(w)|}, \quad \psi \in \mathcal{Q}(X') \setminus \{0\}. \quad (2.3)$$

It is not hard to prove that

$$\text{ord}_p \varphi = \text{ord}_{f_0(p)} \psi \quad \text{for all } p \in X. \quad (2.4)$$

2.3. Geometric picture

The Teichmüller theorem allows one to get a clear geometric picture of the map f_0 . To see it, we consider a point p at which $\varphi(p) \neq 0$ and, hence, also $\psi(f_0(p)) \neq 0$. In a neighborhood of such a point, we may choose a square root $\sqrt{\varphi}$ and we can choose local holomorphic coordinates ζ such that $\zeta(p) = 0$ and $d\zeta = \sqrt{\varphi} dz$, and similarly for $f_0(p)$ and ψ . In these coordinates,

$$\zeta(z) = \int_0^z \sqrt{\varphi(z)} dz, \quad \omega(w) = \int_0^w \sqrt{\psi(w)} dw, \quad (2.5)$$

f_0 then becomes the *affine stretching*

$$\omega = \frac{\zeta + k\bar{\zeta}}{1 - k},$$

where, writing $\zeta = \xi + i\eta$, $\omega = \xi' + i\eta'$, we have that

$$\xi' = K\xi, \quad \eta' = \eta, \quad K = \frac{1+k}{1-k}.$$

The lines $\zeta^{-1}(\mathbb{R})$ (*horizontal*) and $\zeta^{-1}(i\mathbb{R})$ (*vertical*) near p correspond to $\varphi > 0$ ($\Leftrightarrow \arg \varphi = 0$) and $\varphi < 0$ ($\Leftrightarrow \arg \varphi = \pi$), respectively. They define on

$$X - \{\text{zeros of } \varphi\}$$

(transverse) *horizontal* and *vertical foliations*. Thus, in these coordinates, the map f_0 is precisely the Grötzsch map.

We may express this in the following way: From the relations (2.2)–(2.5), letting

$$\zeta' = \int_{z_0}^z \sqrt{\varphi} dz + k \int_{z_0}^z \sqrt{\bar{\varphi}} d\bar{z}, \quad (2.6)$$

the coordinate ζ' define on X a new conformal (analytic) structure. We may extend this conformal structure over the points p for which $\varphi(p)$ has a zero of order m by defining

$$\zeta' = \left(\int_{z_0}^z \sqrt{\varphi} dz + k \int_{z_0}^z \sqrt{\bar{\varphi}} d\bar{z} \right)^{2/(m+2)} \quad (2.7)$$

at these points.

Denote by (X, φ, k) or X^{μ_0} the Riemann surface obtained from X by placing on X this new conformal structure. This surface X^{μ_0} is called the *Teichmüller deformation* of X .

We may then factor the map $f_0: X \rightarrow X'$ into two parts: the first part is the map $X \rightarrow X^{\mu_0}$ which is the identity pointwise, but which is quasiconformal with dilatation K on the given conformal structures. The second part is the map $X^{\mu_0} \rightarrow X'$, and is holomorphic.

Let us mention two *special cases*:

(a) *torus* X . Every torus can be uniformized by a free parabolic group

$$X = \mathbb{C}/\Gamma, \quad \Gamma = \mathbb{Z} \oplus \mathbb{Z} = \{m\omega_1 + n\omega_2: (m, n) \in \mathbb{Z}^2\},$$

where ω_1, ω_2 are two complex numbers such that

$$\operatorname{Im} \frac{\omega_2}{\omega_1} > 0.$$

In other words, the torus X is obtained from the parallelogram spanned by the periods ω_1, ω_2 , by identifying opposite sides.

All holomorphic quadratic differentials are reduced to

$$\varphi = c dz^2, \quad c = \text{const} \in \mathbb{C},$$

where the coordinate z ranges over the complex plane. It follows that each extremal map f_0 is affine on $\mathbb{C}$. The map f_0 becomes unique provided that we impose the additional condition that $f_0(0) = 0$. f_0 then takes the form $f_0(z) = az + b\bar{z}$.

(b) *annulus* $A_{1\lambda} = \{z: 1 < |z| < \lambda\}, \lambda > 1$. We can write

$$A_{1\lambda} = \mathbb{C}^*/\Gamma = \langle \gamma^n \mid \gamma: z \mapsto \lambda z \rangle.$$

All holomorphic quadratic differentials are of the form

$$\varphi(z) = \frac{c}{z^2} dz^2, \quad c \text{ is real.}$$

After changing coordinates by the log function, this case is reduced to rectangles and thereby to the Grötzsch case.

2.4. The deformation (Teichmüller) space

As was already mentioned above, two marked Riemann surfaces X_1, X_2 of the same type (g, n, m) are quasiconformally equivalent. We define the Teichmüller distance between X_1, X_2 by

$$d(X_1, X_2) = \frac{1}{2} \inf \log K(w),$$

taking the infimum over all quasiconformal homeomorphisms of X_1 onto X_2 .

If X_1 and X_2 are conformally equivalent, then $d(X_1, X_2) = 0$. So actually, $d(X_1, X_2)$ is a complete metric on the set $\mathbb{T}(X)$ of the deformation surfaces (X, φ, k) of a given surface X of type (g, m, n) . This set $\mathbb{T}(X)$ is called the *Teichmüller space* of X .

Because of quasiconformal equivalence, one can speak about an abstract Teichmüller space $\mathbb{T}(g, n, m)$ of surfaces of type (g, n, m) , by considering $\mathbb{T}(X)$ as $\mathbb{T}(g, n, m)$ with base point X .

The spaces $\mathbb{T}(g, n, 0)$ and $\mathbb{T}(g, 0, 0)$ are customarily denoted by $\mathbb{T}(g, n)$ and $\mathbb{T}(g)$, respectively.

The nature of these spaces is described by the following:

THEOREM 2.2 (Teichmüller theorem, part 2). *Every (marked) Riemann surface X' of given finite analytic type (g, n, m) is conformally equivalent to some (X, φ, k) .*

Equivalently, the Teichmüller space $\mathbb{T}(g, n, m) = \mathbb{T}(X)$ is homeomorphic to $\mathbb{R}^{6g-6+2n+3m}$, or equivalently to the unit ball in this Euclidean space.

This is one of the basic facts in the theory of Teichmüller spaces. This theory provides extremely fruitful methods for solving variational problems for quasiconformal maps. We will see this in the next sections.

SKETCH OF THE PROOF OF THEOREM 2.2. As was mentioned above,

$$\dim_{\mathbb{R}} Q(X) = 6g - 6 + 2n + 3m =: d.$$

We define on $Q(X)$ some norm, for instance $\|\varphi\|_{Q(X)} = \iint_X |\varphi(z)| dx dy$. Take a base $\varphi_1, \dots, \varphi_d$ in $Q(X)$ as a real vector space, and set $\xi = (\xi_1, \dots, \xi_d)$,

$$\mu_{\xi}(z) = \begin{cases} \|\xi\| \frac{\sum_1^d \xi_j \bar{\varphi}_j}{|\sum_1^d \xi_j \varphi_j|}, & \xi \neq 0, \\ 0, & \xi = 0, \end{cases}$$

where $\|\xi\| = \|\sum_1^d \xi_j \varphi_j\|_{Q(X)} < 1$. The correspondence $\xi \rightarrow X^{\mu_{\xi}}$ is injective and continuous. This implies that the Teichmüller space $\mathbb{T}(X)$ of the surface X is homeomorphic to the ball $Q(X) \simeq \mathbb{R}^d$. $\square$

These were the *first moduli for surface of type* (g, n, m) after Riemann (1857).

Every extremal Beltrami differential (2.2) defines for $\varphi = \text{const}$ a geodesic (in the Teichmüller metric) ray, taking $0 \leq k \leq 1$.

The main difference between the real and complex theories of Teichmüller spaces follows from the fact that the first deals with these rays, i.e., real geodesics, while the complex theory deals with the complex *Teichmüller disks* $\{t\tilde{\varphi}/|\varphi|: t \in \Delta\}$ and more general complex holomorphic images of the disk Δ in $\mathbb{T}(X)$.

One of the main topics in Teichmüller space theory is studying the behavior of these geodesics as $k \rightarrow 1$, respectively, $|k| \rightarrow 1$ (a *completion of the Teichmüller space*), in other words, to develop the asymptotic geometry at infinity ($k = 1$). The theory of quasiconformal maps cannot be applied, because $\|\mu\|_\infty = 1$, and we have degeneration.

But it is possible to apply differential geometric, ergodic and probabilistic methods, and in this way to introduce the *boundary* of $\mathbb{T}(X)$. One may then extend the metrics to this boundary, etc.

It is not hard to define the (transverse) measures on the horizontal and vertical foliations introduced in Section 2.3, for instance, letting for a line interval $\alpha \subset \zeta^{-1}(\mathbb{R})$ (or $\alpha \subset \zeta^{-1}(i\mathbb{R})$),

$$\mu(\alpha) = \text{Var}(\text{Im } \zeta^{-1}(\alpha));$$

here ζ is the canonical parameter and Var denotes the total variation.

The remarkable Thurston theorem states that for a closed surface of genus $g > 1$, the horizontal foliations on the surfaces (X, φ, k) *do not depend on* k (while the vertical foliations degenerate as $k \rightarrow 1$), which implies that the Teichmüller space $\mathbb{T}(X)$ admits a compactification $\mathbb{T}(X)$, whose boundary points are the φ -horizontal measured foliations (X, φ, k) .

There are some other (in fact, equivalent) compactifications of $\mathbb{T}(X)$ (see, e.g., [Ab2, Ber4, Ker1, Mas1, Wo1]).

2.5. Topics in complex metric geometry of Teichmüller spaces

We introduce briefly some important notions and results from Teichmüller space theory and from Finsler geometry, adapting them to the special cases which will appear here in applications to Geometric Function Theory.

2.5.1. The Teichmüller spaces are both *deformation spaces* of conformal structures of Riemann surfaces and *complex* (in general, Banach) *manifolds*. Accordingly, they have long been studied from these two points of view which, in fact, are closely related.

Like deformation spaces, the Teichmüller spaces involve the Beltrami coefficients and quasiconformal maps, holomorphic quadratic differentials and generalized foliations of Riemann surfaces, the moduli spaces of these surfaces, the Finsler structure, as well as other notions.

Like complex manifolds (in addition to providing an important class of such manifolds), the Teichmüller spaces admit invariant metrics and holomorphic contractions, pluricomplex potential description, etc.

Both these approaches, in addition to being interesting in themselves, have applications to Geometric Function Theory.

For a detailed exposition of the Teichmüller space theory, we refer the reader to the books [Ab2,Ah2,IT,Kru5,Leh2,Na].

2.5.2. Let Γ be an arbitrary torsion free Fuchsian group acting discontinuously on the disks Δ and Δ^* (and hence with the invariant unit circle). Consider the (complex) Banach space $\text{Belt}(\Gamma) = L_\infty(\Delta, \Gamma)$ of the *Beltrami differentials* (measured $(-1, 1)$ -forms) with respect to Γ , supported in Δ , namely

$$L_\infty(\Delta, \Gamma) = \{\mu \in L_\infty(\mathbb{C}): \mu|_{\Delta^*} = 0, (\mu \circ \gamma)\bar{\gamma}'/\gamma' = \mu, \gamma \in \Gamma\},$$

and its unit ball

$$\mathbb{M}(\Delta, \Gamma) = \{\mu \in L_\infty(\Delta, \Gamma): \|\mu\|_\infty < 1\}. \quad (2.8)$$

The quasiconformal self-maps of $\widehat{\mathbb{C}}$, satisfying the Beltrami equation $\bar{\partial}w = \mu \partial w$ with $\mu \in \mathbb{M}(\Delta, \Gamma)$ and normalized by means of condition $w(z)|_{\Delta^*} = z + a_1 z^{-1} + \dots$, will be denoted by w^μ . The correspondence $\mu \leftrightarrow w^\mu$ is one-to-one, and these maps are compatible with the group Γ , which means that for any $\gamma \in \Gamma$, $w^\mu \circ \gamma (w^\mu)^{-1} = \chi^\mu(\gamma)$ is again a Möbius transformation of $\widehat{\mathbb{C}}$. Moreover, w^μ conjugates Γ with the quasi-Fuchsian group $\Gamma^\mu = \chi^\mu(\Gamma)$.

The Teichmüller space $\mathbb{T}(\Gamma)$ of the group Γ (and of the Riemann surface Δ/Γ) is the set of equivalence classes $[\mu]$ of elements from $\mathbb{M}(\Delta, \Gamma)$ with respect to the relation $\mu \sim \nu$ if $w^\mu|_{\partial\Gamma} = w^\nu|_{\partial\Gamma}$ (then, in fact, w^μ equals w^ν on Δ^*).

For the trivial group $\Gamma = 1 = \{I\}$, the notations $L_\infty(\Delta)$ and $\mathbb{M}(\Delta)$ are more customary than $L_\infty(\Gamma, 1)$ and $\mathbb{M}(\Delta, 1)$.

Therefore, we assume any bounded measurable function on Δ to be extended by zero to Δ^* (respectively, elements of $L_\infty(w^\mu(\Delta))$ are assumed to be zero on $w^\mu(\Delta)$). For the group Γ^μ , we have a similar space $L_\infty(w^\mu(\Delta), \Gamma^\mu)$.

The Banach ball $\mathbb{M}(\Gamma)$ is the set of all conformal structures on Δ , hence $\mathbb{T}(\Gamma)$ is the space of equivalent conformal structures under the above natural identification.

The quotient space $\mathbb{T}(1) = \mathbb{T}$ is the *universal Teichmüller space*. A universal holomorphic covering $\Delta \rightarrow \Delta/\Gamma$ yields canonical embedding of $\mathbb{T}(\Gamma)$ into $\mathbb{T}$.

There are some natural intrinsic (complete) metrics on $\mathbb{T}(\Gamma)$. The first of them is the *Teichmüller metric*

$$\begin{aligned} \tau_{\mathbb{T}}(\phi(\mu), \phi(\nu)) \\ = \frac{1}{2} \inf \{ \log K(w^{\mu_*} \circ (w^{\nu_*})^{-1}): \mu_* \in \phi^{-1}(\mu), \nu_* \in \phi^{-1}(\nu) \}, \end{aligned} \quad (2.9)$$

where ϕ is the canonical projection

$$\phi(\mu) = [\mu]: \mathbb{M}(\Delta, \Gamma) \rightarrow \mathbb{T}(\Gamma). \quad (2.10)$$

This metric is generated by the Finsler structure on $\mathbb{T}(\Gamma)$ (in fact, on the tangent bundle $\mathcal{T}(\mathbb{T}(\Gamma))$ of $\mathbb{T}(\Gamma)$).

We recall that if $\mathcal{M}$ is a (complex or real) pathwise connected Banach manifold modeled by a Banach space E , then the *Finsler structure* on $\mathcal{T}(\mathcal{M})$ is a function

$$F_{\mathcal{M}}(\mathbf{x}, \xi) : \mathcal{T}(\mathcal{M}) \rightarrow \mathbb{R}^+ \cup \{0\}$$

satisfying the following conditions:

(i) For any fixed $\mathbf{x} \in \mathcal{M}$, the function $F_{\mathcal{M}}(\mathbf{x}, \cdot)$ defines a norm on the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$ which is equivalent to the initial norm on E :

$$A_1(\mathbf{x})\|\xi\| \leq F_{\mathcal{M}}(\mathbf{x}, \xi) \leq A_2(\mathbf{x})\|\xi\|.$$

(ii) This function is Lipschitz continuous when varying the first argument

$$|F_{\mathcal{M}}(\mathbf{x}, \xi) - F_{\mathcal{M}}(\mathbf{y}, \xi)| \leq A_3\|\mathbf{x} - \mathbf{y}\|\|\xi\|.$$

In particular, $F_{\mathcal{M}}$ is convex with respect to ξ .

The structure $F_{\mathcal{M}}(\mathbf{x}, \xi)$ defines on $\mathcal{M}$ the *Finsler metric* d_F as follows. For any piecewise C^1 -curve $\alpha : [0, 1] \rightarrow \mathcal{M}$, its F -length is defined by

$$L_F(\alpha) = \int_0^1 F_{\mathcal{M}}(\alpha(t), \alpha'(t)) dt;$$

then, for any pair of points $\mathbf{x}, \mathbf{y} \in \mathcal{M}$,

$$d_F(\mathbf{x}, \mathbf{y}) = \inf_{\alpha} L_F(\alpha),$$

where the infimum is taken over all such curves joining $\mathbf{x}$ and $\mathbf{y}$.

The *Finsler structure* on $\mathbb{T}(\Gamma)$ is defined by

$$\begin{aligned} F_{\mathbf{T}}(\phi(\mu), \phi'(\mu)v) \\ = \inf \left\{ \|v_*(1 - |\mu|^2)^{-1}\|_{\infty} : \phi'(\mu)v_* = \phi(\mu)v; \mu \in \mathbb{M}(\Delta, \Gamma); \right. \\ \left. v, v_* \in L_{\infty}(w^{\mu}(\Delta), \Gamma^{\mu}) \right\}. \end{aligned}$$

It is obtained by descending the Finsler form

$$F_{\mathbf{M}}(\mu, v) = \|v(1 - |\mu|^2)^{-1}\|_{\infty} \quad (2.11)$$

of the ball (2.8) by the projection (2.10).

A Beltrami differential $\mu \in \mathbb{M}(\Delta, \Gamma)$ is called *extremal* if

$$\tau_{\mathbb{T}}(\phi(\mu), \mathbf{0}) = K(w^{\mu}) = \varrho(0, \|\mu\|_{\infty}).$$

Here ϱ is the hyperbolic metric on the unit disk of curvature -4 (i.e., with the differential length element $ds = (1 - |z|^2)^{-1}|dz|$).

A differential μ is called *locally* (or *infinitesimally*) *extremal* if

$$F_{\mathbb{T}}(\mathbf{0}, \phi'(\mathbf{0})\mu) = \|\mu\|_{\infty}.$$

The well-known extremality criterion states that a differential $\mu \in \mathbb{M}(\Delta, \Gamma)$ is extremal if and only if the inequality

$$\|\mu\|_{\infty} = \sup \left\{ \left| \iint_{\Delta/\Gamma} \mu \varphi \, dx \, dy \right| : \varphi \in A_1(\Delta, \Gamma), \|\varphi\| = 1 \right\} \quad (2.12)$$

holds; here $A_1(\Delta, \Gamma)$ is the complex Banach space of holomorphic integrable $(2, 0)$ -forms $f(z) \, dz^2$, i.e., such that $(\varphi \circ \gamma)(\gamma')^2 = \varphi$, $\gamma \in \Gamma$, with L_1 -norm $\|\varphi\| = \iint_{\Delta/\Gamma} |\varphi| \, dx \, dy$ ($z = x + iy$). See, e.g., [EKK, Ga2, Ha, Kru5, RS1, Re1, St4].

If the supremum in (2.12) is attained on some $\varphi_0 \in A_1(\Delta, \Gamma)$, $\|\varphi_0\| = 1$, the extremal Beltrami differential takes the form

$$\mu(z) = \|\mu\|_{\infty} \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}$$

and is called also the *Teichmüller differential*. This holds, for example, for all the extremal differentials provided Γ is infinitely generated and is of the first kind.

The Finsler geometry provides an alternative proof of Teichmüller's theorem and its generalizations, see [OB, KK].

2.5.3. On the other hand, the space $\mathbb{T}(\Gamma)$ is a complex Banach manifold since as a quotient it inherits the complex structure of the ball $\mathbb{M}(\Delta, \Gamma)$. The projection ϕ is a holomorphic map with respect to this structure.

Thus $\mathbb{T}(\Gamma)$ possesses the invariant *Kobayashi metric* $d_{\mathbb{T}}(\mathbf{x}, \mathbf{y})$ which is the largest of the semi-metrics on $\mathbb{T}(\Gamma)$ that contract the holomorphic maps $\Gamma \rightarrow \mathbb{T}(\Gamma)$:

$$d(h(z'), h(z'')) \leq \varrho(z', z''), \quad h \in \text{Hol}(\Delta, \mathbb{T}(\Gamma)).$$

The *differential (infinitesimal) Kobayashi metric* $D_{\mathbb{T}}(\mathbf{x}, \xi)$ on $\mathcal{T}(\mathbb{T}(\Gamma))$ is defined by

$$\begin{aligned} D_{\mathbb{T}}(\mathbf{x}, \xi) &= \inf \{ |t| : f \in \text{Hol}(\Gamma, \mathbb{T}(\Gamma)), f(0) = \mathbf{x}, df(0)t = \xi \} \\ &= \inf \left\{ \frac{1}{r} : f \in \text{Hol}(\Delta_r, \mathbb{T}(\Gamma)), f(0) = \mathbf{x}, f'(0)t = \xi \right\}, \end{aligned}$$

where $\Delta_r = \{z : |z| < r\}$.

The well-known result of Royden [Ro2], extended to the Banach manifolds in [FV], states that the Kobayashi metric $d_{\mathcal{M}}$ of any complex Banach manifold $\mathcal{M}$ is restored by its infinitesimal form $D_{\mathcal{M}}$.

The fundamental Royden–Gardiner theorem [Ro1, Ga2] states that for any Teichmüller space the Kobayashi and Teichmüller metrics are equal:

THEOREM 2.3. (i) *The Kobayashi metric of the space $\mathbb{T}(\Gamma)$ equals its Teichmüller metric, and*

$$d_{\mathbb{T}}(\mathbf{x}, \mathbf{y}) = \tau_{\mathbb{T}}(\mathbf{x}, \mathbf{y}) = \inf\{\varrho(h^{-1}(\mathbf{x}), h^{-1}(\mathbf{y})): h \in \text{Hol}(\Delta, \mathbb{T}(\Gamma))\}. \quad (2.13)$$

(ii) *The infinitesimal version of (i)*

$$D_{\mathbb{T}}(\mathbf{x}, \xi) = F_{\mathbb{T}}(\mathbf{x}, \xi) = \inf\{(1 - |h_1^{-1}(x)|^2)^{-1}: h \in \text{Hol}(\Delta, \mathbb{T}(\Gamma))\}. \quad (2.14)$$

We will make use of this result in an essential way. Another proof of this theorem is given in [EKK]; for the proof of the infinitesimal part see also [EGL].

2.5.4. Another important biholomorphically invariant metric on complex manifolds is the *Carathéodory metric*. It is defined for $\mathbb{T}(\Gamma)$ by

$$c_{\mathbb{T}}(\mathbf{x}, \mathbf{y}) = \sup\{\rho(h(x), h(y)): h \in \text{Hol}(\Delta, \mathbb{T}(\Gamma))\}.$$

It easily follows from the general properties of invariant metrics and extremal quasiconformal maps that

$$c_{\mathbb{T}}(\mathbf{x}, \mathbf{y}) \leq d_{\mathbb{T}}(\mathbf{x}, \mathbf{y}) \leq \tau_{\mathbb{T}}(\mathbf{x}, \mathbf{y}).$$

Kra [Kr4] gave a sufficient condition for holomorphic Teichmüller disks

$$\Delta_{\varphi} = \{\Phi(t\bar{\varphi}/|\varphi|): t \in \Delta\}$$

in the finite-dimensional spaces $\mathbb{T}(X)$, which provides the coincidence of invariant metrics: this is true for the Abelian disks Δ_{φ} with $\varphi = \theta^2$, where θ is a holomorphic Abelian differential on the surface X (of a finite analytical type). The proof of Kra's theorem essentially relies on the intrinsic properties of closed Riemann surfaces.

An analogous (and even somewhat more general) result for the universal Teichmüller space $\mathbb{T}$ is established in [Kru9] by theoretic-functional arguments based on Grunsky's coefficient inequality and will be stated in the sequel to this paper. Shiga and Tanigawa [ShT] gave a condition on a Fuchsian group Γ , in terms of Poincaré series, under which such a phenomenon occurs. It seems that this fact must be true for all Teichmüller space $\mathbb{T}(\Gamma)$. A somewhat different approach was recently provided in [Sh4].

In any case, due to the rather unexpected applications of the Carathéodory metric to the solution of variational problems of Geometric Function Theory which have been discovered and which will be described in the following sections, it would of great importance, first, to establish whether such a result is valid for the Teichmüller space $\mathbb{T}(\Delta_n)$ for the disks with $n \geq 1$ punctures, and second, to find other quite sufficient conditions for invariant metrics to coincide on the disks Δ_{φ} .

A new complete almost plurisubharmonic metric on the finitely-dimensional Teichmüller space, which descends to the moduli space, is introduced by McMullen [McM3]. This metric closely relates to the Weil–Petersson form defining the symplectic structure on these spaces.

2.5.5. We will apply also the holomorphic *Bers embedding* of $\mathbb{T}(\Gamma)$ into the complex Banach space $Q^*(\Gamma)$ of holomorphic $(2, 0)$ -forms φ on Δ^* satisfying

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi, \quad \gamma \in \Gamma; \quad \varphi(z) = O(|z|^{-4}) \quad \text{as } z \rightarrow \infty,$$

with hyperbolic sup-norm

$$\|\varphi\|_{Q^*(\Gamma)} = \sup_{\Gamma^*} (|z|^2 - 1)^2 |\varphi(z)|.$$

The elements of $Q^*(\Gamma)$ are the *Schwarzian derivatives*

$$S_w(z) = \left(\frac{w''}{w'} \right)' - \frac{1}{2} \left(\frac{w''}{w'} \right)^2, \quad z \in \Delta^*,$$

of locally univalent functions w on Δ^* (compatible with Γ).

The image of the space $\mathbb{T}(\Gamma)$ in such an embedding is a bounded domain that consists of the Schwarzian derivatives of univalent functions in Δ^* with quasiconformal extension, also compatible with Γ .

We conclude this section with the following remark. The definition of the Teichmüller space $\mathbb{T}(X) = \mathbb{T}(\Gamma)$ for a Riemann surface X , presented above, involves its uniformization. In many cases it is convenient to do without it and represent $\mathbb{T}(X)$ as a set of homotopy classes of quasiconformal homeomorphisms of X modulo ideal boundary of X .

2.6. General variational problems for quasiconformal maps of Riemann surfaces of finite type

2.6.1. Let X and X' be two homeomorphic oriented marked Riemann surfaces of a given finite type (g, n, ℓ) , representing the points $\mathbf{X}, \mathbf{X}'$ of the Teichmüller space $\mathbb{T}(g, n, \ell)$ endowed by uniformizing complex parameters z and w , which vary on the universal coverings $\widehat{X}$ and $\widehat{X}'$, respectively, of these surfaces. Let $Q(X, X', q)$ be the class of quasiconformal homeomorphisms $f: X \rightarrow X'$ with dilatations $K(f) \leq q < \infty$, provided $q \geq \tau_T(\mathbf{X}, \mathbf{X}')$. Fix $r \geq 1$ distinct points $p_1, \dots, p_r$ on X whose values of z are $z_1, \dots, z_p$.

Let us consider the following general problem going back to Belinskii (1958).

Problem B. In the class $Q(X, X', q)$ find the maximum of the real functional $\mathcal{F}(f) = F(w_1, \dots, w_r)$, where $w_j = f(z_j)$ and the function F is defined on the Cartesian product $(\widehat{X}')^r$ of copies of $\widehat{X}'$.

Assume that X and X' are represented in Δ by their Fuchsian groups Γ and Γ' , i.e., $X = \Delta/\Gamma$, $X' = \Delta/\Gamma'$, so the points $z_1, \dots, z_r$ can be chosen in one fundamental polygon P of Γ in Δ . Let us assume also that F is a C^1 -function on $(\widehat{X}')^r$ with P of Γ in Δ . Let us assume also that F is a C^{-1} -function on $(\widehat{X}')^r$ with $\text{grad } F \neq 0$.

The most interesting and natural case is when F is Γ' -automorphic in each w_j , i.e., descends to a function on $(X')^r$, though the following discussion also holds for the more

general case in which the function F may not be invariant under the change of uniformizing parameter.

The existence of solutions of this problem easily follows from the compactness argument. The following theorem establishes the properties of the extremal homeomorphisms.

THEOREM 2.4 [Kru1]. *Suppose that $X = \Delta/\Gamma$ and $X' = \Delta'/\Gamma'$ are homeomorphic marked surfaces of finite type and that $w = f_0(z)$ is a quasiconformal homeomorphism maximizing $\mathcal{F}[f]$ in the class of homeomorphisms $f: \bar{X} \xrightarrow{\text{onto}} \bar{X}'$ with $K(f) \leq q$. Then there exists a constant α in $[0, 2\pi]$ and a quadratic differential $\psi_0(w)dw^2 \in A_1(\Delta', \Gamma')$ such that the Beltrami coefficient $\mu_{f_0^{-1}}(w)$ in the disk Δ' has the form*

$$\begin{aligned} |\mu_{f_0^{-1}}(w)| &= \frac{q-1}{q+1}, \\ \arg \mu_{f_0^{-1}}(w) &= -\arg[e^{i\alpha}\psi_*(w) + \psi_0(w)], \end{aligned} \quad (2.15)$$

where

$$\psi_*(w) = \sum_{j=1}^r F_{w_j} \sum_{\gamma \in \Gamma} \frac{\gamma'^2(w)}{w_j - \gamma w}, \quad w_j = f_0(z_j). \quad (2.16)$$

If X and X' are bordered surfaces, then the differential $(e^{i\alpha}\psi_* + \psi_0)dw^2$ assumes real values on the boundary of X' .

SKETCH OF THE PROOF. Let f_0 be an extremal homeomorphism maximizing $F(f)$ on $Q(X, X', q)$, and $f_0(z_j) = w_{0j}$, $j = 1, \dots, r$. Denote the images of w_{0j} on X' by p'_j .

By Teichmüller's theorem, there exists a unique homeomorphism $f_1 \in Q(X, X', q)$ with the smallest dilatation $K(f_1) = K_1$ among all homeomorphisms $f: X \rightarrow X'$, which move the points $p_1, \dots, p_r$ into $p'_1, \dots, p'_r$, respectively. We have

$$\mu_1(w) := \mu_{f_1^{-1}}(w) = k_1 \frac{\overline{\psi_1(w)}}{|\psi_1(w)|}, \quad k_1 = \frac{K_1 - 1}{K_1 + 1},$$

where $\psi_1 dw^2 \neq 0$ is a meromorphic quadratic differential (defined up to a positive factor) on X' that has at most simple poles at the points $p'_1, \dots, p'_r$ and is holomorphic elsewhere. Obviously, $\mathcal{F}(f_1) = \mathcal{F}(f_0)$.

It is not hard to establish that $K_1 = q$. To this end, take a fundamental polygon P' of the surface X' in Δ containing the points $w_{01}, \dots, w_{0r}$. Since the differential $\psi_* dw^2$ constructed according to (2.16) does not belong to $A_1(\Delta', \Gamma')$ the distance from ψ_* to $A_1(\Delta', \Gamma')$ in $L_1(\Delta', \Gamma')$ is equal to $d > 0$. By the Hahn–Banach theorem, there exists a linear functional $m_0(\psi)$ in $L_1(\Delta', \Gamma')$ such that

$$m_0(\psi) = 0, \quad \psi \in A(\Delta', \Gamma'); \quad \|m_0\| = 1, \quad m_0(\psi_*) = d. \quad (2.17)$$

This functional is represented by a corresponding Beltrami coefficient $v_0(w) d\bar{w}/dw$ on X' so that

$$m_0(\psi) = \iint_{P'} v_0(w) \psi(w) du dv, \quad \|v_0\|_\infty = 1, \quad (2.18)$$

and defines an automorphism (variation) of the surface X' with complex dilatation $\varepsilon v_0(w) + O(\varepsilon^2)$, which can be represented in the form

$$\omega = H(w, \varepsilon) = w + \frac{\varepsilon}{\pi} \iint_{P'} v_0(\zeta) \sum_{\gamma \in \Gamma'} \frac{\gamma'^2 \zeta}{w - \gamma \zeta} d\xi d\eta + O(\varepsilon^2), \quad (2.19)$$

where ε is a sufficiently small real parameter. If $K(f_1)$ were less than q , this variation would be admissible for sufficiently small ε , but, on the other hand, we have

$$dF = 2 \operatorname{Re} \sum_{j=1}^r F_{w_j} dw_j = 2\varepsilon \pi^{-1} \operatorname{Re} m_0(\psi_*) = 2\varepsilon \pi^{-1} d > 0, \quad (2.20)$$

in contradiction to equality $\mathcal{F}(f_1) = \max\{\mathcal{F}(f): f \in Q(X, X', q)\}$.

To establish the second equality in (2.15), consider the subspace Ω in $L_1(\Delta', \Gamma')$ consisting of elements of the form

$$\psi' = \lambda \psi_* + \varphi, \quad \varphi \in A_1(\Delta', \Gamma'), \lambda = \text{const},$$

and define

$$\sup_{\psi \in \Omega, \|\psi\|=1} \left| \iint_{P'} \mu_1(w) \psi(w) du dv \right| = k'_1.$$

Our goal is to show that $k'_1 = k_1$, which implies (2.15).

Suppose that $k'_1 < k_1$ and choose t in such a way that $0 < t < k_1 - k'_1$. Consider the functional

$$m_t(\psi) = \iint_{P'} [\mu_1(w) - t v_0(w)] \psi(w) du dv,$$

whose norm on the subspace Ω is estimated by $\|m_t\|_\Omega \leq k'_1 + t = k_t < k_1$, and take its Hahn–Banach extension

$$\ell_t(\pi) = \iint_{P'} \rho_t(w) \psi(w) du dv$$

from Ω to $L_1(\Delta, \Gamma')$, with $\rho_t \in L_\infty(\Delta, \Gamma')$ and

$$\|\rho_t\|_\infty = \|\ell_t\|_{L_1(\Delta, \Gamma')} = \|m_t\|_\Omega \leq k_t. \quad (2.21)$$

Then $v_t = \mu_1 - \rho_t \in A_1(\Delta, \Gamma')^\perp \setminus \{0\}$, i.e., is locally trivial on X' and defines a variation $w^{\varepsilon v_t}$ of X' , for which $\tau_T(w^{\varepsilon v_t}(X'), X') = O(\varepsilon^2)$ and, on the other hand, similar to (2.20)

$$\mathcal{F}(w^{\varepsilon v_t} \circ f_1) - \mathcal{F}(f_1) = \frac{2}{\pi} \varepsilon t d + O(\varepsilon^2) > 0.$$

Exact estimating of the L_∞ -norm of the complex dilatation of composition $w^{\varepsilon v_t} \circ f_1$ requires rather long delicate arguments using (2.20) and the equality

$$\begin{aligned} & \mu_{f_1^{-1} \circ (w^{\varepsilon v_t})^{-1}}(\omega(w)) \bar{\omega}_w / \omega_w \\ &= \mu_{f^{-1}}(w) - \varepsilon v(w) + \varepsilon \overline{v(w)} \mu_{f^{-1}}^2 + O(\varepsilon^2), \quad \omega = w^{\varepsilon v}, \end{aligned} \quad (2.22)$$

which implies

$$\|\mu_{f_1^{-1} \circ (w^{\varepsilon v_t})^{-1}}\|_\infty < k_1 - \varepsilon \eta(k_1, k'_1, t) > 0,$$

where $\eta(k_1, k'_1, t) = \text{const} > 0$, and we again reach a contradiction to extremality of f_1 for $\mathcal{F}$.

For details we refer to the book [Kru5, Chapter 2]. Similar arguments are applied there for proving Teichmüller's theorem. $\square$

2.6.2. Remarks and additions. (1) When X and X' are either the Riemann sphere $\widehat{\mathbb{C}}$ or the disk Δ (or the half-plane), the functions ψ_0 and ψ_* in (2.15) and (2.16) become rational on $\widehat{\mathbb{C}}$; moreover, in the second case $\psi_0 dw^2$ and $\psi_* dw^2$ are real on $S^1 = \partial \Delta$.

In particular, if the number of fixed points b_j is three, when we shift to the double of the disk, one obtains the case examined by Belinskii [Bel3, Bel4]. Then the set of admissible variations is broadened and we can express the function $\psi_0(w)$ and the constant $\varkappa$ explicitly in terms of $w_1, w_2, \dots, w_r$.

For example, if the conditions of normalization have the form $f(0) = 0$ and $f(1) = 1$, formulas (2.15) and (2.16) take the forms

$$|\mu_{f_0^{-1}}(w)| = (q-1)(q+1), \quad \arg u_{f_0^{-1}}(w) = -\arg B(w),$$

where

$$B(w) = \sum_{j=1}^1 \left[\frac{F_{w_j} w_j (w_j - 1)}{w(w-1)(w_j - w)} + \frac{\bar{F}_{w_j} \bar{w} (\bar{w}_j - 1)}{w(w-1)(1 - \bar{w}_j w)} \right].$$

(2) Volynets [Vol] showed that in fact the constant $\varkappa$ in (2.15) equals π .

(3) Theorem 2.5 establishes the qualitative properties of the extremal mappings of Problem B and provides a solution of that problem in terms of the Beltrami coefficient $\mu_{f^{-1}}(w)$ of the inverse map. One has to determine the function (differential) ψ_0 as well as to construct the mapping function by its complex dilatation. In general, this is quite laborious.

In the case when $\varepsilon = q - 1$ is small, one can approximately construct the maps using the variational formulas of Section 1.3, by replacing in (2.15), (2.16) w with z and w_j with z_j . In either case Theorem 2.4 yields an asymptotically sharp bounds and shows that

$$\max \mathcal{F}(f) = \mathcal{F}(\text{id}) + O(\varepsilon), \quad \varepsilon \rightarrow 0.$$

(4) Sheretov and his followers extended the above results to the classes of maps whose dilatations are bounded by a nonconst function: $|\mu_f(z)| \leq q(z) < q_0 < 1$, see, e.g., [She2, She3], cf. Section 2.7.5.

(5) Theorem 2.4 is generalized by Sallinen [Sa] to arbitrary smooth real functionals $\mathcal{F}[f]$. His proof relies on the arguments concerning the general implicit function theorem.

2.6.3. Another generalization is due to Ryazanov [Rya1]. Extremal quasiconformal deformations of divisor classes on compact Riemann surfaces of genus $g > 0$ are studied in [Ya1].

2.7. Back to tori and annuli

We describe briefly certain specific features arising in the cases when the surfaces X and X' are tori or annuli, i.e., of the types $(1, 0, 0)$ and $(0, 0, 2)$, respectively.

2.7.1. Any torus $T^2 = S^1 \times S^1$ can be represented by a lattice

$$\Gamma = \{\gamma: z \mapsto z + m + n\tau; (m, n) \in \mathbb{Z}^2\} \subset SL(2, \mathbb{Z}), \quad \text{Im } \tau > 0,$$

i.e., is conformally equivalent to $X = \mathbb{C}/\Gamma$. This value of τ is the *modulus* of the marked torus X , and the moduli space for tori is $U/SL(2, \mathbb{Z}) \simeq \mathbb{C}$.

The variation formula for quasiconformal homeomorphisms of tori is obtained from (1.10) and assumes the form

$$\begin{aligned} f^\mu(z) = z + \frac{1}{2\pi i} \iint_{\mathbb{C}/\Gamma} \mu(t) [\zeta(t - z; 1, \tau) + (z - 1)\zeta(t; 1, \tau) \\ - z\zeta(t - 1; 1, \tau)] dt \wedge d\bar{t} + O(\|\mu\|_\infty^2), \quad |z| \leq R < \infty, \end{aligned} \quad (2.23)$$

where

$$\begin{aligned} \zeta(u) &= \zeta(u; 2\omega_1, 2\omega_2) \\ &= \frac{1}{u} + \sum_{(m, n) \neq (0, 0)} \left[\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 + 2n\omega_2)^2} \right] \end{aligned}$$

is the Weierstrass ζ -function with quasiperiods $2\omega_1$ and $2\omega_2$; in our case $2\omega_1 = 1$ and $2\omega_2 = \tau$. Since $f(\tau) = \tau'$, one gets from (2.23), using the Legendre relation

$$\zeta(\omega_1)\omega_2 - \zeta(\omega_2)\omega_1 = \pi i/2,$$

the following formula for the change of modulus τ of a marked torus X with the variation f^μ :

$$\tau' - \tau = \iint_{\mathbb{C}/\Gamma} \mu(t) dt \wedge d\bar{t} + O(\|\mu\|_\infty^2).$$

All holomorphic quadratic differentials on the torus $X = \mathbb{C}/\Gamma$ are of the form $c dz^2$, $c \in \mathbb{C}$ and constitute one-dimensional complex space. Thus every extremal quasiconformal map $f_0: X \rightarrow X'$ has the Beltrami coefficient

$$\mu_{f_0}(z) = k_0 e^{-i \arg c_0} \equiv \text{const}, \quad z \in \mathbb{C},$$

and therefore is an affine map of the plane $\mathbb{C}$ defined up to a linear map of $\mathbb{C}$.

As a sequence, we again obtain the result of Grötzsch on extremal quasiconformal maps of the rectangles with correspondence of all vertices given in Section 2.1.

Theorem 2.4 now says that the maximum of the functional $\mathcal{F}[f] = F(w_1, \dots, w_n)$ on the class of quasiconformal homeomorphisms of a marked torus $X = \mathbb{C}/\Gamma$ onto the torus $X' = \mathbb{C}/\Gamma'$ with $K(f) \leq q$ is attained on the maps f_0 whose inverses have the Beltrami differentials $\mu_{f_0^{-1}}$ of the form

$$\begin{aligned} |\mu_{f_0^{-1}}(w)| &= \frac{q-1}{q+1}, \\ \arg \mu_{f_0^{-1}}(w) &= -\arg[c_1 \psi_*(w) + c_2], \quad w \in \mathbb{C}, \end{aligned}$$

where c_1 and c_2 are two constants,

$$\psi_*(w) = \sum_{j=1}^r F_{w_j}[\zeta(w - w_j; 1, \tau') - \zeta(w; 1, \tau')],$$

and τ' is the modulus of the torus X' .

If we do not fix the orientation of a torus, we must add, in addition to

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \in SL(2, \mathbb{Z}), \tag{2.24}$$

the transformations $\tau \mapsto (a - b\bar{\tau})/(c - d\bar{\tau})$.

2.7.2. The case of maps of an annulus (and hence of any doubly-connected region) can be reduced to maps of a torus by passing to the double, or it can be examined independently. Recall that the holomorphic differentials in the annulus $\mathcal{R} = \{h \leq |z| \leq 1\}$ ($0 < h < 1$) that assume real values on its boundary are of the form $c dz^2/z^2$, where c is a real constant (and the set of such differentials forms a one-dimensional real space).

The formula for quasiconformal variations f^μ of the annulus $\mathcal{R}$, with $f^\mu(1) = 1$, assumes the form (see, e.g., [Kru5]):

$$f^\mu(z) = z - \frac{z}{4\pi i} \iint_G \left\{ \frac{\mu(t)}{t^2} \left[K_h\left(\frac{t}{z}\right) - K_h(t) \right] - \frac{\overline{\mu(t)}}{t^2} [K_h(z\bar{t}) - K_h(\bar{t})] \right\} dt \wedge d\bar{t} + O(\|\mu\|_\infty^2). \quad (2.25)$$

Here

$$\begin{aligned} K_h(t) &= \zeta\left(\frac{\omega_1}{\pi i} \log t\right) - \frac{\zeta(\omega_1)}{\pi i} \log t \\ &= \frac{\pi i}{\omega_1} \left[\frac{1+t}{1-t} + 2 \sum_{k=1}^{\infty} \left(\frac{h^{2k} t}{1-h^{2k} t} - \frac{h^{2k} t^{-1}}{1-h^{2k} t^{-1}} \right) \right], \end{aligned} \quad (2.26)$$

$\zeta(u) = \zeta(u; 2\omega_1, 2\omega_2)$, where the imaginary quasiperiod $2\omega_2$ is defined by

$$\frac{\omega_2}{\omega_1} = \frac{1}{\pi i} \log h,$$

and the estimate of the remainder term in (2.26) is uniform in any annulus $0 < R \leq |z| \leq R' < \infty$. The expansion (2.25) is well known in the theory of elliptic functions.

It follows from (2.25) and (2.26) that the annulus $\mu(\mathcal{R})$ has an inner radius

$$h' = h \left(1 + \frac{1}{2\pi i} \operatorname{Re} \iint_a \frac{\mu(t)}{t^2} dt \wedge d\bar{t} \right) + O(\|\mu\|_\infty^2); \quad (2.27)$$

thus for the conformal *modulus*

$$\operatorname{mod} \mathcal{R} = \frac{1}{2\pi} \log \frac{1}{h}$$

of $\mathcal{R}$, we get

$$\operatorname{mod} f^\mu(\mathcal{R}) = \operatorname{mod} \mathcal{R} + \frac{1}{4\pi^2 i} \iint_{\mathcal{R}} \frac{\mu(t)}{t^2} dt \wedge d\bar{t} + O(\|\mu\|_\infty^2).$$

2.7.3. Let us look at some applications of the obtained results.

First consider the ordered quadruples $a = (a_1, a_2, a_3, a_4)$ of distinct points on $\widehat{\mathbb{C}}$. Their cross-ratios

$$\alpha = \frac{a_3 - a_1}{a_2 - a_1} : \frac{a_3 - a_4}{a_2 - a_4} \in \mathbb{C} \setminus \{0, 1\} =: \mathbb{C}^* \quad (2.28)$$

are invariant under Möbius transformations. Let $\mathbb{C}(a) = \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$. The following Teichmüller theorem [Te1] has various applications in the theory of quasiconformal maps.

THEOREM 2.5. *There is a K -quasiconformal automorphism of $\widehat{\mathbb{C}}$ moving the ordered quadruple (a_1, a_2, a_3, a_4) into the ordered quadruple (a'_1, a'_2, a'_3, a'_4) if and only if their cross-ratios α and α' satisfy $\rho_{\mathbb{C}^*}(\alpha, \alpha') \leq \frac{1}{2} \log K$, where $\rho_{\mathbb{C}^*}(\cdot, \cdot)$ is the hyperbolic metric on $\mathbb{C}^*$ of Gauss' curvature -4 .*

One can require in this theorem much more, namely, that the desired quasiconformal automorphism $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ belongs to a given homotopic class of homeomorphisms of the punctured spheres $\mathbb{C}(a) \rightarrow \mathbb{C}(a')$. The different proofs and some applications of theorem can be found in [Ah2, Ag1, Ho1, Ho2, Kr3, KK, LVV]. We provide here another application, following [Kru5].

It suffices to consider the quadruples $(0, 1, \alpha, \infty)$, applying additional fractional-linear transformations of $\widehat{\mathbb{C}}$ to the initial ones. Let $\widetilde{\mathbb{C}}(\alpha)$ be the two-sheeted covering of $\widehat{\mathbb{C}}$ with branch points $0, 1, \alpha, \infty$; it is conformally equivalent to a torus X . This conformal isomorphism $\widetilde{\mathbb{C}}(\alpha) \leftrightarrow X$ is realized by the elliptic integral of the first kind

$$u = \int_{z_0}^z \frac{dt}{\sqrt{t(t-1)(t-\alpha)}}, \quad (2.29)$$

where z_0 is a fixed point distinct from $0, 1, \alpha$ and ∞ , and a fixed branch of the square root in a neighborhood of z_0 is chosen. We take a canonical dissection of $\widetilde{\mathbb{C}}(\alpha)$ formed by the twice converted cuts along a Jordan arc γ_1 connecting the points 0 and 1 and along a Jordan arc γ_2 connecting the points 0 and α . The image of the dissected surface $\widetilde{\mathbb{C}}(\alpha)$ under the map (2.29) is a topological quadrilateral G in the plane $\mathbb{C}_u$ with pairwise identified opposite sides. Its conformal modulus $\tau = \omega_2/\omega_1$, where

$$\omega_j = \int_{\gamma_j} \frac{dz}{\sqrt{z(z-1)(z-\alpha)}}.$$

Now, let $a' = (a'_1, a'_2, a'_3, a'_4)$ be another ordered quadruple; it is equivalent to $(0, 1, \alpha', \infty)$. Any quasiconformal automorphism $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ moving a into a' (i.e., with fixed points $0, 1, \infty$ and moving α into α') is lifted to a quasiconformal homeomorphism $\tilde{f}: \widetilde{\mathbb{C}}(\alpha) \rightarrow \widetilde{\mathbb{C}}(\alpha')$ with $K(\tilde{f}) = K(f)$. Applying the conformal maps of both tori $\widetilde{\mathbb{C}}(\alpha)$ and $\widetilde{\mathbb{C}}(\alpha')$, one obtains that

$$q = \min_f K(f) = \frac{|\tau' - \bar{\tau}| + |\tau' - \tau|}{|\tau' - \bar{\tau}| - |\tau' - \tau|} \quad (2.30)$$

(where $2\omega'_1$ and $2\omega'_2$ are the corresponding periods for $\widetilde{\mathbb{C}}(\alpha')$ and $\tau' = \omega'_2/\omega'_1$). Hence, the extremal map f_0 minimizing $K(f)$ corresponds to the affine map

$$u' = \frac{(\tau' - \bar{\tau})u + (\tau - \tau')\bar{u}}{\tau - \bar{\tau}} \quad (2.31)$$

of the plane $\mathbb{C}_u$. Only these maps determine the boundary points of the non-Euclidean disk

$$\Delta(\alpha') = \left\{ \alpha' \in \mathbb{C}^*: \rho_{\mathbb{C}^*}(\alpha, \alpha') = \frac{1}{2} \log q \right\}$$

in Theorem 2.6.

2.7.4. When $\alpha = -r$ (for $0 < r < \infty$), the integral (2.29) with $z_0 = 0$ maps the upper half-plane $\{\operatorname{Im} z > 0\}$ onto the rectangle with vertices $0, K(k), K(k) + iK'(k)$ and $iK'(k)$, where

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

is the complete elliptic integral of the first kind with modulus $k = \sqrt{r/(r+1)}$ and

$$K'(k) = K(\sqrt{1-k^2}).$$

If in addition $\alpha' = -\rho$, where $0 < \rho < \infty$, we have from (2.30) that

$$q = \begin{cases} \lambda(\rho)/\lambda(r) & \text{for } \rho \leq r, \\ \lambda(r)/\lambda(\rho) & \text{for } \rho \geq r, \end{cases} \quad (2.32)$$

where

$$\lambda(t) = \frac{K'(\sqrt{t/(t+1)})}{K(\sqrt{t/(t+1)})}, \quad 0 < t < \infty.$$

We note that $\lambda(t)$ is a monotonically decreasing function.

Let us now estimate

$$\max_{|z|=r} |f^\mu(z) - z| = L_0(r)$$

in the class of automorphisms $w = f^\mu(z)$ of $\widehat{\mathbb{C}}$ with $K(f^\mu) \leq q$. It can be shown (see [LVV]) that the mapping $f^{\mu_0}(z)$ that satisfies

$$\max_{f^\mu} L(r) = L_0(r)$$

maps the real axis into itself and

$$L_0(r) = |f^{\mu_0}(-r) + r|, \quad f^{\mu_0}(-r) < 0,$$

that is, $L_0(r)$ is attained at $z = -r$.

From (2.32), we get the following result of Belinskii [Bel3].

THEOREM 2.6. *In the class of quasiconformal automorphisms $w = f^\mu(z)$ such that $K(f^\mu) \leq q$, we have the sharp estimate*

$$|f^\mu(z) - z| \leq \lambda^{-1} \left(\frac{1}{q} \lambda(r) \right) - r, \quad r = |z|. \quad (2.33)$$

Here, an arbitrary circle $|z - z_0| = R$ is deformed in such a way that

$$\frac{\max_z |f^\mu(z) - f^\mu(z_0)|}{\min_z |f^\mu(z) - f^\mu(z_0)|} \leq \lambda^{-1} \left(\frac{1}{q} \lambda(1) \right). \quad (2.34)$$

It remains only to prove inequality (2.34). To this end, note that, by applying a translation and a similarity transformation, we reduce the problem to the case of the circle $|z| = 1$. We may assume that $f^\mu(1) = \min_{|z|=1} |f^\mu(z)|$. Then (2.34) follows from (2.33).

Note that estimates of this kind were first obtained by Lavrentiev [La1] for q close to 1.

One can derive from Theorem 2.7 some sharp estimates for other functionals, e.g., for the ratio $f^\mu(z_2)/f^\mu(z_1)$ for an arbitrary fixed pair $z_1, z_2 \in \mathbb{C}^*$ (cf. [Ag1, Va2]). We shall show this below.

2.7.5. A simple consequence of formulas (2.22) and (2.23) is the following assertion, proven by various authors, wherein C_r denotes the circle $|z| = r$.

THEOREM 2.7. *In the class of quasiconformal homeomorphisms f of the annulus $\mathcal{R} = \{h < |z| < 1\}$ onto annuli $\mathcal{R}' = \{h' < |w| < 1\}$ such that $f(C_1) = C_1$ with dilatations $K(z)$ such that $1 \leq K(z) \leq K_0(z)$, where $K_0(z)$ is a given measurable function bounded in G , the maximum (resp. minimum) value of $\text{mod } G'$ is attained with a mapping having dilatation $K(z) = K_0(z)$ that maps infinitesimally small ellipses in G into infinitesimally small ellipses whose major (resp. minor) axes are located along the rays $\arg w = \text{const}$. The extremal map is unique up to rotations.*

In the case $f(C_h) = C_1$, rotations are replaced with the maps a_1/w or a_2/z (where a_1 and a_2 are constants). In particular, for $K_0(z) = q = \text{const}$, we have $h^q \leq h' \leq h^{1/q}$, that is, a q -quasiconformal map does not change the modulus of the annulus by a factor bigger than q [Gro2].

The problem of describing Riemann surfaces (in particular, multiple-connected regions) obtained from a given surface by quasiconformal homeomorphisms with a given dilatation $K(z) = (1 + |\mu(z)|)(1 - |\mu(z)|) = K_0(z)$ (or $K(z) \leq K_0(z)$) and different functions $\theta(z) = (1/2) \arg \mu(z)$ was posed already by Teichmüller in [Te1, p. 15], and later reformulated by Volkovyskiy [Vo]. This problem has been studied by Andreian Cazacu (see, e.g., [AC1, AC2]), Kühnau [Ku2], Sheretov [Sh1], Ioffe [Io].

2.8. Extremal quasiconformal maps: General theory

2.8.1. Teichmüller's Theorem 2.2 concerns the situation in which the dimension of the space of integral holomorphic quadratic differentials is finite. In view of Ahlfors well-

known homotopy, two homeomorphisms f and f' of a Riemann surface X onto a Riemann surface X' are homotopic on X if and only if the isomorphisms $\pi_1(X) \rightarrow \pi_1(X')$ induced by these maps differ from inner automorphisms of these fundamental groups, or equivalently, the same holds for the induced isomorphisms of the deck transformation groups of the universal coverings $\widehat{X}$ and $\widehat{X}'$.

This naturally leads to extremal quasiconformal maps in the equivalence class $[f]$ of a given quasiconformal automorphism f of the disk Δ (or some other domain in $\mathbb{C}$), having the same boundary values $\omega = f|_{\partial\Delta}$, or in other words, minimizing the dilatation $K(f')$ on the set of quasiconformal extensions to Δ of a given quasisymmetric homeomorphisms of $\partial\Delta$ (or of $\mathbb{R} = \partial\mathbb{H}^2$). This question goes back to [Te1].

2.8.2. In 1962 there appeared Strebel's famous example on extremal maps of a chimney region. Fix $K > 1$ and consider the affine stretching $f_0(z) = x + iKy$ ($z = x + iy$) of the plane region

$$G = \{z: y < 0\} \cup \{z: |x| < 1, y > 0\}$$

onto itself. This map f_0 is K -quasiconformal, with $\mu_{f_0}(z) = \text{const} \neq 0$ and, moreover, it is extremal in its class $[f_0]$. However, this class contains infinitely many distinct extremal maps, for example, each map $f(z) = x + ih(y)$ with $h(0) = 0$, $h(y) = Ky$ for $y > 0$ and $|h'(y)| < K$ for $y < 0$ is extremal in $[f_0]$ (see [St1]).

Belinskii constructed in 1962 the example (published in [Kru4]) of an extremal quasiconformal map arising as a variable stretching of a strip, which brings out even better the features (substantial points) of the case in which the entire boundary correspondence is prescribed. Consider the map $f_p(z) = p(x) + iy$ of the strip $\Pi = \{z: -\infty < x < \infty, 0 < y < 1\}$, where $p(x)$ increases monotonically on $[-\infty, \infty]$ from 1 to some $p_0 < \infty$. Let f be an arbitrary quasiconformal self-map of Π , which coincides with f_p for $y = 0$ and $y = 1$. Theorem 2.6 provides the existence of a constant $M = M(K(f)) < \infty$ such that for all $x_0 \in (-\infty, \infty)$, the image of the segment $\{x = x_0, 0 < y < 1\}$ under the map f is located in the rectangle $\{f_p(x_0) - M < x' < f_p(x_0) + M, 0 < y' < 1\}$. Therefore, f maps the rectangle $\mathcal{R}_0 = \{x_0 < x < 2x_0, 0 < y < 1\}$ onto the topological quadrilateral $\mathcal{R}'_0$ with the vertices $f_p(jx_0)$, $f_p(jx_0 + i)$, $j = 1, 2$, whose modulus equals $p_0x_0(1 + O(1))$ as $x_0 \rightarrow \infty$. Consequently,

$$K(f) \geq \sup_{0 < x_0 < \infty} (\text{mod } \mathcal{R}'_0 / \text{mod } \mathcal{R}_0) = p_0,$$

thus f_p is extremal (though also not unique) in its class $[f_p]$.

2.8.3. Of course, an extremal map in the class $[\omega]$ for a quasisymmetric $\omega: S^1 \rightarrow S^1$ can be regarded as a limit of the extremal quasiconformal automorphisms f_n of the disk carrying the distinguished points $z_1, \dots, z_n \in \partial\Delta$ into $\omega(z_1), \dots, \omega(z_n)$ as $n \rightarrow \infty$, provided the set $\{z_n\}$ is dense on $\partial\Delta$. Each of these "polygonal" extremal maps f_n has the Beltrami coefficient $\mu_n(z) = k_n \overline{r_n(z)} / |r_n(z)|$, where r_n is a rational function with simple poles and $r_n(z) dz^2$ is real on $\partial\Delta$. One can normalize these r_n , setting $\|r_n\|_{A_1(\Delta)} = k_n$.

As $n \rightarrow \infty$, then f_n converge to f_0 uniformly on the closed disk $\bar{\Delta}$ and the dilatations $K(f_n) \nearrow K(f_0)$. However, $r_n(z)$ may converge to zero (on compact subsets of Δ), i.e., form a *degenerate* sequence.

The same can occur for extremal maps of open Riemann surfaces by exhausting these surfaces by bordered subsurfaces of finite analytical type.

2.8.4. We now present certain fundamental results concerning general extremal quasiconformal maps. It is natural to regard the equivalence classes $[f]$ as the points of the Teichmüller space $\mathbb{T}(\Gamma)$.

An element $\mu_0 \in \mathbb{M}(\Gamma)$ is called *extremal* if

$$\tau(\mu_0, \mathbf{0}) = \tau_{\mathbb{T}(\Gamma)}(\phi(\mu_0), \phi(\mathbf{0})) = \inf\{\tau(\mu, \mathbf{0}) : \mu \in \phi^{-1}(\mu_0)\}; \quad (2.35)$$

here ϕ is again the canonical projection $\mathbb{M}(\Gamma) \rightarrow \mathbb{T}(\Gamma)$.

THEOREM 2.8. *An element $\mu_0 \in \mathbb{M}(\Gamma)$ is extremal if and only if*

$$\sup\left\{\left|\iint_{\Delta/\Gamma} \mu_0 \varphi dx dy\right| : \varphi \in \mathcal{Q}(\Gamma), \|\varphi\| = 1\right\} = \|\mu_0\|_\infty. \quad (2.36)$$

This result is called *the Hamilton–Krushkal–Reich–Strebel theorem*. The necessity of (2.36) was first proved in a more special case in Krushkal’s papers in 1965 [Kru1, Kru2, Kru3], and independently by Hamilton in 1966. The sufficiency of this condition was established by Reich and Strebel in [RS1] and its sequels by using their fundamental inequality: *let f be a quasiconformal map of a Riemann surface X onto a Riemann surface X' and let K_0 be the dilatation of an extremal map in the class of f , then for all $\varphi \in \mathcal{Q}(X)$ with $\|\varphi\| = 1$, we have*

$$\frac{1}{K_0} \leq \iint_X \frac{|1 - \mu(\varphi/|\varphi|)|^2}{1 - |\mu|^2} |\varphi| dx dy, \quad \mu = \mu_f, \quad (2.37)$$

or, equivalently,

$$\left|\iint_X \frac{\mu\varphi}{1 - |\mu|^2} dx dy\right| \leq \frac{k_0}{1 + k_0} + \iint_X \frac{|\mu|^2}{1 - |\mu|^2} |\varphi| dx dy, \quad (2.38)$$

where $K_0 = (K_0 - 1)/(K_0 + 1)$. For the proof of this inequality, see, e.g., [Ber8, RS1, Ga4, GaL].

An element $\mu_0 \in \mathbb{M}(\Gamma)$ is called *locally extremal* if equality (2.35) holds for the *Finsler lengths* of μ_0 and $\phi(\mu_0)$; i.e., $F_M(\mathbf{0}, \mu_0) = F_{\mathbb{T}(\Gamma)}(\phi(\mathbf{0}), \phi'(\mathbf{0})\mu_0)$. This is also equivalent to (2.36).

A remarkable observation of Reich and Chen is that for any extremal quasiconformal deformation $F(z, t) : \Delta \times [0, 1] \rightarrow \mathbb{C}$ the $\bar{\partial}$ -derivative $\bar{\partial}F$ is bounded and satisfies (2.36), see [RC, Sh3].

2.8.5. The extremal Beltrami differentials are naturally connected with geodesics in Teichmüller spaces. When $\mathbb{T}(\Gamma)$ is finite-dimensional, every extremal μ is uniquely extremal and has Teichmüller form

$$\mu = k|\varphi|/\varphi, \quad 0 \leq k < 1, \varphi \in \mathcal{Q}(\Gamma) \setminus \{\mathbf{0}\}. \quad (2.39)$$

Moreover, every geodesic segment is uniquely determined by its endpoints, and every holomorphic isometry $f: \Delta \rightarrow \mathbb{T}(\Gamma)$ with $f(0) = 0$ has the form $f(t) = \phi(t\mu/\|\mu\|_\infty)$, with μ given by (2.39). The image of such a holomorphic isometry is called the *Teichmüller disk*.

When $\mathbb{T}(\Gamma)$ is infinite-dimensional, an extremal μ is not necessarily uniquely extremal, and there always exist two points that are the endpoints of infinitely many distinct geodesic segments. This was proved by Li Zhong (see [Li1]) when the group Γ is trivial and by Tanigawa (see [Ta]) in the general case; see also [Sh2]. However, Li Zhong proved (see Theorem 3 in [Li1]) that the geodesic segment joining 0 and $\Phi(\mu)$ is unique if μ is uniquely extremal and $|\mu|$ is constant. The following theorem from [EKK] includes the converse of Li Zhong's result. It also implies the statements in the preceding paragraph about uniqueness of geodesic segments and holomorphic isometries in the finite-dimensional case.

THEOREM 2.9 [EKK]. *Suppose μ in $\mathcal{M}(\Gamma)$ is extremal and nonzero. The following four conditions are equivalent:*

- (a) μ is uniquely extremal and $|\mu| = \|\mu\|_\infty$ a.e.,
- (b) there is only one geodesic segment joining $\mathbf{0}$ and $\Phi(\mu)$,
- (c) there is only one holomorphic isometry $g: \Delta \rightarrow \mathcal{M}(\Gamma)$ such that $g(0) = \mathbf{0}$ and $\phi(g(\|\mu\|_\infty)) = \Phi(\mu)$, and
- (d) there is only one holomorphic map $g: \Delta \rightarrow \mathcal{M}(\Gamma)$ such that $g(0) = \mathbf{0}$ and $\Phi(g(\|\mu\|_\infty)) = \Phi(\mu)$.

For the proof see [EKK].

A characterization of extremal complex dilatations by their angular map distribution is given, e.g., in [HO,MSu,Or,OS].

An excellent exposition of extremal quasiconformal maps of plane domains considered as the complements of arbitrary (closed) sets is given in Earle and Zhong's paper [EL1]; it concerns the dynamical approach.

For another characterization of extremal dilatations and extensions to more general Kleinian groups see, e.g., [Ber3,Kru5,RS1,Sh1,She2,She3].

2.8.6. Many important results concerning extremal quasiconformal maps and Teichmüller spaces rely on *Strebel's frame mapping condition* [St2], which involves the boundary dilatation. For simplicity, we restrict ourselves here by the case of homeomorphisms of the disk. Generalizations to Riemann surfaces are given in [Ga2,EL2].

Let h be a quasymmetric homeomorphism h of S^1 . Consider its local quasiconformal extensions $\tilde{h}$ into the neighborhoods of the points $z_0 \in S^1$ (equivalently, to the rings $\{1 - \delta \leq |z| \leq 1\}$, $\delta > 0$), which are called the *frame maps* for h . Set

$$H(h) = \inf\{K(\tilde{h}): \tilde{h} \text{ frame}\} \quad \text{and} \quad K(h) = \inf\{K(w): w \in [h]\},$$

where $[h]$ is again the class of quasiconformal extensions of h onto $\overline{\Delta}$.

Strebel's condition says that if $H(h) < K(h)$, then no sequences $\{\varphi_n\}$ maximizing the left-hand side of (2.26) may degenerate (converge to zero on compact subsets of Δ), and, consequently, the set $[h]$ contains a unique Teichmüller extremal extension w_0 of h with

$$\mu_{w_0} = k(h)|\varphi_0|/\varphi_0, \quad k(h) = \frac{K(h) - 1}{K(h) + 1}, \quad \varphi_0 \in A_1(\Delta) \setminus \{0\}.$$

Earle and Li Zhong [EL2] have proved that Strebel's frame mapping condition is also necessary for uniqueness and existence of Teichmüller extremal map in $[h]$.

2.8.7. The features of extremal quasiconformal maps were investigated by many authors. Various deep applications have been found recently for this theory. One of them is a beautiful connection of holomorphic quadratic differentials and extremal Beltrami coefficients with Thurston's theory of measured laminations.

The reader can find the details, for example, in [GaM,HuM,MSt,RS1].

2.9. A new general variational principle

2.9.1. A quite different general variational principle has been discovered recently which is intrinsically related to the Carathéodory metric on Teichmüller spaces and which provides a new approach to solving the extremal problems for quasiconformal maps and for univalent functions with quasiconformal extension. We present this variational principle in this section by solving a general problem of the distortion theory for quasiconformal maps. In the sequel to this paper [Kru8], this principle will be applied to solving the coefficient problem for univalent functions with quasiconformal extension, a problem which has been open for quite a long time. The details can be found in [Kru8,Kru9,Kru14].

Let us illustrate this principle here by solving a general distortion problem for quasiconformal maps of the whole plane.

Consider again the class Q of all quasiconformal automorphisms f^μ of $\widehat{\mathbb{C}}$ (endowed with the topology of convergence in the spherical metric on $\widehat{\mathbb{C}}$), whose Beltrami coefficients μ range over the Banach ball

$$B(\mathbb{C}) = \{\mu \in L_\infty(\mathbb{C}): \|\mu\| < 1\}$$

of conformal structures on $\widehat{\mathbb{C}}$. We assume the maps f^μ satisfy some normalization, for example, chosen so that the points $0, 1, \infty$ remain fixed. Let

$$Q(k) = \{f \in Q: k(f) \leq k\}, \quad 0 \leq k \leq 1.$$

Consider on Q a complex holomorphic (Gateaux differentiable) functional J , for simplicity of the form

$$J(f) = F(f(z_1), f(z_2), \dots, f(z_n)), \quad (2.40)$$

where $z_1, z_2, \dots, z_n$ are given points in $\mathbb{C}_* = \mathbb{C} \setminus \{0, 1\}$ and $F(w_1, \dots, w_n)$ is a holomorphic function of n variables in an appropriate domain of $\mathbb{C}^n$, $w_j = f(z_j)$. The case of an arbitrary complex functional can be reduced to (2.40), e.g., by using n -tuples $(w_1, \dots, w_n)$ as local complex coordinates on the Teichmüller space $\mathbb{T}(0, n+3)$ of the spheres with $n+3$ punctures.

The problem consists of determining the domain of the range values of this functional on the class $Q(k)$.

We have already seen that the application of the most variational methods to quasiconformal maps is associated with great difficulties. In fact they only propose in general the form for the Beltrami coefficient $\mu_{f_0^{-1}}$ of the maps f_0^{-1} which are inverse for extremal ones for J :

$$\mu_{f_0^{-1}}(w) = k e^{i\alpha} \overline{\psi(w)} / |\psi(w)|,$$

where $\alpha \in \mathbb{R}$,

$$\psi(w) = J'_{f_0}(g(f_0, w)) \equiv \sum_{j=1}^n \frac{F_{w_j} w_j^0 (w_j^0 - 1)}{w(w-1)(w-w_j^0)}, \quad w_j^0 = f(z_j)$$

is the value of the Gateaux derivative of the functional F on the kernel

$$g(f, w) = \frac{f(f-1)}{w(w-1)(w-f)}$$

of the variation of maps of the class Q with the pointed normalization. In this case the values w_j^0 itself are not known, and in most cases it is difficult to find them. The same difficulties also arise for analytic functions with quasiconformal extension. Excepting special cases the variational method gives, as yet, only asymptotically exact (for small k) estimates of the type

$$|J(f) - J(I)| \leq \frac{k}{\pi} \iint_{\mathbb{C}} |J'_I(g(I, z))| dx dy + O(k^2), \quad (2.41)$$

where I is the identity maps.

The main task here is to find extremal maps f_0 themselves or, at least, their Beltrami coefficients μ_{f_0} .

2.9.2. We now present a new approach to variational problems. It is based on the properties of Teichmüller spaces and of the Carathéodory metric on these spaces. It appears that in many cases one can solve the problem posed above without any calculations; in fact, everything depends on the *initial* rational function

$$\varphi_0(z) = J'_I(g(I, z)) = \frac{1}{z(z-1)} \sum_{j=1}^n \frac{F_{z_j} z_j (z_j - 1)}{z - z_j} \quad (2.42)$$

or, equivalently, on the quadratic differential $\varphi_0(z) dz^2$ on the punctured sphere

$$X_0 = \widehat{\mathbb{C}} \setminus \{0, 1, \infty, z_1, \dots, z_n\}. \quad (2.43)$$

THEOREM 2.10. *Let the Carathéodory metric on the space $\mathbb{T}(0, n+3)$ coincide with the Teichmüller–Kobayashi metric of this space on the holomorphic disk $\Delta_{\varphi_0} = \{\Phi(t \frac{\varphi_0}{|\varphi_0|}) : t \in \Delta\}$ in $\mathbb{T}(0, n+3)$. Then there exists a number $k_0(F)$ such that for all $k \leq k_0(F)$ the inequality*

$$\max_{\|\mu\| \leq k} |J(f^\mu) - J(I)| \leq \max_{|t|=k} |J(f^{t\tilde{\varphi}_0/|\varphi_0|}) - J(I)| \equiv d(k) \quad (2.44)$$

holds. Hence, the range of values $J(Q(k))$ is contained in the closed disk $\{|\zeta - J(I)| \leq d(k)\}$.

Here Φ is a canonical projection of the ball $B(\mathbb{C}) = \{\mu \in L_\infty(\mathbb{C}) : \|\mu\| < 1\}$ onto $\mathbb{T}(0, n+3)$, similar to (2.10).

The following theorem, which proves a convenient sufficient condition for applications, is obtained from the above theorem on the basis of Kra's theorem mentioned in Section 2.6.4.

THEOREM 2.11. *Let the function φ_0 have on X_0 zeros of only even order. Then for $k \leq k_0(F)$ the inequality (2.43) holds.*

Thus, the geometric picture is the following: for small k , for the map maximizing $|F|$, the directions of the major axes of characteristic ellipses in the plane z are only determined by their initial state with the possible rotation on the constant angle, depending on k_0 only.

To apply Theorem 2.11 effectively, one needs to have an explicit lower bound for the indicated value $k_0(F)$. This point is discussed in [Kru9], [Kru12, Theorem 4.4], [Kru14].

2.9.3. We shall use the following simple facts about projections with norm 1, see [EK].

Let V be a complex Banach space with norm $\|\cdot\|$ differentiable on $V \setminus \{\mathbf{0}\}$ and suppose that

$$A(v, w) = \lim_{t \rightarrow \infty} \frac{\|v + w\| - \|v\|}{t} \quad \text{for all } v \in V \setminus \{\mathbf{0}\}, w \in V,$$

for every fixed $v \neq \mathbf{0}$ it is a bounded linear functional $w \rightarrow A(v, w)$ on V .

LEMMA 2.12. *Let W be a nontrivial closed (complex) subspace of V and*

$$W' = \{w \in V : A(v, w) = 0 \text{ for all } v \in W \setminus \{\mathbf{0}\}\}.$$

A projection P_W with norm 1 of V onto W exists if and only if W' is a complementary subspace for W , that is, $V = W \oplus W'$.

If V is finite-dimensional, this is equivalent to

$$\dim W + \dim W' = \dim V.$$

(Of course, a similar assertion is also true for the case of real spaces V and W .)

The next lemma is a slight modification of the corresponding lemmas of Royden and Earle–Kra from [Ro1,EK].

Let $a_1, a_2, \dots, a_n$ be distinguished points of the plane C ($n \geq 3$), and let φ and ψ be rational functions belonging to $L_1(C)$ that are holomorphic and nonzero at points $z \in C \setminus \{a_1, \dots, a_n\}$. We denote the orders of the functions φ and ψ at points a_j by α_j and β_j , respectively ($\alpha_j = \text{ord}_{a_j} \varphi$, $\beta_j = \text{ord}_{a_j} \psi$), and their orders at the point $z = \infty$ by α_0 and β_0 (here $\alpha_j, \beta_j \geq -1$ for $j = 1, \dots, n$ and $\alpha_0, \beta_0 \geq 3$). Consider for real t the function

$$h(t) = \iint_{\mathbb{C}} |\varphi(z) + t\psi(z)| dx dy.$$

LEMMA 2.13. *The function $h(t)$ is differentiable for t close to 0, and*

$$h'(0) = \text{Re} \iint_{\mathbb{C}} \psi(z) \frac{\overline{\varphi(z)}}{|\varphi(z)|} dx dy. \quad (2.45)$$

Moreover, if $\alpha_j \leq 2\beta_j + 1$ for $j = 1, \dots, n$ and $\alpha_0 \leq 2\beta_0 - 3$, then the second derivative $h''(0)$ exists. If, for some $j_s = j_0, j_1, \dots, j_m$ ($0 \leq m \leq n$, $j_0 = 0$), we have $\alpha_{j_s} > 2\beta_{j_s} + 1$ when $s \neq 0$ or $\alpha_{j_s} > 2\beta_{j_s} - 3$, then

$$h(t) = h(0) + th'(0) + \sum_{s=0}^m c_{j_s} \delta_{j_s}(t) + o\left(\max_s \delta_{j_s}(t)\right), \quad (2.46)$$

where all the c_{j_s} are positive constants and

$$\delta_{j_s}(t) = \begin{cases} |t|^2 \log \frac{t}{|t|} & \text{when } \alpha_{j_s} = 2\beta_{j_s} + 2 \text{ for } s \neq 0, \\ & \alpha_{j_s} = 2\beta_{j_s} - 2 \text{ for } s = 0, \\ |t|^{1 + \frac{2+\beta_{j_s}}{\alpha_{j_s}-\beta_{j_s}}} & \text{when } \alpha_{j_s} > 2\beta_{j_s} + 2 \text{ for } s \neq 0, \\ |t|^{1 + \frac{\beta_{j_0}-2}{\alpha_{j_0}-\beta_{j_0}}} & \text{when } \alpha_{j_0} > 2\beta_{j_0} + 2. \end{cases} \quad (2.47)$$

Equality (2.43) is a direct corollary of the inequality

$$\left| \frac{|\psi - t\psi| - |\psi|}{t} \right| \leq |\psi|$$

and of Lebesgue's dominant convergence theorem. As for (2.46) and (2.47), observe that since the integral $\iint_E |\varphi + t\psi| dx dy$ is an infinitely differentiable function of t for any

compactum $E \subset C \setminus \{a_1, \dots, a_n\}$, it suffices to estimate the contributions of integrals of the form

$$I_j(t) = \iint \left(|(z - a_j)^\alpha + t\omega_j(z)| - |z - a_j|^\alpha \right. \\ \left. - t \operatorname{Re} \left[\omega_j(z) \left(\frac{z - a_j}{|z - a_j|} \right)^\alpha \right] \right) dx dy, \\ I_0(t) = \iint \left(|z^{-\nu} + t\omega_0(z)| - |z|^{-\nu} - t \operatorname{Re} [\omega_0(z)|z|^\nu / z^\nu] \right) dx dy,$$

over sufficiently small disks $\{|z - a_j| < r_j\}$, $j = 1, \dots, n$, and over $\{|z| > 1/r_0\}$, respectively, with holomorphic ω_j and $\alpha \geq -1$, $\nu \geq 3$. This is done similarly to [Ro1].

SKETCH OF THE PROOF OF THEOREM 2.10 (A complete proof is given in [Kru8]). The proof is accomplished in several stages.

(a) Put

$$\mu_0(z) = \overline{\varphi_0(z)} / |\varphi_0(z)|$$

and assume (for simplicity of writing) that $F(\operatorname{id}) = 0$. Consider the sphere (2.43) as a basepoint of the Teichmüller space $\mathbb{T}(0, n+3)$.

The equality $c_T(X_0, X) = \tau_{\mathbb{T}}(X_0, X)$ on Δ_{φ_0} implies the existence of a holomorphic retraction h of the space $\mathbb{T}(0, n+3) = \mathbb{T}(X_0)$ onto the disk Δ_{φ_0} ; its differential

$$dh(X_0): T_{X_0}^* \mathbb{T}(X_0) \rightarrow T_{X_0} \Delta_{\varphi_0}$$

is a projector with the norm 1.

The operator P conjugated to $dh(X_0)$ acts on the cotangent space $T_{X_0}^* \mathbb{T}(X_0) = Q(X_0)$, where $Q(X_0)$ is the space of holomorphic quadratic differentials on X_0 with L_1 -norm, leaving the complex line $W = \{t\varphi_0: t \in \mathbb{C}\}$ fixed. By Lemma 2.12, applied to $Q(X_0)$ and P , one concludes that $Q(X_0) = W \oplus W'$, where

$$W' = \left\{ \varphi \in Q(X_0): \iint_{\mathbb{C}} \varphi \frac{\bar{\varphi}_0}{|\varphi_0|} dx dy = 0 \right\}. \quad (2.48)$$

Choose in W' a basis $\varphi_1, \dots, \varphi_{n-1}$ and note that φ_j are of the form

$$\varphi_j(z) = \sum_{m=1}^n \frac{c_{jm}}{z(z-1)(z-z_m)}, \quad j = 1, \dots, n-1; \quad (2.49)$$

together with φ_0 they constitute a basis in the whole space $Q(X_0)$.

(b) By using variational techniques and the above-mentioned properties of the norm

$$r_j(t) = \|\varphi_0 + t\varphi_j\|_{L_1}, \quad t \in \mathbb{R},$$

we prove the following important

□

LEMMA 2.14. *For sufficiently small $k \leq k_0(F)$ the map f_0 maximizing $|F|$ on $Q(k)$ has the following property: Its Beltrami coefficient μ_{f_0} is orthogonal to the functions (2.45):*

$$\langle \mu_{f_0}, \varphi_j \rangle \equiv \iint_{\mathbb{C}} \mu_{f_0}(z) \varphi_j(z) dx dy = 0, \quad j = 1, \dots, n-1. \quad (2.50)$$

PROOF. Applying the integral representation (1.9) to $f^\mu \in Q(k)$, we obtain that for small k ,

$$\max_{Q(k)} |J(f^\mu)| = k\pi^{-1} \iint_{\mathbb{C}} |\varphi_0(\zeta)| d\xi d\eta + O(k^2), \quad k \rightarrow 0. \quad (2.51)$$

Now, for each $j = 1, \dots, n-1$, we define on $Q(k)$ the functional

$$J_j(f^\mu) = J(f^\mu) + t \sum_{m=1}^n \frac{c_{jm}(f^\mu(t_m) - z_m)}{z_m(z_m - 1)}, \quad t \in \mathbb{C},$$

where the constant c_{jm} are taken from (2.49). Similarly to (2.51) we have

$$\max_{Q(k)} |J_j(f^\mu)| = k\pi^{-1} \iint_{\mathbb{C}} |\varphi_0(\zeta) + t\varphi_j(\zeta)| d\xi d\eta + O(k^2), \quad j = 1, \dots, n-1, \quad (2.52)$$

where the bound of the remainder terms is uniform in t when $|t| \leq t_0$, and t_0 is sufficiently small.

Applying Lemma 2.13 to the corresponding functions

$$h_j(t) = \iint_{\mathbb{C}} |\varphi_0(\zeta) + t\varphi_j(\zeta)| d\xi d\eta, \quad j = 1, \dots, n-1,$$

one concludes that $h'_j(0) = 0$ for every j ; therefore, either $h_j(t)$ near $t = 0$ must be of the form

$$h_j(t) = h_j(0) + \sum_{s=0}^n c_{js} \delta_{js} + o\left(\max_x \delta_{js}(t)\right), \quad (2.53)$$

where $c_{js} > 0$ and δ_{js} has the form (2.46), or if $\text{ord}_{z_s} \varphi_j > 1$ for some s , then instead of the terms $c_{js} \delta_{js}(t)$ the quantities of order t^2 appear. In any case by (2.52) and (2.53),

$$\max_{Q(k)} |J_j(f^\mu)| = \max_{Q(k)} |J(j^\mu)| + k_0 + O(k^2 t) + O(k^2),$$

by appropriate choices of $k \rightarrow 0$ and $t \rightarrow 0$. Comparison with (2.49) implies (2.50).

(c) Let us now assume that, for any $k_0 > 0$, there is a $k \in (0, k_0)$ for which $\mu_0 \neq t \frac{\bar{\varphi}_0}{|\varphi_0|}$, $|t| = k$. Consider the lifting $\tilde{h}(\mu) = h(\Phi(\mu)) \frac{\bar{\varphi}_0}{|\varphi_0|}$ of the retraction h onto the ball $B(\mathbb{C})$ and denote

$$\mu_* = \frac{\mu_{f_0}}{\|\mu_{f_0}\|}, \quad dh(0)(t\mu_*) = d(k) \frac{\bar{\varphi}_0}{|\varphi_0|}.$$

Then it follows from our assumption, by the Schwarz lemma, that

$$|\alpha(k)| < k, \quad (2.54)$$

and

$$v_0 = t\mu_\infty - \alpha(k) \frac{\bar{\varphi}_0}{|\varphi_0|} \neq 0.$$

On the other hand, from (2.48), (2.50) and from the properties of the operator P we get the equality $\langle v_0, \varphi \rangle = 0$ for all $\varphi \in W(X_0)$ which means that $v_0 \in Q(X_0)^\perp$.

(d) The general Theorem 2.1 on extremal quasiconformal maps implies that the Beltrami coefficient μ_{f_0} must be of the form

$$\mu_{f_0} = k \bar{\psi}_0 / |\psi_0| \quad \text{with } \psi_0 \in Q(X_0) \setminus \{0\}$$

(and f_0 is the extremal Teichmüller map in the class of automorphisms $f^\mu : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $f^\mu(z_j) = f_0(z_j)$, $j = 1, \dots, n$, homotopic to f_0 on X_0). Therefore, for each nonzero $v \in Q(X_0)^\perp$ we have $\|\mu_{f_0}\|_\infty < \|\mu_{f_0} + v\|_\infty$, thus

$$\|t\mu_* - v_0\|_\infty = |\alpha(k)| > \|t\mu_*\|_\infty = |t|,$$

and we reach contradiction with (2.54). This completes the proof of Theorem 2.10. $\square$

2.10. Examples

2.10.1. Many distortion bounds in the theory of quasiconformal maps as well as some other important results are based on the properties of the functional

$$F(f^\mu) = f^\mu(z_0), \quad z_0 \in \mathbb{C} \setminus \{0, 1\};$$

in this case

$$\varphi_0(z) = \frac{z_0(z_0 - 1)}{z(z - 1)(z - z_0)} \quad (2.55)$$

and the maps f_0 with the Beltrami coefficients $\mu_0 = t\bar{\varphi}_0/\varphi_0$ are extremal on $Q(k)$ for all $k = |t| \in [0, 1)$.

This fact provides the key results in various topics of geometric complex analysis. Let us take a more general functional

$$F(f) = w_0 + H(w_0, w_1, \dots, w_n),$$

where $w_j = f(z_j)$, $j = 0, 1, \dots, n$, and H is a holomorphic function in a domain containing the origin in $\mathbb{C}^{n+1}$, with the development

$$H(w_0, \dots, w_n) = \sum_{|m| \geq 2} c_{m_0, \dots, m_n} w_0^{m_0} \cdots w_n^{m_n}, \quad |m| = m_0 + \dots + m_n.$$

Then $F'_I(g(I, z)) = \varphi_0(z)$ again has the form (2.44) and by Theorem 2.11 there exists a number $k_0 = k_0(z_0, \dots, z_n) > 0$ such that for $k \leq k_0$ the set $F(Q(k))$ is located in the closed disk with the center z_0 of radius

$$d(k) = \max_{|t|=k} |F(f^{t\tilde{\varphi}_0/|\varphi_0|}) - z_0|;$$

in particular,

$$\max_{Q(k)} |F(f^\mu) - z_0| = d(k).$$

2.10.2. Suppose that

$$J(f) = \frac{f(z_0)}{z_0}, \quad z \in \mathbb{C}_x.$$

Then in (2.42)

$$\varphi_0(z) = \frac{z_0 - 1}{z(z - 1)(z - z_0)},$$

and the assumption of Theorem 2.11 is again satisfied. Consequently, when $k < k_0(z_0)$,

$$\max_{Q(k)} \left| \frac{f(z_0)}{z_0} - 1 \right| = \max_{|f|=k} |J(f^{t\tilde{\varphi}_0/|\varphi_0|}) - 1|.$$

2.10.3. Consider the functional

$$J(f^\mu) = \frac{f^\mu(z_1) - f^\mu(z_2)}{z_1 - z_2}, \quad z_1, z_2 \in \mathbb{C} \setminus \{0, 1\}, z_1 \neq z_2.$$

We then have

$$\varphi_0(z) = J'_I(g(I, z)) = \frac{1}{(z_1 - z_2)z(z - 1)} \left[\frac{z_1(z_1 - 1)}{z - z_1} - \frac{z_2(z_2 - 1)}{z - z_2} \right].$$

If z_1 and z_2 are such that $z_1 + z_2 = 1$, then

$$\varphi_0(z) = \frac{\text{const}}{z(z-1)(z-z_1)(z-z_2)} \quad (2.56)$$

and thus the condition of Theorem 2.10 is fulfilled. Hence, for such z_1, z_2 , for $k \leq k_0(J)$, we obtain the estimate

$$\max_{\|\mu\| \leq k} \left| \frac{f^\mu(z_1) - f^\mu(z_2)}{z_1 - z_2} - 1 \right| \leq \max_{|t|=k} \left| \frac{f^{t\bar{\varphi}_0/|\varphi_0|}(z_1) - f^{t\bar{\varphi}_0/|\varphi_0|}(z_2)}{z_1 - z_2} - 1 \right|$$

with φ_0 defined by (2.56). This is apparently the first general sharp bound for the maps with two distinguished points.

2.10.4. Similarly, one can consider, for example, the functionals

$$J(f) = f(z_1) + f(z_2) + f(z_3) + (z_1 + z_2 + z_3),$$

$$J(f) = \frac{f(z_1)f(z_2)f(z_3)}{z_1 z_2 z_3}, \quad z_1, z_2, z_3 \in \mathbb{C}_*,$$

and obtain sharp estimates for the distortion under the maps with three or more distinguished points, for appropriate $k \leq k_0(J)$.

Other two-points sharp distortion estimates are established in [Ag1, Va1, Va2]. The last estimates concern the values $w^\mu(r_1), w^r(r_2)$ for special real r_1, r_2 and by applying the general Theorems 2.3 and 2.10 can be extended to arbitrary $r_1, r_2 \in \mathbb{C}$.

2.11. Extremal quasiconformal embeddings

2.11.1. A natural extension of the problems investigated in the previous sections concerns quasiconformal embeddings f of a given Riemann surface $\mathcal{R}$ with border into a Riemann surface $\mathcal{R}'$ so that the images $\overline{f(\mathcal{R})}$ are the proper subsets of $\mathcal{R}'$. Originally, the quasiconformal embeddings of a simply connected subdomain $D \subsetneq \widehat{\mathbb{C}}$ into $\widehat{\mathbb{C}}$ were considered, see [Bi1, Schi, Re1, Scho].

We point out that in certain extremal problems of Belinskii type the factorization formula $f = \phi \circ F$ for the solutions of the Beltrami equation (locally from Sobolev's space W_1^2) by means of holomorphic functions ϕ does not work in the sense that the composition of an extremal quasiconformal homeomorphism F_0 of a given Riemann surface $\mathcal{R}_0$ onto a fixed Riemann surface $\mathcal{R}_*$ followed by an extremal conformal map ϕ_0 among conformal embeddings of $\mathcal{R}_*$ into the prescribed Riemann surface $\mathcal{R}'$ does not need to be an extremal embedding of $\mathcal{R}_0$.

We present here somewhat more general results.

2.11.2. Let $S_q(\mathcal{D}, a, b)$ be the class of q -quasiconformal homeomorphisms of a hyperbolic simply-connected or, more generally, finitely-connected domain $\mathcal{D} \subset \mathbb{C}$, which

assume given finite-distinct values $b_1, \dots, b_n$ at given points $a_1, \dots, a_m$, respectively; $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m)$, $m \geq 2$. For sufficiently large q this class is trivially nonempty.

Similarly as in Section 2.6, we pose the following problem.

In the class $S_q(\mathcal{D}, a, b)$, find the maximum of the real functional $\mathcal{F}[f] = F(w_1, \dots, w_n)$, where $w_k = f(z_k) = u_k + iv_k$, z_k are fixed points of the domain $\mathcal{D}$ distinct from $a_1, \dots, a_m$ ($k = 1, \dots, m$) and the function F is continuously differentiable in all variables u_k, v_k with $\text{grad } F \neq 0$.

THEOREM 2.15. *The functional $\mathcal{F}[f]$ achieves its maximum within the class $S_q(\mathcal{D}, a, b)$ for some mapping $w = f_0(z)$ with the following properties:*

(1) *there exist a constant α , $0 \leq \alpha \leq 2\pi$, and a rational function $\varphi_0(w)$, $\varphi_0(\infty) = 0$, possibly having simple poles at the points $b_1, \dots, b_m$, such that the complex dilatation $\mu(w)$ of the inverse map $z = f_0^{-1}(w)$ has the form*

$$\mu(w) = \frac{q-1}{q+1} \frac{\overline{\varphi(w)}}{|\varphi(w)|}, \quad \varphi(w) = e^{i\alpha} \sum_{k=1}^n \frac{F_{w_k}}{w - w_k} + \varphi_0(w); \quad (2.57)$$

(2) *the function $w = f_0(z)$ maps the region $\mathcal{D}$ onto the w -plane with cuts along a finite number of analytic arcs satisfying the inequality*

$$\varphi(w) dw^2 \geq 0. \quad (2.58)$$

SKETCH OF THE PROOF. Let $w = f(z)$ be an extremal function. From the result of Section 2.6, it follows that the extremal region $f(\mathcal{D}) = \Delta$ does not have exterior points, the Lebesgue area of its boundary is equal to zero and the complex dilatation $\mu(w)$ has the form (2.57) where $\varphi_0(w)$ is a rational function such that $\varphi(w)$ can be written in the form

$$\begin{aligned} \varphi(w) = & \sum_{k=1}^n \frac{B_k}{(w - b_{m-1})(w - b_m)(w - w_k)} \\ & + \sum_{j=1}^{m-2} \frac{B_{n+j}}{(w - b_{m-1})(w - b_m)(w - b_j)} \end{aligned}$$

with complex constants B_k , $k = 1, \dots, m+n-2$.

Now assume for simplicity that the domain $\mathcal{D}$ is simply connected. We represent the extremal map f as a composition $f = g \circ h$, where $\zeta = h(z)$ is a quasiconformal map of the region $\mathcal{D}$ onto the disk $\{|\zeta| < 1\}$, $f(a_{m-1}) = 0$, with the same complex dilatation as $f(z)$, and $w = g(\zeta)$ is a conformal map of the disk $\{|\zeta| < 1\}$ onto the region Δ . Let $h(a_j) = c_j$, $j = 1, \dots, m-2, m$, and $h(z_k) = \zeta_k$, $k = 1, \dots, n$. The function $w = g(\zeta)$ yields the maximum of the functional $F(w_1, \dots, w_n)$, where now $w_k = g(\zeta_k)$, within the class $S(c, b)$, $c = (c_1, \dots, c_{m-2}, 0, c_m)$, of conformal maps $w = g(\zeta)$ of the disk $|\zeta| < 1$ with normalization $g(0) = b_{m-1}$, $g(c_j) = b_j$, $j = 1, \dots, m-2, m$. Without loss of generality, we can assume $b_{m-1} = 0$.

We first solve the problem of maximization of F in the class $S(c, b; \rho)$ of conformal maps $w = g(\zeta)$ of the disk $|\zeta| < 1$ satisfying the condition $g(0) = 0$, $g(c_m) = b_m$, $|g(c_j) - b_j| \leq \rho$, $j = 1, \dots, m-2$, where ρ is positive and sufficiently small. Suppose that for the extremal function $w = g_\rho(\zeta)$ in the class $S(c, b; \rho)$ we have m_1 , $1 \leq m_1 \leq m-2$, equalities $|g_\rho(c_{j\ell}) - b_{j\ell}| = \rho$, $\ell = 1, \dots, m_1$, where we can assume (after changing the indexing if necessary) that $j_\ell = 1, \dots, m_1$. Let $g_\rho(c_j) = b_j + \rho \varepsilon_j$, $|\varepsilon_j| = 1$.

We fix arbitrary points t_ℓ , $|t_\ell| < 1$, $\ell = 1, \dots, m$, distinct from the points c_j , $j = 1, \dots, m$, and ζ_k , $k = 1, \dots, n$, and arbitrary complex constants $A_1, \dots, A_{m_1}$. Using the variation

$$g^*(\zeta) = g(\zeta) + \lambda \sum_{\ell=1}^{m_1} [A_\ell P(\zeta, t_\ell) + \bar{A}_\ell \overline{Q(\zeta, t_\ell)}] + O(\lambda^2), \quad \lambda > 0, \quad (2.59)$$

with

$$P(\zeta, t_\ell) = \frac{g(\zeta)g(t_\ell)(b_m - g(\zeta))}{(g(\zeta) - g(t_\ell))(b_m - g(t_\ell))} - \frac{g^2(t_\ell)}{t_\ell g'^2(t_\ell)} \left[\frac{\zeta g'(\zeta)}{\zeta - t_\ell} - \frac{c_m g'(c_m)g(\zeta)}{b_m(c_m - t_\ell)} \right], \quad (2.60)$$

$$Q(\zeta, t_\ell) = \frac{g^2(t_\ell)}{t_\ell g'^2(t_\ell)} \left[\frac{\bar{\zeta}^2 \overline{g'(\zeta)}}{1 - t_\ell \bar{\zeta}} - \frac{\bar{c}_m^2 \overline{g'(c_m)g(\zeta)}}{\bar{b}_m(1 - t_\ell \bar{b}_m)} \right] \quad (2.61)$$

(cf. [Gol, Section 14]), we get

$$\begin{aligned} |g_\rho^*(c_j) - b_j| &= |g_\rho(c_j) - b_j| \left\{ 1 + \frac{\lambda}{\rho} \operatorname{Re} \sum_{\ell=1}^{m_1} A_\ell [\bar{\varepsilon}_j P(c_j, t_\ell) + \varepsilon_j Q(c_j, t_\ell)] \right\} \\ &\quad + O(\lambda^2), \quad j = 1, \dots, m_1, \\ dF &= 2\lambda \operatorname{Re} \sum_{\ell=1}^{m_1} A_\ell \sum_{k=1}^n F_{w_k} [P(\zeta_k, t_\ell) + Q(\zeta_k, t_\ell)]. \end{aligned}$$

This yields that the functions $L_0(\zeta) = \sum_{k=1}^n F_{w_k} [P(\zeta_k, \zeta) + Q(\zeta_k, \zeta)]$, $L_j(\zeta) = \bar{\varepsilon}_j P(c_j, \zeta) + \varepsilon_j Q(c_j, \zeta)$, $j = 1, \dots, m_1$, are linearly dependent, i.e.,

$$\sum_{j=0}^{m_1} \lambda_j L_j(\zeta) = 0, \quad \sum_{j=0}^{m_1} |\lambda_j| > 0. \quad (2.62)$$

The first equality in (2.62) can be regarded as a differential equation for the extremal function $g_\rho(\zeta)$, from which we obtain, in the usual way, that the function $g_\rho(\zeta)$ maps the disk $|\zeta| < 1$ onto the w -plane with cuts along a finite number of analytic arcs. Let us normalize the constants $\lambda_0, \lambda_1, \dots, \lambda_{m_1}$ so that $\sum_{j=0}^{m_1} |\lambda_j| = 1$. Letting $\rho \rightarrow 0$, one derives a limits function g_0 with similar properties.

Making use of the boundary variation constructed above in Section 1.4 (which we actually need to apply here only for the case $\theta = \pi/2$), we shall prove that the angle θ' made by the path $\arg dw = -\arg \phi(w)$ at each point w of the boundary cuts of the region Δ_0 distinct from their ends is equal to zero and consequently these cuts satisfy the inequality (2.59). The function $f_0 = g_0 \circ h$ is the desired one.

For $m = 2$, as was shown in [Bel4], the constant a and the function $\phi_0(w)$ can be explicitly expressed in terms of $w_1, \dots, w_n$.

The assertion of the theorem carries over completely to maps of finitely-connected regions (with the same normalization as above). For example, consider the map $w = f(z)$ such that if γ_1 and γ_2 are two nonintersecting closed Jordan curves separating the boundary components of the original doubly-connected region $\mathcal{D}$, where γ_1 lies inside γ_2 , then $f(\gamma_1)$ lies inside $f(\gamma_2)$. Then the above reasoning remains valid except for the following features. Take a function $h(z)$ which quasiconformally maps the region $\mathcal{D}$ onto a corresponding ring $r < |z| < R$. Without loss of generality, we can assume that $b_{m-1} = 0$, $b_m = 1$. Instead of (2.59)–(2.61), we now apply conformal variation of the annuli given by

$$\begin{aligned}
 g^*(\tau) = g(\tau) + \lambda \tau g'(\tau) & \left\{ \sum_{\ell=1}^{m_1+1} A_\ell \frac{g(\tau)g(\tau) - 1}{\tau g'(\tau)(g(\tau) - g(\tau_\ell))} \right. \\
 & + \frac{\omega}{\pi i} \sum_{\ell=1}^{m_1+1} C_\ell \left[K\left(\frac{t_\ell}{\tau}\right) - K\left(\frac{t_\ell}{c_{m-1}}\right) \right] \\
 & + \frac{w}{\pi i} \sum_{\ell=1}^{m_1+1} \bar{C}_\ell \left[\bar{K}\left(\frac{\tau \bar{t}_\ell}{R^2}\right) - \bar{K}\left(\frac{c_{m-1} \bar{t}_\ell}{R^2}\right) \right] \Big\} \\
 & - \lambda \frac{\omega}{\pi i} c_m g'(c_m) g(\tau) \left\{ \sum_{\ell=1}^{m_1+1} C_\ell \left[K\left(\frac{t_\ell}{c_m}\right) - K\left(\frac{t_\ell}{c_{m-1}}\right) \right] \right. \\
 & + \sum_{\ell=1}^{m_1+1} \bar{C}_\ell \left[\bar{K}\left(\frac{c_m \bar{t}_\ell}{R^2}\right) - \bar{K}\left(\frac{c_{m-1} \bar{t}_\ell}{R^2}\right) \right] \Big\} + o(\lambda), \tag{2.63} \\
 C_\ell = A_\ell & \frac{g(t_\ell)(g(t_\ell) - 1)}{t_\ell^2 g'^2(t_\ell)},
 \end{aligned}$$

where

$$K(x) = \zeta\left(\frac{\omega}{\pi i} \log x\right) - \frac{\zeta(\omega)}{\pi i} \log x,$$

$\zeta(u)$ is the Weierstrass ζ -function with real and imaginary pseudoperiods 2ω and $2\omega'$, $\omega'/i\omega = (1/\pi) \log(1/R)$; $t_1, \dots, t_{m_1+1}$ are fixed points of the annulus $r < |z| < R$, $A_1, \dots, A_{m_1+1}$ are complex constants and the number m_1 has the same meaning as above.

Formula (2.63) is obtained from the corresponding analog of Golusin's variation for conformal maps of a ring (see [Al]). In this case, the constants $C_1, \dots, C_{m_1+1}$ are subject

to the relation $\operatorname{Re} \sum_{\ell=1}^{m_1+1} C_\ell = 0$; therefore the function $g(r)(g(r) - 1)/(r^2 g'^2(r))$ should be considered together with the analogs of the functions $L_j(r)$, $j = 0, 1, \dots, m$ for the annulus. $\square$

2.11.3. The above problems are extended by Golubev and Graf [GoG] to the embeddings of the Riemann surfaces of finite type (g, n, m) with $m > 0$.

Let $X_1, \dots, X_p$ be a finite collection of such surfaces represented as the orbit spaces Δ/G_j by Fuchsian groups Γ_j acting in Δ , $j = 1, \dots, p$. Regard $X = \bigcup_{j=1}^p X_j$ as the topological space with the standard topology of the union of topological spaces.

Let us consider quasiconformal embeddings f of X into a finite Riemann surface $Y = \Delta/\Gamma$, i.e., such that their restrictions $f|_{X_j}$ are quasiconformal maps $X_j \rightarrow Y$ so that $f(X_j) \cap f(X_\ell) = \emptyset$ for $j \neq \ell$. Of course, the Riemann surfaces $X_1, \dots, X_p$ can be simultaneously uniformized by a Kleinian group $G < \operatorname{PSL}(2, \mathbb{C})$. Then μ_f is lifted to a Beltrami $(-1, 1)$ form $\hat{\mu}(z) d\bar{z}/dz$ with respect to G supported on the discontinuity set $\Omega(G)$ of G .

Let $E_\tau([f]; X, Y)$ be the set of quasiconformal embeddings $\tilde{f}: X \rightarrow Y$ homotopic on each $\mathcal{R}_j$ to a given embedding $f: X \rightarrow Y$ and with complex dilatations $\mu_{\tilde{f}} = \tilde{f}_{\bar{z}}/\tilde{f}_z$ bounded by a given function $\tau(z)$ on $\mathcal{R}_2$ so that $|\mu_{\tilde{f}}(z)| \leq \tau(z) < \tau_0 < 1$. The problem again consists of maximization of a functional $\mathcal{F}(f) = F(f(z_1), \dots, f(z_n))$, where $(z_1, \dots, z_n)$ is a fixed tuple on X and $F(w_1, \dots, w_p)$ is now a C^1 smooth real function on the n -product of M with $\operatorname{grad} F \neq 0$.

THEOREM 2.16 [GoG]. *A nonconformal embedding $f_0: X \rightarrow Y$ is an extremal of $\mathcal{F}(f)$ in $E_\tau[f], X, Y$ if and only if f_0 is a Teichmüller-type map on each X_j determined by holomorphic quadratic differentials $\{\varphi_j dz^2\}_{j=1}^p$ on X with at most simple poles at $z_1, \dots, z_n$, and by ψdw^2 on Y so that*

$$\mu_{f_0}(z) = \tau(z) \frac{\overline{\varphi_j(z)}}{|\varphi_j(z)|} \quad \text{on } X_j; \quad \mu_{f_0^{-1}}(w) = -\tau \circ f_0^{-1}(w) \frac{\overline{\psi(w)}}{|\psi(w)|}. \quad (2.64)$$

Here

$$\psi(w) = c_0 \sum_{j=1}^p F_{w_j} \sum_{\gamma \in \Gamma_j} \frac{\gamma'^2(w)}{w_j - \gamma^2} + \psi_0(w), \quad (2.65)$$

where $\psi_0 dw^2$ is holomorphic on M and $c_0 > 0$. The extremal domains (surfaces) $f_0(X_j)$, $j = 1, \dots, p$, are obtained from Y by cutting along horizontal trajectories of ψ .

The central point in the proof is, of course, to establish the sufficiency of the conditions (2.64) and (2.65) for extremality, which provides some kind of uniqueness of the extremal. This can be obtained by combining the proof of uniqueness in Teichmüller's theorem with the ideas from [Io] extended to the dilatations with nonconstant bound; for details see [GoG].

2.12. Quasiconformality in the mean

2.12.1. Another interesting and useful generalization of the theory of extremal quasiconformal maps is to extend it to the maps whose dilatations $K(z)$ are bounded only in some integral sense. This direction was established already by several authors in the beginning 1950s, first of all by Ahlfors [Ah1] and Belinskii (see [Bel4, Chapter 3]).

Ahlfors did this in his proof of Teichmüller's theorem, reducing it to minimization of the integrals

$$I_m(f) = \iint_{\Delta} \left(\frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} \right)^m dx dy$$

and sending n to infinity. His approach was extended by others.

Belinskii applied the maps with integrable dilatations in his strengthening and extension of Teichmüller–Wittich's theorem. One of his results says:

THEOREM 2.17 [Bel4]. *Let f map quasiconformally the closed disk $\bar{\Delta}$ onto itself, with $f(0) = 0$, and*

$$\iint_{\Delta} [K_f(z) - 1] dx dy \leq \varepsilon.$$

Then

$$||f(z) - z| \leq \lambda_0(\varepsilon), \quad (2.66)$$

where $\lambda_0(\varepsilon)$ depends only on ε , and $\lambda_0(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

If, in addition, $f(1) = 1$, then

$$|f(z) - z| \leq \lambda(\varepsilon) \quad \text{with } \lambda(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

2.12.2. A systematic study of extremal problems for the maps quasiconformal in the mean, was begun by Biluta [Bi2]. In particular, he proved that if the punctured disk $\{0 < |z| \leq 1\}$ is mapped quasiconformally onto the disk $\{|w| \leq 1\}$ with radial cut from 0 to ρ , $\rho < 1$, and

$$\iint_{0 < |z| < 1} (K_f(z) - 1) dx dy \leq \varepsilon,$$

then $\rho \leq \rho(\varepsilon)$, where

$$\rho(\varepsilon) = \frac{4}{e} \sqrt{\frac{\varepsilon}{\pi}} + O(\varepsilon^{5/2}).$$

This gives the exact order of the function $\lambda_0(\varepsilon)$ in (2.66).

2.12.3. A rather complete theory of extremal quasiconformal maps in the mean was developed by Kühnau [Ku9, Ku10], see also [KK].

Let G_0 be a subdomain of $\widehat{\mathbb{C}}$ containing the point at infinity and $\phi(p, z)$ be a real function defined on $[1, \infty) \times G$, where G is a bounded subregion of G_0 . The restrictions to dilatations $p(z) = K_f(z)$ are of the form

$$I_\phi(p) = \iint_G \phi(p(z), z) dx dy \leq M < \infty, \quad (2.67)$$

and $1 \leq p(z) \leq p_*(z) < M_* < \infty$ in $G_0 \setminus G$ (with constant M and M_* and prescribed $p_*(z)$). Moreover, it is assumed that $p_*(z) = 1$ (thus the maps are conformal) in some neighborhood of ∞ . Assume also that the derivatives ϕ_p and ϕ_{pp} are continuous jointly in p and z , and let $\phi_p > 0$. Let $\mathcal{A}_\phi$ denotes the class of the maps

$$g(z) = z + A_1 z^{-1} + \dots : G_0 \rightarrow \widehat{\mathbb{C}}$$

satisfying the above conditions.

We illustrate Kühnau's method on the solving the problem: *find* $\max \operatorname{Re} A_1$ on $\mathcal{A}_\phi$.

The existence of extremals g_0 of I_ϕ remains an open problem, because the class $\mathcal{A}_\phi$ is not compact. Kühnau established the necessary conditions for extremality and some sufficient conditions, which are different.

THEOREM 2.18 [Ku9, KK]. (a) *Any extremal map g_0 (if any) with the dilatation p_0 is a solution of the equation*

$$w\bar{z} = \frac{p_0(z) - 1}{p_0(z) + 1} \bar{w}_z, \quad z \in G_0,$$

with p_0 satisfying $p_0(z) = p_*(z)$ on $G_0 \setminus G$ and such in G that $I_\phi(p_0) = M$.

(b) *The Jacobian $J(g_0)$ of g_0 is related to ϕ and p_0 by*

$$\begin{aligned} J(g_0) &= c p_0(z) \phi_p(p_0(z), z) \quad \text{if } p_0(z) \not\equiv 1, \\ J(g_0) &\leq c p_0(z) \phi_p(p_0(z), z) \quad \text{if } p_0(z) \equiv 1, \end{aligned}$$

with a constant $c > 0$.

(c) *The extremal domain $g_0(G_0)$ is bounded by the straight line intervals parallel to the real axis.*

(d) *For C^2 smooth ϕ , we have also that*

$$p_0(z) \frac{\phi_{pp}(p_0(z), z)}{\phi_p(p_0(z), z)} \geq -2 \quad \text{on the set } \{z \in G: p_0(z) > 1\}.$$

(e) *If $\phi_{pp}(q, z) > 0$ for all $z \in G$ and all $q \geq 1$, then there is only one extremal function g_0 , i.e., the conditions (a)–(d) are both necessary and sufficient for extremality of a map g_0 .*

The proof of this theorem is variational. For example, to establish (a), (b) and (c), one has to apply the comparison maps satisfying the equation

$$\overline{w_z} = \frac{\tilde{p}(z) - 1}{\tilde{p}(z) + 1} \tilde{w}_z$$

with

$$\tilde{p}(z) = \begin{cases} p_0(z) - \varepsilon_1, & z \in \Delta(z_1, r), \\ p_0(z) + \varepsilon_2, & z \in \Delta(z_2, r), \\ p_0(z), & z \in G_0 \setminus (\Delta(z_1, r) \cup \Delta(z_2, r)) \end{cases}$$

where $\Delta(z_1, r)$ and $\Delta(z_2, r)$ are two (sufficiently small) nonoverlapped disks in G , centered at the given points $z_1, z_2 \in G$, respectively, with a radius r , and $p_0(z) > 1$.

2.12.4. Theorem 2.18 provides various interesting applications to fluid mechanics, minimal surface theory and harmonic map theory. For example, the function $V^2 = |\text{grad}(\text{Re } g_0)|^2$ is related to the dilatation p_0 by $V^2 = p_0 J(g_0)$, and the equation in the assertion (b) of Theorem 2.18 is equivalent to

$$V^2 = c p_0^2 \phi_p(p_0(z), z),$$

which can be resolved in the form $p_0 = P(V^2, z)$. On this way, one obtains that the real part of the extremal function $g_0(z) = u + iv$ must be a solution of the nonlinear system

$$u_x = P(u_x^2 + u_y^2, z) v_y, \quad u_y = -P(u_x^2 + u_y^2, z)$$

of the gas dynamics equations. For details and other applications, we refer to [KK, Part 2, Chapter 5].

2.12.5. Another variational method for the maps quasiconformal in the mean was developed by Gulyanskii and Ryazanov. In fact, it is a nice modification and extension of their approach described in Section 3.2. It would be interesting to combine the convexity of the function ϕ in the integrability condition of type (2.67) with general variation formulas in order to establish the extremals more explicitly. We refer to [GuR2, GuR3, Rya2].

3. Nonlinear quasiconformal maps

3.1. Lavrentiev–Lindelöf variational principle for strongly elliptic systems

3.1.1. In order to solve many problems either in mathematics or in its applications, one must apply in general the maps which are the solutions of the systems of partial differential

equations of the first order. These equations need not be necessarily linear. Consider a general system of equations of such type

$$L_j(x, y, u, v, u_x, u_y, v_x, v_y) = 0, \quad j = 1, 2, \quad (3.1)$$

for two real functions $u(x, y), v(x, y)$ of two variables $(x, y) \in \mathbb{R}^2$. Following Lavrentiev [La2], we call any generalized homeomorphic solution of the system (3.1) in a region $D \subset \mathbb{C}$ to be a quasiconformal map corresponding to this system.

When regarded in such a way, quasiconformal maps provide a geometric theory of elliptic systems of partial differential equations.

It turned out that many basic results on conformal maps can be extended to solutions of so-called *strongly elliptic* systems (3.1), not necessarily linear ones. Roughly speaking, strong ellipticity means that the linearized system is uniformly elliptic, though in [La2] some additional geometric condition is required.

Of course, nonlinearity constitutes a rather strong obstacle to the existence of global solutions of (3.1) in the given domains. The greatest progress has been made for linear systems, which, in fact, can be reduced to the Beltrami equation.

3.2. Main theorem for strips

Lavrentiev was one of the founders of the theory of quasiconformal maps. In this survey we are only concerned with his variational principle, which extends one of the basic principles in the theory of conformal maps. This principle involves both inner and boundary variations.

Lavrentiev's research was motivated mostly by solving problems in fluid dynamics; however, the principle and its various applications are of great interest in their own right.

We restrict ourselves to the maps of strips and first formulate the principle for conformal maps. Let $D(C_0, C)$ be either a curvilinear lune or a strip bounded by two distinct smooth Jordan arcs C_0 and C with the same endpoints a_1, a_2 ; in particular, one of these points or both points a_1, a_2 can lie at infinity. Let f be a conformal map of $D(C_0, C)$ onto the horizontal strip

$$H = \{w = u + iv: 0 < v < 1\}$$

so that $f(a_1) = -\infty$, $f(a_2) = \infty$. Thereby f is determined uniquely up to an additive real constant, which plays no essential role here. We distinguish in $D(C_0, C)$ the inverse images C_v of the level lines $\{v = \text{const}\}$, $0 < v < 1$.

Let $D(\tilde{C}_0, \tilde{C})$ be a similar domain bounded by arcs $\tilde{C}_0$ and $\tilde{C}$ with the same endpoints a_1, a_2 , and let $\tilde{f}$ be a conformal map of $D(\tilde{C}_0, \tilde{C})$ onto H ; similarly, $\tilde{C}_v = \tilde{f}^{-1}(\{v = \text{const}\})$.

The generalization of Lindelöf's principle says:

THEOREM 3.1. *For any domain $D(\tilde{C}_0, \tilde{C})$ contained in $D(C_0, C)$ (so that $\tilde{C}_0$ has the same endpoints a_1, a_2) and its corresponding map $\tilde{f}: D(\tilde{C}_0, \tilde{C}) \rightarrow H$, we have:*

- (a) for each v , $0 < v < 1$, the domain $D(\tilde{C}_v, C)$ is contained in $D(C_v, C)$;
- (b) the boundary derivatives at any point $z_0 \in C$ satisfy $|\tilde{f}'(z_0)| \geq |f'(z_0)|$;
- (c) if C_0 and $\tilde{C}_0$ have a point z_1 in common, then $|\tilde{f}'(z_1)| \leq |f'(z_1)|$;
- (d) if the defining functions of the lines C_0 , $\tilde{C}_0$ and C are of the form

$$y = h_0(x), \quad y = \tilde{h}_0(x), \quad y = h(x), \quad 0 \leq x \leq 1,$$

respectively, then

$$|\tilde{f}'(\tilde{z}_2)| > |f'(z_2)|,$$

where $z_2 = x_2 + ih_0(x_2)$ is the point of maximal deformation of $h_0(x) - \tilde{h}_0(x)$ (i.e., the point where this function attains its maximal value), and $\tilde{z}_2 = x_2 + i\tilde{h}_0(x_2)$.

Moreover, equality in (a), (b), (c) occurs only when $\tilde{C} = C$.

3.2.1. Lavrentiev has extended his principle (in a weaker form) to $C^{2+\alpha}$ smooth solutions of strongly elliptic systems (3.1) of the form

$$L_1(u_x, u_y, v_x, v_y) = 0, \quad L_2(u_x, u_y, v_x, v_y) = 0,$$

i.e., when L_1, L_2 do not depend explicitly from x, y, u, v (thus it holds for linear equations of such form). This allows him to establish, for example, the main existence and uniqueness theorems for homeomorphic solutions of these systems and provide various hydrodynamical applications.

The existence theorem remains for solutions of an arbitrary strongly elliptic system (3.1), but the variational principle fails in general case. Nevertheless, this principle remains in force also for the systems not depending explicitly from the desired functions u, v .

Roughly speaking, the generalized principle is similar to Theorem 3.1 (after some additional assumptions on the pre-images of the level lines $\{v = \text{const}\}$). For details we refer to Lavrentiev's book [La2].

Bojarski and Iwaniec (as well as some other authors) approached the nonlinear elliptic systems of type (3.1) from a different point of view; see, e.g., [BoI].

A wide circle of interesting questions related to nonlinearity remains still open. For example, not much is known for the general nonlinear Beltrami equation.

4. Quasilinear Beltrami equation

4.1. Gutlyanskii–Ryazanov's method

An extension of the main variational method to quasilinear quasiconformal maps is given in [GuR1]. Our exposition follows this paper.

4.1.1. Consider the quasilinear Beltrami equation

$$\partial_{\bar{z}} w = \mu(z, w) \partial_z w \quad (4.1)$$

with $\mu : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$|\mu(z, w)| \leq k(z, w) \leq k < 1. \quad (4.2)$$

We assume that

$$\mu(\cdot, w) \in L_\infty^\Delta = \{v \in L_\infty : \text{supp } v \subset \Delta\}$$

for each $w \in \mathbb{C}$, while k is a so-called *Carathéodory function*, i.e., $k(\cdot, w) : \mathbb{C} \rightarrow \mathbb{C}$ is a measurable function for every $w \in \mathbb{C}$, and $k(z, \cdot) : \mathbb{C} \rightarrow \mathbb{C}$ is continuous for almost every $z \in \mathbb{C}$.

Let us consider the homeomorphic generalized solutions of the equation (4.1) normalized by

$$w(z) = z + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty. \quad (4.3)$$

The class of such maps is denoted by $\Sigma_{K(z, w)}$, where $K(z, w) = (1 + k(z, w))/(1 - k(z, w))$. This class is also sequentially compact in the topology of locally uniform convergence in $\mathbb{C}$.

The central point is to construct admissible variations. This differs essentially from the situation in the linear case investigated above. However, it becomes possible when the Beltrami coefficient k satisfies the following additional restrictions:

- (a) $k(z, w) \geq k_0 > 0$ for $z \in \Delta$,
- (b) $k(z, w + \Delta w) = k(z, w) + 2 \operatorname{Re}[\partial_w k(z, w) \Delta w] + o(|\Delta w|)$, and moreover, $\partial_w k(z, w)$ is a Carathéodory function such that $|\partial_z k(z, w)| \leq C < \infty$.

Let again

$$T_\rho(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\rho(\zeta)}{\zeta - z} d\xi d\eta, \quad \rho \in L_p(\mathbb{C}), p > 2, \zeta = \xi + i\eta.$$

Then we have the following theorem.

THEOREM 4.1. *Let μ be the Beltrami coefficient of a map $f \in \Sigma_{K(z, w)}$. Fix $\delta \in (0, k_0)$, and let $\sigma(z)$ be any function from $L_\infty(\Delta)$ such that $\operatorname{ess\,inf} \sigma(z) > 0$ on the set*

$$E(\delta) = \{z \in \Delta : |\mu(z)| > k(z, f) - \delta\}.$$

Then, for small t , the class $\Sigma_{K(z, w)}$ contains the variations of f of the form:

$$f_t = f - t\phi \circ f + o(t), \quad t \rightarrow 0, \quad (4.4)$$

where

$$\phi(w) = \varphi(w)T(g/\varphi), \quad \varphi = \exp T(h), \quad (4.5)$$

with

$$h \circ f(z) = \begin{cases} 2\gamma(z)k(z, f) \partial_{\bar{w}}(z, f) / \overline{\mu(z)}, & z \in E(\delta), \\ 0, & z \notin E(\delta), \end{cases} \quad (4.6)$$

and

$$g \circ f(z) = \begin{cases} -\sigma(z)\gamma(z) / \overline{\mu(z)}, & z \in E(\delta), \\ \sigma(z)\gamma(z), & z \notin E(\delta). \end{cases} \quad (4.7)$$

Here

$$\gamma(z) = (1 - |\mu(z)|^2)^{-1} \partial_z f / \partial_{\bar{z}} f.$$

SKETCH OF THE PROOF. Set $\mu_t(z) = \mu(z) + ta(z)$, with $a \in L_\infty(\Delta)$ to be determined and $t \rightarrow 0$, then the solution f_t of the Beltrami equation $\partial_z w = \mu_t \partial_{\bar{z}} w$, normalized by (4.3), depends holomorphically on t . To satisfy inequality $|\mu_t(z)| \leq k(z, f)$ for small $|t|$ and almost all $z \in \Delta$, one gets for a a system containing one differential and one functional equation. Solving the system is reduced to solving the equation $\partial_{\bar{w}} \phi = h\phi + g$ under the assumption $T(\partial_{\bar{w}} \phi) = \phi$; the coefficient h and g are defined by (4.6) and (4.7). One can prove that such ϕ is unique and must be of the form (4.5). $\square$

4.1.2. Now let J be a real upper semicontinuous Gateaux differentiable functional on $\Sigma_{k(z,w)}$, then J attains its maximum on this class, and there is a complex Borel measure κ on $\mathbb{C}$ with compact support so that for any $f_t = f + tH + o(t)$, we have

$$J(f_t) = J(f) + t \operatorname{Re} \iint_{\mathbb{C}} H d\kappa + o(t), \quad t \rightarrow 0.$$

Suppose also that the variation kernel $1/(w - f(z))$ is locally integrable with respect to the product measure $dm_2 \otimes d\kappa$, where m_2 is the Lebesgue measure on $\mathbb{C}$, and that, for any m_2 -measurable set $E \subsetneq \mathbb{C}$, the function

$$\mathcal{A}_E(w) := \frac{1}{\pi} \iint_{\mathbb{C}} [\exp T(h\chi_E) \circ f_0](z) \frac{d\kappa(z)}{w - f_0(z)}$$

does not vanish almost everywhere on $\mathbb{C}$. Here h is given by (4.6) and χ_E denotes the characteristic function of E .

The next theorem provides the necessary conditions for the extremality.

THEOREM 4.2. *The map $f_0 \in \Sigma_{K(z,w)}$ maximizing J on this class has the Beltrami coefficient μ_0 , which satisfies almost everywhere the relations*

$$|\mu_0(z)| = k(z, f_0(z)) \quad (4.8)$$

and

$$\mu_0(z)(\partial_z f_0)^2(z) \left[\frac{\mathcal{A}}{\exp T(h)} \circ f_0 \right](z) \geq 0, \quad (4.9)$$

where h is again given by (4.6) and

$$\mathcal{A}(w) = - \iint_{\mathbb{C}} \frac{\exp T(h) \circ f_0(\zeta)}{w - f_0(\zeta)} d\kappa(\zeta). \quad (4.10)$$

If $k(z, w) = k_0$ for almost all $z \in \Delta$, the relations (4.8)–(4.10) take the known form $|\mu(z)| = k_0$ in Δ , and

$$\mu_0(z)(\partial_z f_0)^2(z) \iint_{\mathbb{C}} \frac{d\kappa(\zeta)}{f(z) - f(\zeta)} \geq 0.$$

5. A glimpse at further methods and developments

We mention briefly certain important fields and directions closely connected with the theory of extremal quasiconformal maps. Because of the framework of this chapter, we cannot present the details; for these we refer the reader to the bibliography below which of course is not complete.

- *Extremal properties of holomorphic quadratic differentials and measured foliations on Riemann surfaces.*

This is now a rather complete chapter of the theory of Riemann surfaces concerning the ergodic properties of certain dynamical systems and foliations on the surfaces with many new ideas. The main point is that every integrable holomorphic quadratic differential on a Riemann surface X of a finite analytic type determines a measured foliation on X in the sense of Thurston, and vice versa. The existence relies on minimization of the Dirichlet norm of quadratic differentials. A good introduction to this field can be found, e.g., in [GaL,HuM,MSt,St4,Ren3,Th2].

- *Interaction between harmonic and quasiconformal maps. Harmonic flows on Teichmüller spaces.*

This is now in fact a separate chapter in the theory of quasiconformal maps. Harmonic maps minimizing the energy integrals (in fact the Dirichlet integral in a corresponding metric) produce intrinsically a holomorphic quadratic differential associated with an extremal quasiconformal map. We refer, e.g., to the papers [Mi1,She1,Wo1].

A new approach to uniqueness of an extremal quasiconformal map in its class based on harmonicity is provided in [BLMM], see also [Re3].

The question of *uniqueness* of extremal map has a long history and was treated by many authors (see, e.g., [Re5,RS2,St1,St2,St6,MaMa]).

Recently Markovic [Mark] has established that the affine map is uniquely extremal in its class of the maps of punctured plane

$$\mathbb{C} \setminus \{z_{mn} = m + in; m, n \in \mathbb{Z}\}.$$

His proof provides an extension of the standard technique involving the Hahn–Banach extension; it relates also to certain problems concerning the Poincaré theta-operator treated earlier by Kra and McMullen (see [McM1]).

- *Univalent holomorphic functions with quasiconformal extensions.*

This subject is now an important part of modern geometric function theory and has deep applications, for example, in the Teichmüller space theory. It exploits various methods including the general methods of complex analysis and complex geometry. An exposition will be given separately.

- *Extremal maps with weighted dilatations.*

The papers of Teichmüller and Volkovyskiy cited above gave rise to the study of the maps whose dilatations are bounded by a nonconstant function. Such maps have been investigated by many authors (see, e.g., [AC1,GuR1,Io,Kru3,She1,She2]). An essential advance in this direction by various methods is due to Kühnau, see [Ku2–Ku8,KK].

- *Special cases of regular and nonregular functionals on classes of quasiconformal maps of plane domains and Riemann surfaces: holomorphic, plurisubharmonic, etc.*

There are special regular types of functionals $\mathcal{F}(f^\mu)$ whose distortion bounds can be established in a way related to the classical Schwarz lemma and its generalizations. It is given by Lehto's majorant principle and its improvements, which rely on contracting the hyperbolic metric by the holomorphic map $\mu \mapsto \mathcal{F}(f^\mu)$ from the unit ball of the Beltrami coefficients into $\mathbb{C}$. See, e.g., [KK,Leh1,Leh2]. The details will be presented in a separate survey paper devoted to holomorphic maps with quasiconformal extensions.

Such an approach can be extended partially to nonregular plurisubharmonic functionals, cf. [Kru10].

- *Dynamical approach.*

Given a quasiconformal map w^μ , one can produce a dynamical system (taking, for example, the maps $w^{t\mu}$, $t \in [0, 1]$) and a corresponding semigroup of deformations whose generator defines in fact a variation of these deformations. Moreover, such an approach works also in case of quasiconformal maps of the space.

Reimann [Rei1] and Semenov [Se1,Se2] have established in this way some important existence theorems as well as the sharp distortion estimates for quasiconformal maps (even in the space). Other applications are given, e.g., in [KL,Res,SF].

- *Holomorphic motions and iterations, applications to extremal quasiconformal maps.*

A holomorphic motion of a set $E \subset \widehat{\mathbb{C}}$ is an isotopy $f(z, t)$ of E with complex time parameter t varying holomorphically on a region in $\widehat{\mathbb{C}}$ or, more generally, in a Banach space. It turns out that this holomorphy forces a strong regularity in both arguments; moreover, it provides the extension of motion to ambient space preserving all properties.

Holomorphic motions were introduced by Mañé, Sad and Sullivan in [MSS]. They have been important in the study of dynamical systems, Kleinian groups, Teichmüller spaces and

their applications, in many problems of geometric function theory concerning conformal and quasiconformal maps where the holomorphic embeddings of the disk occur.

Holomorphic motions are intrinsically connected with quasiconformal maps, which is revealed by the remarkable lambda-lemma of Mañé–Sad–Sullivan. This lemma and its improvements have become a power tool. Its various applications can be found, e.g., in [MSS,Sul1,Sul2,BR,ST,Sl,As,EKK,EM,Kr5,Kru12,Kru13,Mart,MSu,Pom,Shi1,Shi2,Su1,Su2].

- *Quasiconformal reflections.*

One of the important applications of quasiconformality relies on quasiconformal reflections. For example, Teichmüller’s theory and Fredholm eigenvalues theory are naturally connected with such reflections.

Recall that a quasireflection across a closed Jordan curves $L \subset \widehat{\mathbb{C}}$ is an orientation reversing quasiconformal automorphism of $\widehat{\mathbb{C}}$ which preserves this curve pointwise fixed (interchanging the interior and exterior of L).

This notion was introduced by Ahlfors in his celebrated paper [Ah1] and can be extended to much more general subsets of the complex plane. Further applications provided by Kühnau concern, for example, his extension of Dirichlet’s principle to quasiconformal maps, Fredholm eigenvalues and Grunsky’s coefficient conditions.

Another approach involves holomorphic motions and the polynomial approximation of holomorphic functions. We refer, e.g., to survey papers [Ku11,Ku13,Kru19] and to [KK, Part 2].

For an extension to higher dimensions see, e.g., [MMPV,Yan].

- *The method of extremal lengths (or moduli).*

This method has its origin in an older method in geometric function theory, known as the length–area principle. A systematical use of extremal lengths was originated by Beurling and Ahlfors. The method relies on the geometric definition of quasiconformality and has become now one of the basic methods in various fields, first of all, in solving the extremal problems for conformal and quasiconformal maps. A preference of the method of the extremal lengths is that it became rather universal and provides easily and naturally the uniqueness of solution in many extremal problem, which is complicated by applying another methods. Note also that in the multidimensional case, other methods do not work.

- *New phenomena arising in infinite-dimensional Teichmüller spaces.*

Certain properties of these spaces are at odds with the Teichmüller metric.

- *Extension of the theory of extremal quasiconformal maps to solutions of general elliptic systems in two variables.*

This remains an important open problem. The only extension that is known is an extension of Schiffer’s variational method for conformal maps to (injective) solutions of elliptic systems

$$\partial_{\bar{z}} w = \nu \partial_z w + \mu \partial_z \bar{w}, \quad \|\nu\|_{\infty} + \|\mu\|_{\infty} < 1,$$

given by Renelt, see [Ren4,Ren5].

- *Asymptotic Teichmüller theory.*

This new approach to the Teichmüller space theory was recently developed by Earle, Gardiner and Lakic, for details see, e.g., the book [GaL].

The features of quasiconformal automorphisms of the disk (half-plane) whose boundary values are asymptotically conformal quasisymmetric functions are interesting by themselves. Thus an interesting problem is to discover the appropriate (intrinsic) variational methods for such maps.

- *Multidimensional generalizations.*

A rigidity of quasiconformal deformations in $\mathbb{R}^n$ for $n > 2$ provides, first of all, a strong obstruction for existence of variations. The theory of extremal quasiconformal maps of the space domains is rather poorly developed. See, e.g., [Ag2, AF, AVV, Fe2, Fe3, Ge1, Iw, Ku1, Res, Vai, Vu1, Vu2].

The main obstruction to extend the Teichmüller space theory to n -dimensional manifolds for $n \geq 3$ is the strong rigidity of hyperbolic space forms, due to Mostow's famous rigidity theorem ([Mos], see also [Marg]).

References

- [Ab1] W. Abikoff, *Degenerating families of Riemann surfaces*, Ann. of Math. **105** (1977), 29–44.
- [Ab2] W. Abikoff, *Real Analytic Theory of Teichmüller Spaces*, Lecture Notes in Math., Vol. 820, Springer, Berlin (1980).
- [Ab3] W. Abikoff, *The uniformization theorem*, Amer. Math. Monthly **88** (1981), 574–592.
- [Ag1] S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **413** (1968), 1–12.
- [Ag2] S. Agard, *On the extremality of affine mappings with small dilatation*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 95–111.
- [AF] S. Agard and R. Fehlmann, *On the extremality and unique extremality of affine mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **11** (1986), 87–110.
- [Ah1] L.V. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. **3** (1953–1954), 1–58.
- [Ah2] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton (1966).
- [Ah3] L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [AB] L.V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 383–404.
- [AW] L.V. Ahlfors and G. Weill, *A uniqueness theorem for the Beltrami equation*, Proc. Amer. Math. Soc. **13** (1962), 975–978.
- [Al] I.A. Aleksandrov, *Parametric Continuations in the Theory of Univalent Functions*, Nauka, Moscow (1976) (in Russian).
- [AVV] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Mappings*, Wiley, New York (1997).
- [AC1] C. Andreian Cazacu, *Sur une problème de L.I. Volkovyski*, Rev. Roumaine Math. Pures Appl. **10** (1965), 43–63.
- [AC2] C. Andreian Cazacu, *Problèmes extrémaux des représentations quasiconformes*, Rev. Roumaine Math. Pures Appl. **10** (1965), 409–429.
- [AC3] C. Andreian Cazacu, *On extremal quasiconformal mappings*, Rev. Roumaine Math. Pures Appl. **22** (1977), 1359–1365.
- [As] K. Astala, *Area distortion for quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [AsIM] K. Astala, T. Iwaniec and G. Martin, *Elliptic Equations and Quasiconformal Mappings in the Plane*, Syracuse Univ. (2002).
- [AsM] K. Astala and G. Martin, *Holomorphic motions*, Papers on Analysis, Rep. Univ. Jyväskylä Dept. Math. Stat. **83**, Univ. Jyväskylä (2001), 27–40.
- [Bel1] P.P. Belinskii, *Distortion under quasiconformal mappings*, Dokl. Akad. Nauk SSSR **91** (1953), 997–998 (in Russian).

- [Bel2] P.P. Belinskii, *On the solution of extremal problems of quasiconformal mappings by the method of variations*, Dokl. Akad. Nauk SSSR **121** (1958), 199–201 (in Russian).
- [Bel3] P.P. Belinskii, *Solution of extremal problems in the theory of quasiconformal mappings by the variational method*, Sibirsk. Mat. Zh. **1** (1960), 303–330 (in Russian).
- [Bel4] P.P. Belinskii, *General Properties of Quasiconformal Mappings*, Nauka, Novosibirsk (1974) (in Russian).
- [Ber1] L. Bers, *Quasiconformal mappings and Teichmüller theorem*, Analytic Functions, L. Ahlfors, H. Behnke, H. Grauert, L. Bers et al., eds, Princeton Univ. Press, Princeton (1960), 89–119.
- [Ber2] L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
- [Ber4] L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups, I*, Ann. of Math. **91** (1970), 570–600.
- [Ber3] L. Bers, *Extremal quasiconformal mappings*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud., Vol. 66, Princeton University Press, Princeton (1971), 27–52.
- [Ber5] L. Bers, *Uniformization, modules and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [Ber6] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
- [Ber7] L. Bers, *An extremal problem for quasiconformal mappings and a theorem of Thurston*, Acta Math. **141** (1978), 73–98.
- [Ber8] L. Bers, *A new proof of fundamental inequality for quasiconformal mappings*, J. Anal. Math. **36** (1979), 15–30.
- [Ber9] L. Bers, *Finitely dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. **5** (1981), 131–172.
- [BK] L. Bers and I. Kra (eds), *A Crash Course on Kleinian Groups*, Lecture Notes in Math., Vol. 400, Springer, Berlin (1974).
- [BR] L. Bers and H.L. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.
- [Bi1] P. Biluta, *Some extremal problems for mappings that are quasiconformal in mean*, Sibirsk. Mat. Zh. **6** (1965), 717–726 (in Russian).
- [Bi2] P. Biluta, *The solution of extremal problems for a certain class of quasiconformal mappings*, Siberian Math. J. **10** (1969), 533–540.
- [Bi3] P. Biluta, *An extremal problem for quasiconformal mappings of regions of finite connectivity*, Siberian Math. J. **13** (1972), 16–22.
- [BiK] P. Biluta and S.L. Krushkal, *On the question of extremal quasiconformal mappings*, Soviet Math. Dokl. **11** (1971), 76–79.
- [Bo] B.V. Bojarski, *Generalized solutions of a first order system of the elliptic type with discontinuous coefficients*, Mat. Sb. **43** (85) (1957), 451–503 (in Russian).
- [BoI] B.V. Bojarski and T. Iwaniec, *Quasiconformal mappings and non-linear elliptic equations in two variables, I, II*, Bull. Polish Acad. Sci. Ser. Math. Astron. Phys. **22** (1974), 473–478, 479–484.
- [BLMM] V. Božin, N. Lakic, V. Marković and M. Mateljević, *Unique extremality*, J. Anal. Math. **75** (1998), 299–338.
- [Da] V.A. Danilov, *Estimates of distortion of quasiconformal mapping in space C_α^m* , Siberian Math. J. **14** (1973), 362–369.
- [Di] B. Dittmar, *Extremalprobleme quasikonformer Abbildungen der Ebene als Steuerungsprobleme*, Z. Anal. Anwendungen **5** (1986), 563–573.
- [DE] A. Douady and C.J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.
- [DS] V.V. Dumkin and V.G. Sheretov, *Teichmüller's problem for a class of open Riemann surfaces*, Mat. Zametki **7** (1970), 605–615 (in Russian).
- [Ea1] C.J. Earle, *Reduced Teichmüller spaces*, Trans. Amer. Math. Soc. **126** (1967), 54–63.
- [Ea2] C.J. Earle, *On holomorphic cross-sections in Teichmüller spaces*, Duke Math. J. **36** (1969), 409–415.
- [Ea3] C.J. Earle, *On the Carathéodory metric in Teichmüller spaces*, Discontinuous Groups and Riemann Surfaces, Princeton Univ. Press, Princeton (1974), 99–103.
- [Ea4] C.J. Earle, *The Teichmüller distance is differentiable*, Duke Math. J. **44** (1977), 389–397.
- [Ea5] C.J. Earle, *Teichmüller theory*, Discrete Groups and Automorphic Functions, Proc. Conf., Cambridge, 1975, Academic Press, London (1977), 143–162.

- [Ea6] C.J. Earle, *Some intrinsic coordinates on Teichmüller space*, Proc. Amer. Math. Soc. **83** (1981), 527–531.
- [EE] C.J. Earle and J. Eells, *On the differential geometry of Teichmüller spaces*, J. Anal. Math. **19** (1967), 35–52.
- [EGL] C.J. Earle, F.P. Gardiner and N. Lakic, *Vector fields for holomorphic motions of closed sets*, Contemp. Math. **211** (1997), 193–225.
- [EK] C.J. Earle and I. Kra, *On sections of some holomorphic families of closed Riemann surfaces*, Acta Math. **137** (1976), 49–79.
- [EKK] C.J. Earle, I. Kra and S.L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Trans. Amer. Math. Soc. **343** (1994), 927–948.
- [EL1] C.J. Earle and Li Zhong, *Extremal quasiconformal mappings in plane domains*, Quasiconformal Mappings and Analysis: A collection of Papers Honoring F.W. Gehring, P. Duren et al., eds, Springer, New York (1997), 141–157.
- [EL2] C.J. Earle and Li Zhong, *Isometrically embedded polydisks in infinite dimensional Teichmüller spaces*, J. Geom. Anal. **9** (1999), 51–71.
- [EM] C.J. Earle and S. Mitra, *Analytic dependence of conformal invariants on parameters*, Contemp. Math., Vol. 256, Amer. Math. Soc., Providence, RI (2000).
- [ErH] A. Eremenko and D.H. Hamilton, *On the area distortion by quasiconformal mappings*, Proc. Amer. Math. Soc. **123** (1995), 2793–2797.
- [FK] H.M. Farkas and I. Kra, *Riemann Surfaces*, Springer, Berlin (1980).
- [Fe1] R. Fehlmann, *Über extremale quasikonforme Abbildungen*, Comment. Math. Helv. **56** (1981), 1558–1580.
- [Fe2] R. Fehlmann, *Extremal problems for quasiconformal mappings in space*, J. Anal. Math. **48** (1987), 179–215.
- [Fe3] R. Fehlmann, *On the trace dilatation and Teichmuellers problem in n -space*, Geometric Function Theory and Applications of Complex Analysis to Mechanics: Studies in Complex Analysis and Its Applications to Partial Differential Equations, Vol. 2, R. Kühnau and W. Tutschke, eds, Pitman Res. Notes Math. Ser., Longman, Harlow, UK (1991), 27–39.
- [FeG] R. Fehlmann and F.P. Gardiner, *Extremal problems for quadratic differentials*, Preprint IHES (1995).
- [FV] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam (1980).
- [Ga1] F.P. Gardiner, *An analysis of the group operation in universal Teichmüller space*, Trans. Amer. Math. Soc. **132** (1968), 471–486.
- [Ga2] F.P. Gardiner, *Schiffer's interior variation and quasiconformal mappings*, Duke Math. J. **42** (1975), 371–380.
- [Ga3] F.P. Gardiner, *On partially Teichmüller–Beltrami differentials*, Michigan Math. J. **29** (1982), 237–242.
- [Ga4] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
- [GaL] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Math. Surveys Monogr., Vol. 76, Amer. Math. Soc., Providence, RI (2000).
- [GaM] F.P. Gardiner and H. Mazur, *Extremal length geometry of Teichmüller spaces*, Complex Var. **16** (1991), 209–237.
- [GaS] F.P. Gardiner and D. Sullivan, *Symmetric and quasisymmetric structures on closed curve*, Amer. J. Math. **114** (1992), 683–736.
- [Ge2] F.W. Gehring, *Quasiconformal mappings which hold the real axes pointwise fixed*, Mathematical Essays Dedicated to A.J. McIntyre, Ohio Univ. Press, Athens (1970), 145–148.
- [Ge1] F.W. Gehring, *Extremal mappings of tori*, Certain Problems of Mathematics and Mechanics (On the Occasion of the Seventieth Birthday of M.A. Lavrent'ev), Nauka, Leningrad (1978), 146–158 (in Russian).
- [Ge3] F.W. Gehring, *Characteristic Properties of Quasidisks*, Les Presses de l'Université de Montréal (1982).
- [GoG] A.A. Golubev and S.Yu. Graf, *Extremal problems on the classes of quasiconformal embeddings of Riemann surfaces*, Siberian Adv. Math. **11** (4) (2001), 47–64; Engl. transl. from: Proc. Analysis and Geometry, Novosibirsk, Akademgorodok, 1999, Izdat. Ross. Akad. Nauk Sibirsk. Otd. Inst. Mat., Novosibirsk (2000), 204–226.

- [GoS] A.A. Golubev and V.G. Sheretov, *Quasiconformal extremals of energy integral*, Math. Notes **55** (1994), 580–585.
- [Gol] G.M. Goluzin, *Geometric Theory of Functions of Complex Variables*, Transl. Math. Monogr., Vol. 26, Amer. Math. Soc., Providence, RI (1969).
- [Gro1] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **80** (1928), 367–376.
- [Gro2] H. Grötzsch, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **84** (1932), 114–120.
- [GuR1] V.Ya. Gutlyansky and V.I. Ryazanov, *On the variational method for quasiconformal mappings*, Siberian Math. J. **28** (1987), 59–81.
- [GuR2] V.Ya. Gutlyansky and V.I. Ryazanov, *Quasiconformal mappings with constraints of integral type on the Lavrent'ev characteristic*, Soviet Math. Dokl. **36** (1988), 456–459.
- [GuR3] V.Ya. Gutlyansky and V.I. Ryazanov, *Quasiconformal mappings with integral constraints on the M.A. Lavrent'ev characteristic*, Siberian Math. J. **31** (1990), 205–215.
- [HJ] L. Habermann and J. Jost, *Riemannian metrics on Teichmüller space*, Manuscripta Math. **89** (1996), 281–306.
- [H] D. Hamilton, *BMO and Teichmüller space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 213–224.
- [Ha] R.S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. **138** (1969), 399–406.
- [HaW] R. Hardt and M. Wolf, *Harmonic extensions of quasiconformal maps to hyperbolic spaces*, Indiana Univ. Math. J. **46** (1997), 107–120.
- [HO] A. Harrington and M. Ortel, *Dilatation of an extremal quasiconformal mapping*, Duke Math. J. **43** (1976), 533–544.
- [He] D.A. Hejhal, *On Schottky and Teichmüller spaces*, Adv. in Math. **15** (1975), 133–156.
- [Ho1] E. Hoy, *On a class of mappings being quasiconformal in the mean*, Complex Var. **15** (1990), 19–26.
- [Ho2] E. Hoy, *On a class of mappings being quasiconformal in the mean. II*, Complex Var. **17** (1992), 201–212.
- [Hu] J. Hubbard, *Sur les sections analytiques de la courbe universelle de Teichmüller*, Mém. Amer. Math. Soc., Vol. 4, Amer. Math. Soc., Providence, RI (1976).
- [HuM] J. Hubbard and H. Mazur, *Quadratic differentials and foliations*, Acta Math. **152** (1979), 221–274.
- [IT] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer, Tokyo (1992).
- [Io] M.S. Ioffe, *Extremal quasiconformal embeddings of Riemann surfaces*, Siberian Math. J. **16** (1975), 398–411.
- [Iw] T. Iwaniec, *Nonlinear Differential Forms*, Univ. of Jyväskylä, Jyväskylä (1998).
- [IwM] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press, New York (2001).
- [Je] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin (1965).
- [Jo] J. Jost, *Harmonic maps and curvature computations in Teichmüller theory*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 13–46.
- [Ke1] L. Keen, *Intrinsic moduli of Riemann surfaces*, Ann. of Math. **84** (1966), 404–420.
- [Ker1] S. Kerckhoff, *The asymptotic geometry of Teichmüller space*, Topology **19** (1980), 23–41.
- [Ko1] S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.
- [Ko2] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer-Verlag, New York (1997).
- [Kr1] I. Kra, *On Teichmüller spaces for finitely generated Fuchsian groups*, Amer. J. Math. **91** (1969), 67–74.
- [Kr2] I. Kra, *Automorphic Forms and Kleinian Groups*, Benjamin, Reading, MA (1972).
- [Kr3] I. Kra, *On Teichmüller's theorem on the quasi-invariance of cross ratios*, Israel J. Math. **30** (1981), 152–158.
- [Kr4] I. Kra, *Canonical mappings between Teichmüller spaces*, Bull. Amer. Math. Soc. **4** (1981), 143–179.
- [Kr5] I. Kra, *The Carathéodory metric on Abelian Teichmüller disks*, J. Anal. Math. **40** (1981), 129–143.
- [Kr6] I. Kra, *Quadratic differentials*, Rev. Roumaine Math. Pures Appl. **39** (1994), 751–787.

- [Kru1] S.L. Krushkal, *On the theory of extremal problems for quasiconformal mappings of closed Riemann surfaces*, Soviet Math. Dokl. **7** (1966), 1541–1544.
- [Kru2] S.L. Krushkal, *On Teichmüller's theorem on extremal quasiconformal mappings*, Siberian Math. J. **8** (1967), 730–744.
- [Kru3] S.L. Krushkal, *Extremal quasiconformal mappings with given boundary correspondence*, Soviet Math. Dokl. **8** (1967), 883–885.
- [Kru4] S.L. Krushkal, *Extremal quasiconformal mappings*, Siberian Math. J. **10** (1969), 411–418.
- [Kru5] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Wiley, New York (1979).
- [Kru6] S.L. Krushkal, *On the Grunsky coefficient conditions*, Siberian Math. J. **28** (1987), 104–110.
- [Kru7] S.L. Krushkal, *A new approach to variational problems in the theory of quasiconformal mappings*, Soviet Math. Dokl. **35** (1987), 219–222.
- [Kru8] S.L. Krushkal, *A new method of solving variational problems in the theory of quasiconformal mappings*, Siberian Math. J. **29** (1988), 245–252.
- [Kru9] S.L. Krushkal, *The coefficient problem for univalent functions with quasiconformal extension*, Holomorphic Functions and Moduli I, Vol. 10, D. Drasin et al., eds, Math. Sci. Research Inst. Publications, New York (1988), 155–161.
- [Kru10] S.L. Krushkal, *Extension of conformal mappings and hyperbolic metrics*, Siberian Math. J. **30** (1989), 730–744.
- [Kru11] S.L. Krushkal, *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650–660.
- [Kru12] S.L. Krushkal, *New developments in the theory of quasiconformal mappings*, Geometric Function Theory and Applications of Complex Analysis to Mechanics: Studies in Complex Analysis and Its Applications to Partial Differential Equations, Vol. 2, R. Kühnau and W. Tutschke, eds, Pitman Res. Notes Math. Ser., Longman, Harlow, UK (1991), 3–26.
- [Kru13] S.L. Krushkal, *Quasiconformal extremals of non-regular functionals*, Ann. Acad. Sci. Fenn. Ser. A I Math. **17** (1992), 295–306.
- [Kru14] S.L. Krushkal, *Exact coefficient estimates for univalent functions with quasiconformal extension*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), 349–357.
- [Kru15] S.L. Krushkal, *Univalent functions and holomorphic motions*, J. Anal. Math. **66** (1995), 253–275.
- [Kru16] S.L. Krushkal, *Quasiconformal maps decreasing L_p norm*, Siberian Math. J. **40** (2000), 884–888.
- [Kru17] S.L. Krushkal, *Old and new variational principles in the theory of quasiconformal maps*, Math. Rep. **2** (51) (2000), 485–496.
- [Kru18] S.L. Krushkal, *Univalent functions with quasiconformal extensions (variational approach)*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, Amsterdam (2005), 165–241 (this volume).
- [Kru19] S.L. Krushkal, *Quasiconformal extensions and reflections*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier Science, Amsterdam (2005), 507–553 (this volume).
- [KAG] S.L. Krushkal, B.N. Apanasov and N.A. Gusevsky, *Kleinian Groups and Uniformization in Examples and Problems*, 2nd edn., Amer. Math. Soc., Providence, RI (1992).
- [KK] S.L. Krushkal (Kruschkal) and R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner-Texte Math., Vol. 54, Teubner, Leipzig (1983).
- [KL] J. Krzyz and J. Lawrynowicz, *Quasiconformal mappings of the unit disk with two invariant points*, Michigan Math. J. **14** (1967), 487–492.
- [Ku1] R. Kühnau, *Elementare Beispiele von möglichst konforme Abbildungen im dreidimensionalen Raum*, Wiss. Z. d. Martin-Luther-Univ. Halle-Wittenberg, Math.-Nat. Reihe **11** (1962), 729–732.
- [Ku2] R. Kühnau, *Über gewisse Extremalprobleme der quasikonformen Abbildung*, Wiss. Z. d. Martin-Luther-Univ. Halle-Wittenberg, Math.-Nat. Reihe **13** (1964), 35–39.
- [Ku3] R. Kühnau, *Einige Extremalprobleme bei differentialgeometrischen und quasikonformen Abbildungen*, Math. Z. **94** (1966), 178–192.
- [Ku4] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien*, J. Reine Angew. Math. **231** (1968), 101–113.
- [Ku5] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien. II*, J. Reine Angew. Math. **238** (1969), 61–66.

- [Ku6] R. Kühnau, *Zur analytischen Darstellung gewisser Extremalfunktionen der quasikonformen Abbildung*, Math. Nachr. **60** (1974), 53–62.
- [Ku7] R. Kühnau, *Zur analytischen Gestalt gewisser Extremalfunktionen der quasikonformen Abbildung. II*, Math. Nachr. **92** (1979), 139–143.
- [Ku8] R. Kühnau, *Funktionalabschätzungen bei quasikonformen Abbildungen mit Fredholmschen Eigenwerten*, Comment. Math. Helv. **56** (1981), 297–206.
- [Ku9] R. Kühnau, *Über Extremalprobleme bei im Mittel quasikonformen Abbildungen*, Romanian–Finnish Sem. Complex Analysis, Proc. Bucharest, 1981, Lecture Notes in Math., Vol. 1013, Springer, Berlin (1983), 113–124.
- [Ku10] R. Kühnau, *Eine Extremalcharakterisierung von Unterschallgasströmungen durch quasikonforme Abbildungen*, Complex Analysis, Banach Center Publ., Vol. 11, Warsaw (1983), 199–210.
- [Ku11] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jahresber Deutsch. Math. Verein. **90** (1988), 90–109.
- [Ku12] R. Kühnau, *Interpolation by extremal quasiconformal Jordan curves*, Siberian Math. J. **32** (1991), 257–264.
- [Ku13] R. Kühnau, *Einige neuere Entwicklungen bei quasikonformen Abbildungen*, Jahresber Deutsch. Math. Verein. **94** (1992), 141–169.
- [Kuz] G.V. Kuz'mina, *Methods of geometric function theory, I*, St. Petersburg Math. J. **9** (1998), 455–507.
- [La1] M.A. Lavrentieff, *Sur une classe de représentations continues*, Mat. Sb. **42** (1935), 407–424 (in Russian).
- [La2] M.A. Lavrentiev, *Variational Methods for Boundary Value Problems for Systems of Elliptic Equations*, English transl. from the Russian, Noordhoff, Groningen (1963).
- [Leh1] O. Lehto, *Quasiconformal mappings and singular integrals*, Istituto Nazionale di Alta Matematica, Simposia Mathematica, Vol. 18, Academic Press, New York (1976), 429–453.
- [Leh2] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987).
- [LV] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin (1965).
- [LVV] O. Lehto, K.I. Virtanen and J. Väisälä, *Contribution to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **273** (1959), 3–14.
- [Li1] Li Zhong, *Non-uniqueness of geodesics in infinite-dimensional Teichmüller spaces*, Complex Var. **16** (1991), 261–272.
- [Li2] Li Zhong, *Non-uniqueness of geodesics in infinite-dimensional Teichmüller spaces (II)*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 355–367.
- [MSS] R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193–217.
- [Mar] A. Marden, *On homotopic mappings of Riemann surfaces*, Ann. of Math. **90** (1969), 1–9.
- [MSt] A. Marden and K. Strebel, *Pseudo-Anosov Teichmüller mapping*, J. Anal. Math. **46** (1986), 194–220.
- [Marg] G.A. Margulis, *Isometry of closed manifolds of constant negative curvature with the same fundamental group*, Soviet Math. Dokl. **11** (1970), 722–723.
- [Mark] V. Markovic, *Extremal problems for quasiconformal maps of punctured plane domains*, Trans. Amer. Math. Soc. **354** (2002), 1631–1650.
- [MaMa] V. Markovic and M. Mateljević, *The unique extremal QC mappings and uniqueness of Hahn–Banach extensions*, Mat. Vesnik **48** (1996), 107–112.
- [MSm] D.E. Marshall and W. Smith, *The angular distribution of mass by Bergman functions*, Rev. Mat. Iberoamericana **15** (1999), 131–150.
- [Mart] G. Martin, *The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions*, Proc. Amer. Math. Soc. **125** (1997), 1093–1103.
- [MMPV] O. Martio, V. Miklyukov, P. Ponnusamu and M. Vuorinen, *On some properties of quasiplanes*, Results Math. **42** (2002), 107–113.
- [Mas1] B. Maskit, *On boundaries of Teichmüller spaces and on Kleinian groups, II*, Ann. of Math. **91** (1970), 608–638.
- [Mas2] B. Maskit, *Uniformization of Riemann surfaces*, Contributions to Analysis, L.V. Ahlfors et al., eds, Academic Press, New York (1974), 293–312.
- [Mas3] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin (1987).

- [McM1] C. McMullen, *Amenability, Poincaré series and quasiconformal maps*, Invent. Math. **99** (1989), 95–127.
- [McM2] C. McMullen, *Complex earthquakes and Teichmüller theory*, Preprint (1996).
- [McM3] C. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, Ann. of Math. **151** (2000), 327–357.
- [MSu] C. McMullen and D. Sullivan, *Quasiconformal homeomorphisms and dynamics. III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1998), 351–395.
- [Mi1] Y. Minsky, *Harmonic maps, length and energy in Teichmüller space*, J. Differential Geom. **35** (1992), 151–217.
- [Mi2] Y. Minsky, *Quasiprojections in Teichmüller space*, J. Reine Angew. Math. **473** (1996), 121–136.
- [Mit] S. Mitra, *Teichmüller spaces and holomorphic motions*, J. Anal. Math. **81** (2000), 1–33.
- [Mo] V.N. Monakhov, *Boundary Value Problems with Free Boundaries for Elliptic Systems of Equations*, Transl. from Russian, Amer. Math. Soc., Providence, RI (1983).
- [Mos] G.D. Mostow, *Strong rigidity of locally symmetric spaces*, Ann. Math. Stud., Vol. 78, Princeton Univ. Press, Princeton (1973).
- [Na] S. Nag, *Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).
- [NV] S. Nag and A. Verjovsky, *Diff(S^1) and the Teichmüller spaces*, Comm. Math. Phys. **135** (1991), 401–411.
- [Nak] T. Nakanishi, *The inner radii of finite-dimensional Teichmüller spaces*, Tôhoku Math. J. **41** (1989), 679–688.
- [NakVe] T. Nakanishi and J.A. Velling, *A sufficient condition for Teichmüller spaces to have smallest possible inner radii*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 13–21.
- [NY] T. Nakanishi and H. Yamamoto, *On the outradius of the Teichmüller space*, Comment. Math. Helv. **64** (1989), 288–299.
- [Ne1] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
- [Nev] R. Nevanlinna, *Uniformisierung*, Springer-Verlag, Berlin (1953).
- [OB] B. O’Byrne, *On Finsler geometry and applications to Teichmüller spaces*, Advances in the Theory of Riemann Surfaces, Ann. Math. Stud., Vol. 66, Princeton Univ. Press, Princeton (1971), 317–328.
- [Oht] H. Ohtake, *Partially conformal quasiconformal mappings and the universal Teichmüller space*, J. Math. Kyoto Univ. **31** (1991), 171–180.
- [Or] M. Ortel, *Integral means and a theorem of Hamilton, Reich and Strebel*, Complex Analysis Joensuu, 1978 Proc., I. Laine, O. Lehto and T. Sorvali, eds, Lecture Notes in Math., Vol. 747, Springer, Berlin (1979), 301–308.
- [OS] M. Ortel and W. Smith, *The argument of an extremal dilatation*, Amer. Math. Soc. **104** (1988), 498–502.
- [Pe] I.N. Pesin, *Metrical properties of Q -quasiconformal mappings*, Mat. Sb. **40** (82) (1956), 281–294 (in Russian).
- [Pf] A. Pflüger, *Theorie der Riemannschen Flächen*, Springer-Verlag, Berlin (1957).
- [Pom] Chr. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin (1992).
- [PR] Chr. Pommerenke and B. Rodin, *Holomorphic families of Riemann mapping functions*, J. Math. Kyoto Univ. **26** (1) (1986), 13–22.
- [PRy] V.L. Potëmkin and V.I. Ryazanov, *On mappings with measure restrictions on dilatation*, Siberian Math. J. **39** (1998), 354–357.
- [Re1] E. Reich, *Quasiconformal mappings of the disk with given boundary values*, Advances in Complex Function Theory, W.E. Kirwan and L. Zalcman, eds, Lecture Notes in Math., Vol. 505, Springer, Berlin (1976), 101–137.
- [Re3] E. Reich, *An approximation condition and extremal quasiconformal extensions*, Proc. Amer. Math. Soc. **125** (1997), 1479–1481.
- [Re4] E. Reich, *A quasiconformal extension using the parametric representations*, J. Anal. Math. **54** (1990), 246–258.
- [Re2] E. Reich, *On the mapping with complex dilatation $ke^{i\theta}$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 261–267.
- [Re5] E. Reich, *The unique extremality counterexample*, J. Anal. Math. **75** (1998), 339–347.

- [Re6] E. Reich, *Extremal quasiconformal mappings of the disk*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, R. Kühnau, ed., Elsevier, Amsterdam (2002), 75–136.
- [RC] E. Reich and J.X. Chen, *Extensions with bounded $\bar{\partial}$ -derivative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 377–389.
- [RS1] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, L.V. Ahlfors et al., eds, Academic Press, New York (1974), 375–391.
- [RS2] E. Reich and K. Strebel, *Quasiconformal mappings of the punctured plane*, Complex Analysis – Fifth Romanian–Finnish Seminar, Part I, Bucharest, 1981, Lecture Notes in Math., Vol. 1013, Springer-Verlag, Berlin (1983), 182–221.
- [Rei1] H.M. Reimann, *Ordinary differential equations and quasiconformal mappings*, Invent. Math. **33** (1976), 247–270.
- [Rei2] H.M. Reimann, *Teichmüller Spaces*, The Madrid Lectures, Math. Institut der Universität Bern (1990).
- [Ren1] H. Renelt, *Modifizierung und Erweiterung einer Schifferschen Variationsmethode für quasikonforme Abbildungen*, Math. Nachr. **55** (1973), 353–379.
- [Ren2] H. Renelt, *Extremalprobleme bei quasikonformen Abbildungen unter höheren Normierungen*, Math. Nachr. **66** (1975), 125–143.
- [Ren3] H. Renelt, *Konstruktion gewisser quadratischer Differentiale mit Hilfe von Dirichletintegralen*, Math. Nachr. **73** (1976), 353–379.
- [Ren4] H. Renelt, *Über Extremalprobleme für schlichte Lösungen elliptischer Differentialgleichungssysteme*, Comment. Math. Helv. **54** (1979), 17–41.
- [Ren5] H. Renelt, *Elliptic Systems and Quasiconformal Mappings*, Wiley, New York (1988).
- [Res] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monogr., Vol. 73, Amer. Math. Soc., Providence, RI (1989).
- [Ri] B. Riemann, *Collected Works*, Dover, New York (1953).
- [Ro1] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces, Ann. Math. Stud., Vol. 66, Princeton University Press, Princeton (1971), 369–383.
- [Ro2] H.L. Royden, *Remarks on the Kobayashi metric*, Several Complex Variables, Maryland, 1970, Lecture Notes in Math., Vol. 185, Springer-Verlag, Berlin (1971), 125–137.
- [Rya1] V.I. Ryazanov, *Variational method for the general classes of quasiconformal mappings*, Dokl. Acad. Nauk Ukr. **8** (1982), 25–28 (in Russian).
- [Rya2] V.I. Ryazanov, *On the mappings that are quasiconformal in the mean*, Siberian Math. J. **37** (1996), 335–334.
- [Sa] E.V. Sallinen, *Extremal problems on the classes of quasiconformal homeomorphisms*, Matem. Sbornik **105** (147) (1978), 109–120 (in Russian).
- [Schi] M. Schiffer, *A variational method for univalent quasiconformal mappings*, Duke Math. J. **33** (1966), 395–411.
- [ScS] M. Schiffer and D.C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton (1954).
- [Scho] G. Schober, *Univalent Functions – Selected Topics*, Lecture Notes in Math., Vol. 478, Springer, Berlin (1975).
- [Seh] G.C. Sehtares, *The extremal property of certain Teichmüller mappings*, Comment. Math. Helv. **43** (1968), 98–119.
- [Se1] V.I. Semenov, *Necessary conditions in the extremal problems for the quasiconformal mappings in space*, Siberian Math. J. **21** (1980), 696–701.
- [Se2] V.I. Semenov, *On sufficient conditions for extremal quasiconformal mappings in space*, Sibirsk. Mat. Zh. **22** (1981), 222–224 (in Russian).
- [Se3] V.I. Semenov, *Some dynamical systems and quasiconformal mappings*, Siberian Math. J. **28** (1987), 674–682.
- [Se4] V.I. Semenov, *An extension principle for quasiconformal deformations of the plane*, Russian Acad. Sci. Sb. Math. **77** (1994), 127–137.
- [SSh] V.I. Semenov and S.I. Sheenko, *On certain extremal problems in the theory of quasiconformal mappings*, Siberian Math. J. **31** (1990), 171–173.
- [SSo] M. Seppälä and T. Sorvali, *Geometry of Riemann Surfaces and Teichmüller Spaces*, North-Holland, Amsterdam (1992).

- [SF] Shah Dao-Sing and Fan Le-Le, *On the parametric representation of quasiconformal mappings*, Sci. Sinica **11** (1962), 149–162.
- [Sh1] Shen Yuliang, *Extremal quasiconformal mappings compatible with Fuchsian groups*, Acta Math. Sinica (N.S.) **12** (1996), 285–291.
- [Sh2] Shen Yuliang, *On the geometry of infinite dimensional Teichmüller spaces*, Acta Math. Sinica (N.S.) **13** (1997), 413–420.
- [Sh3] Shen Yuliang, *On the extremality of quasiconformal mappings and quasiconformal deformations*, Proc. Amer. Math. Soc. **128** (1999), 135–139.
- [Sh4] Shen Yuliang, *Pull-back operators by quasisymmetric functions and invariant metrics on Teichmüller spaces*, Complex Var. **42** (2000), 289–307.
- [She1] V.G. Sheretov, *On extremal quasiconformal mappings with given boundary values*, Siberian Math. J. **19** (1978), 942–952.
- [She2] V.G. Sheretov, *On the theory of the extremal quasiconformal mappings*, Mat. Sb. **107** (1978), 146–158.
- [She3] V.G. Sheretov, *Criteria for extremality in a problem for quasiconformal mappings*, Mat. Zametki **39** (1986), 14–23 (in Russian).
- [She4] V.G. Sheretov, *Quasiconformal extremals of the smooth functionals and of the energy integral on Riemann surfaces*, Siberian Math. J. **29** (1988), 942–952.
- [Shi1] H. Shiga, *On analytic and geometric properties of Teichmüller spaces*, J. Math. Kyoto Univ. **24** (1985), 441–452.
- [Shi2] H. Shiga, *Characterization of quasidisks and Teichmüller spaces*, Tôhoku Math. J. **37** (2) (1985), 541–552.
- [ShT] H. Shiga and H. Tanigawa, *Grunsky's inequality and its application to Teichmüller spaces*, Kodai Math. J. **16** (1993), 361–378.
- [SI] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.
- [Sp] G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, MA (1957).
- [St1] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises*, Comment. Math. Helv. **36** (1962), 306–323.
- [St2] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Anal. Math. **30** (1976), 464–480.
- [St3] K. Strebel, *On lifts of extremal quasiconformal mappings*, J. Anal. Math. **31** (1977), 191–203.
- [St4] K. Strebel, *Quadratic Differentials*, Springer-Verlag, Berlin (1984).
- [St5] K. Strebel, *Extremal quasiconformal mappings*, Results Math. **10** (1986), 168–210.
- [St6] K. Strebel, *On the extremality and unique extremality of certain Teichmüller mappings*, Complex Analysis, J. Hersch and A. Huber, eds, Birkhäuser, Basel (1988), 225–238.
- [St7] K. Strebel, *On certain extremal quasiconformal mappings*, Mitt. Math. Sem. Giessen **228** (1996), 39–50.
- [St8] K. Strebel, *On the dilatation of extremal quasiconformal mappings of polygons*, Comment. Math. Helv. **74** (1999), 143–149.
- [Su1] T. Sugawa, *The Bers projection and the λ -lemma*, J. Math. Kyoto Univ. **32** (1992), 701–713.
- [Su2] T. Sugawa, *Holomorphic motions and quasiconformal extensions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 239–252.
- [Sul1] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I: Solution of Fatou–Julia problem on wandering domains*, Ann. of Math. **122** (2) (1985), 401–418.
- [Sul2] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.
- [ST] D. Sullivan and W.P. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.
- [Ta] H. Tanigawa, *Holomorphic families of geodesic disks in infinite dimensional Teichmüller spaces*, Nagoya Math. J. **127** (1992), 117–128.
- [Tan] M. Taniguchi, *Boundary variation and quasiconformal maps of Riemann surfaces*, J. Math. Kyoto Univ. **32** (4) (1992), 957–966.
- [Te1] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Math. Naturw. Kl. **22** (1940), 1–197.
- [Te2] O. Teichmüller, *Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen Riemannschen Flächen*, Abh. Preuss. Akad. Wiss. **4** (1943), 3–42.

- [Te3] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern*, Deutsche Math. **7** (1944), 309–336.
- [Te4] O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Math. **7** (1944), 336–343.
- [Te5] O. Teichmüller, *Veränderliche Riemannsche Flächen*, Deutsche Math. **9** (1944), 344–359.
- [Th1] W.P. Thurston, *Zipers and univalent functions*, The Bieberbach Conjecture: Proceedings of the Symposium on the Occasion of Its Proof, A. Baernstein II et al., eds, Amer. Math. Soc., Providence, RI (1986), 185–197.
- [Th2] W.P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1, Princeton Mathematical Series, Vol. 35, Princeton (1997).
- [To] M. Toki, *On non-starlikeness of Teichmüller spaces*, Proc. Japan Acad. Ser. A **69** (1993), 58–60.
- [Ts] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959).
- [Tu2] P. Tukia, *Quasiconformal extension of quasimetric mappings compatible with a Fuchsian group*, Acta Math. **154** (1985), 153–193.
- [Vai] J. Väisälä, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer, Berlin (1971).
- [Va1] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, J. Math. Sci. **106** (2001), 3487–3517; Transl. from Itogi Nauki i Tekhniki, Seria Sovremennaya Matematika i Ee Prilozheniya: Tematicheskie Obzory, Vol. 71, Complex Analysis and Representation Theory-2 (2000).
- [Va2] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Lecture Notes in Math., Vol. 1788, Springer-Verlag, Berlin (2002).
- [Ve] I.N. Vekua, *Generalized Analytic Functions*, Fizmatgiz, Moscow (1959) (in Russian); English transl.: *Generalized Analytic Functions*, Internat. Ser. Pure Appl. Math., Vol. 25, Pergamon Press, New York (1962).
- [Vel1] J.A. Velling, *Degeneration of quasicircles: Inner and outer radii of Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 191–202.
- [Vel2] J.A. Velling, *Mapping the disk to convex subregions*, Analysis and Topology, C. Andreian Cazacu, O. Lehto and Th.M. Rassias, eds, World Scientific, Singapore (1998), 719–723.
- [VS1] E.B. Vinberg and O.V. Schwarzman, *Riemann surfaces*, J. Soviet Math. **14** (1980), 985–1020.
- [Vo] L.I. Volkovyskiy, *On conformal moduli and quasiconformal mappings*, Some Problems of Mathematics and Mechanics, Novosibirsk (1961), 65–68 (in Russian).
- [Vol] I.A. Volynets, *Some extremal problems for quasiconformal mappings*, Metric Questions in the Theory of Functions and Mappings, Vol. 5, Naukova Dumka, Kiev (1974), 26–39 (in Russian).
- [Vu1] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin (1988).
- [Vu2] M. Vuorinen, *Conformally invariant extremal problems and quasiconformal maps*, Quart. J. Math. Oxford **43** (2) (1992), 501–514.
- [Wo1] M. Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), 449–479.
- [Wo2] M. Wolf, *High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space*, Topology **30** (1991), 517–540.
- [WW] M. Wolf and S. Wolpert, *Real analytic structures on the moduli space of curves*, Amer. J. Math. **114** (1992), 1079–1102.
- [Wol1] S. Wolpert, *The Fenchel–Nielsen deformation*, Ann. of Math. **115** (1982), 501–528.
- [Wol2] S. Wolpert, *On the Weil–Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), 969–997.
- [Ya1] E. Yakubov, *On extremal quasiconformal mappings of closed Riemann surfaces*, Soviet Math. Dokl. **15** (1974), 1411–1414.
- [Ya2] E. Yakubov, *On extremal quasiconformal mappings of bicolour maps*, Izv. USSR Acad. Sci. **1** (1976), 25–28 (in Russian).
- [Yan] S. Yang, *Quasiconformal reflection, uniform and quasiconformal extension domains*, Complex Var. **17** (1992), 277–286.
- [Zh1] I.V. Zhuravlev, *Univalent functions and Teichmüller spaces*, Soviet Math. Dokl. **21** (1980), 252–255.
- [Zh2] I.V. Zhuravlev, *A topological property of Teichmüller space*, Math. Notes **38** (1985), 803–804.
- [Zh3] I.V. Zhuravlev, *On a model of the universal Teichmüller space*, Siber. Math. J. **27** (1986), 691–697.

- [ZVC] H. Zieschang, E. Vogt and H.D. Coldway, *Surfaces and Planar Discontinuous Groups*, Lecture Notes in Math., Vol. 835, Springer-Verlag, Berlin (1980).
- [ZT1] P. Zograf and L.A. Takhtadzhyan, *On Liouville's equation, accessory parameters, and the geometry of Teichmüller space for Riemann surfaces of genus 0*, Math. USSR-Sb. **60** (1988), 143–161.
- [ZT2] P. Zograf and L.A. Takhtadzhyan, *On uniformization of Riemann surfaces and the Weil–Petersson metric on Teichmüller and Schottky spaces*, Math. USSR-Sb. **60** (1988), 297–313.

CHAPTER 3

The Conformal Module of Quadrilaterals and of Rings

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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY,
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1. Definition of the module

For a quadrilateral or a ring in the complex plane the conformal module (or modulus) is a positive number which is very important for several questions in Geometric Function Theory and its applications.

Let $\mathfrak{V}$ be a given simply-connected domain on the Riemann sphere $z = x + iy$ with 4 marked distinct accessible boundary points z_1, z_2, z_3, z_4 (“corners”) in positive orientation ($\mathfrak{V}$ to the left), with two opposite sides Γ_0 (boundary between z_1 and z_2) and Γ_1 (between z_3 and z_4). (In most cases we can restrict ourselves to the case of a closed Jordan curve as boundary, in which case all boundary points are accessible.) Such a configuration is called a *quadrilateral* $\mathfrak{V}$. Let $w = f(z)$ ($w = u + iv$) be the schlicht conformal mapping of $\mathfrak{V}$ onto the rectangle $0 < u < M, 0 < v < 1$, in such a way that z_1, z_2, z_3, z_4 correspond to the corners $0, M, M + i, i$, respectively (see Figure 1). (To get this unique mapping at first we transform $\mathfrak{V}$ onto a half plane and then with an elliptic integral of the first kind onto a rectangle.) The quantity $M = M(\mathfrak{V})$ is called (conformal) *module* of $\mathfrak{V}$. This M is uniquely determined. M is a conformal invariant: There exists a conformal mapping between the two quadrilaterals $\mathfrak{V}$ and $\mathfrak{V}'$ which transforms the corners into the corners and opposite sides into opposite sides, if and only if the modules are the same.

The pre-images of the segments $u = \text{const}, 0 < v < 1$, are called “module-lines”. If we replace the pair of opposite sides by the other pair then evidently we have to replace M by $1/M$.

In physics the module means, for example, the reciprocal electrical resistance (up to a constant multiple) of $\mathfrak{V}$ as a metallic plate or an electrical conductor with electrodes Γ_0 and Γ_1 (with a constant potential there).

Analogously we define the (conformal) module of *rings* $\mathfrak{R}$ (or *ring domains* = doubly-connected domains, where both boundary components are not single points) with a schlicht conformal mapping onto an annulus with radii 1 and $R > 1$: The quantity

$$M = M(\mathfrak{V}) = \text{Module}\{\mathfrak{V}\} = \frac{1}{2\pi} \log R \quad (1.1)$$

is called the *module* of the ring (in some cases also R itself or $\log R$ as “logarithmic module”). This module is also uniquely determined and a conformal invariant.

There are several connections between the modules of quadrilaterals and rings. If we consider for example the configuration of Figure 2: Let $\mathfrak{V}$ be a subdomain of the ring $\mathfrak{R}$

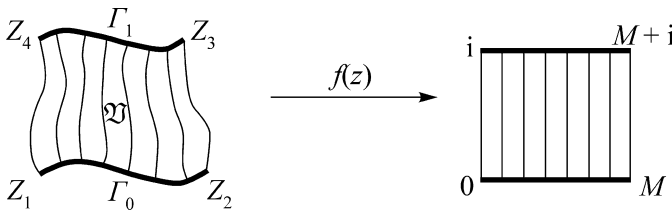


Fig. 1.

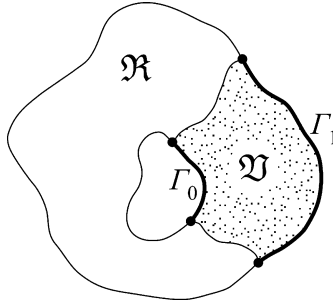


Fig. 2.

with the opposite sides Γ_0 and Γ_1 as parts of the two boundary components of $\mathfrak{R}$. Then we have

$$\text{Module}\{\mathfrak{V}\} \leq \text{Module}^{-1}\{\mathfrak{R}\}, \quad (1.2)$$

with equality if and only if we obtain $\mathfrak{V}$ from $\mathfrak{R}$ by cutting $\mathfrak{R}$ along an arc connecting the two boundary components of $\mathfrak{R}$ and which transforms into a ray in the above mentioned ring mapping of $\mathfrak{R}$.

If $\mathfrak{R}$ is symmetric with respect to the real axis, then we have equality in (1.2), for example, if we obtain $\mathfrak{V}$ by cutting $\mathfrak{R}$ along a suitable segment of the real axis.

2. A boundary value problem: Dirichlet's principle

The evaluation of the above mentioned conformal mapping $w = f(z) = u(x, y) + iv(x, y)$ of the quadrilateral $\mathfrak{V}$ and of the module $M(\mathfrak{V})$ is equivalent to a boundary value problem for the harmonic function $v(x, y)$: This function satisfies $v = 0$ on Γ_0 , $v = 1$ on Γ_1 , the normal derivative $\frac{\partial v}{\partial n} = 0$ on the remaining two parts of the boundary of $\mathfrak{V}$. (If these are not smooth then we consider the normal derivative after a conformal mapping with smooth images.) Thus we obtain the following classical characterization of $M(\mathfrak{V})$ with Dirichlet's principle (cf. [21], [36, p. 434])

$$M(\mathfrak{V}) = \inf_V \iint_{\mathfrak{V}} (V_x^2 + V_y^2) dx dy. \quad (2.1)$$

Here such $V(x, y)$ are admissible which are piecewise continuously differentiable on $\mathfrak{V}$ and satisfy $V = 0$ on Γ_0 , $V = 1$ on Γ_1 .

3. Capacitance

We consider in Figure 1 the function $h(x, y)$ harmonic in $\mathfrak{V}$ which has boundary values 1 on Γ_1 and 0 on the rest of the boundary of $\mathfrak{V}$. This h is the harmonic measure of Γ_1 in $\mathfrak{V}$.

Then

$$c = \frac{1}{4\pi} \int_{\Gamma_0} \frac{\partial h}{\partial n} |dz| \quad (3.1)$$

is called the capacitance between Γ_0 and Γ_1 ; here the derivative in the integral is in the direction of the inward normal.

Between the two values c and M there exists a simple explicit relation; cf. [23], [36, p. 442].

4. Examples, symmetries

Only in a few cases we can relate the value M immediately to the geometry of $\mathfrak{V}$.

For example, in the case of a *rectangle* $\mathfrak{V}$ the value M is obviously the ratio of the two sides. Already the case of a *parallelogram* is much more complicated; cf. [5].

From here we easily obtain $M(\mathfrak{V})$ in the case $\mathfrak{V}$ is a *disk or a half-plane with 4 marked boundary points*. After a Möbius transformation we can assume $\mathfrak{V}$ is the upper half-plane with the 4 boundary points (real values) $\lambda < 0, 0, 1, \infty$. Then the relation between λ and M is given with elliptic integrals (resp. theta series) in [48, pp. 244–245]; cf. also [71, p. 202]. Already in this simple case $\mathfrak{V}$ we can observe that the value $M(\mathfrak{V})$ is often far from what is expected. If, for example, $\mathfrak{V}$ is the upper half-plane with the segment $-1 \leq x \leq 1$ as Γ_0 and the segment $(1 <) a \leq x \leq 3$ as Γ_1 , then according to [21] $M = 0.25$ for $a = 2.99978 \dots \approx 3$. This phenomenon is in this example of course a result of the boundary distortion by conformal mappings in the neighborhood of a corner.

For a *rhombus* $\mathfrak{V}$ we have $M = 1$ because of the symmetry $M = \frac{1}{M}$ (replace Γ_0, Γ_1 by the other pair of boundary parts). In the same manner we obtain $M = 1$ for $\mathfrak{V}$ in Figure 3 [37].

But this is not so obvious because here the symmetry is a little bit hidden. This is again much more the case in the $\mathfrak{V}$ of Figure 4.

A long calculation with the Schwarz–Christoffel integral yields $M = \frac{1}{\sqrt{3}}$. But Hersch [39,40] (cf. also [36, p. 428]) gave this result “without formulae”, using only clever symmetry considerations.

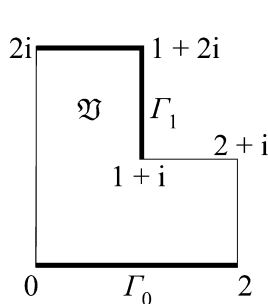


Fig. 3.

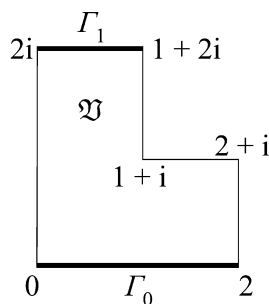


Fig. 4.

5. Numerical calculation of the modulus

With (2.1) in many cases an elegant numerical calculation of $M(\mathfrak{V})$ is possible. This was demonstrated in [21] in the case, when the boundary of $\mathfrak{V}$ is a polygon whose sides are parallel to the two axis, with the uniform length 1 and entire numbers for the coordinates of the corners. In [21] for the function $V(x, y)$ splines were used, for which a simple calculation of the Dirichlet integral is possible. Here the squares of the lattice were divided into triangles with a diagonal. In every triangle $V(x, y)$ is, e.g., linear in x and y . In this way we obtain a linear system for the unknown constants. Because we have the same considerations for the “conjugate quadrilateral” with modulus $\frac{1}{M}$, we obtain an estimate for M from both sides. For the error there is an additional estimate.

Simple numerical calculations of M for quadrilaterals with polygons as boundary can be found in [99] in several examples with the help of the Schwarz–Christoffel formula; cf. software in [43].

For the numerical computation of M in some extreme geometric situations for $\mathfrak{V}$ (“strip-like”) cf. below.

Several other methods for computing the conformal mappings and further references can be found in [21, 22, 25, 28].

6. Grötzsch’s strip method

With the help of the Cauchy–Schwarz inequality we obtain for the area I of a quadrilateral $\mathfrak{V}$ with the infimum l for the length of the module-lines the following fundamental inequality of Grötzsch’s strip method:

$$I \geq l^2 M. \quad (6.1)$$

(Strip = “Flächenstreifen” was the terminology of Grötzsch instead of quadrilateral with an orientation.) We have equality in (6.1) if and only if $\mathfrak{V}$ is a rectangle.

This simple inequality (6.1) was the root for the great success in the papers of Grötzsch in 1928–1934 to solve extremal problems in classes of schlicht conformal mappings, especially for multiply-connected domains. It is possible to obtain (6.1) also with the method of extremal length (compare Section 11 and [3, 45, 46, 77]) of Beurling and Ahlfors. Therefore it is in principle possible to get these results of Grötzsch with the method of extremal length. But Ahlfors remarked in [2]:

Es soll auch nicht vergessen werden, daß die geschickt angewandte, aber unübersichtlich formulierte Parallelstreifenmethode von Grötzsch sich inhaltlich mit großen Stücken der Beurling-schen Theorie deckt.

7. Grötzsch’s principle

Inequality (6.1) yields immediately the inequality

$$\sum_k M(\mathfrak{V}_k) \leq M(\mathfrak{V}) \quad (7.1)$$

if there is in $\mathfrak{V}$ a finite number of quadrilaterals $\mathfrak{V}_k$ whose opposite sides are situated, respectively, on Γ_0 and Γ_1 . We have equality in (7.1) if we obtain the $\mathfrak{V}_k$ by cutting $\mathfrak{V}$ along module-lines. Grötzsch remarked in [32, footnote on p. 370] that (7.1) (in the analogous case of rings) already was known to Koebe. Inequality (7.1) was called [31, Chapter IV] “Grötzsch’s principle”.

Because M corresponds to the reciprocal electrical resistance (= conductance) the case of equality in (7.1) corresponds to the formula for parallel resistors.

8. Simple estimates for the modules

Inequalities (6.1) and (7.1) immediately give rise to simple estimates for M . If we take, for example, in the case of the L -shaped $\mathfrak{V}$ of Figure 3 (with $M = \frac{1}{\sqrt{3}}$) only one $\mathfrak{V}_k$ in (7.1) as the rectangle with one side Γ_1 and one half of Γ_0 as the opposite side, then we get $M > \frac{1}{2}$. If we consider this $\mathfrak{V}$ as a part of an obvious square with the side 2, then we get $M < 1$. If we apply (6.1) directly with $l \geq 2$, we obtain the sharper inequality $M < \frac{3}{4}$ etc.

Estimating the modules is a very important method because in most cases the computation of the exact value of M is very difficult. In what follows we will prescribe some other methods to estimate M .

9. Some estimates of the module of rings with geometric quantities

For a ring domain G with conformal module M (defined by (1.1)), in the finite complex plane, let be

A = area inside the inner boundary component,

A' = area inside the outer boundary component,

D = diameter of the inner boundary component,

D' = diameter of the outer boundary component,

B = minimal width of G

(= width of the smallest parallel strip which contains G).

Then we have the following sharp estimates:

$$\frac{A}{A'} \leq f_1(M), \quad (9.1)$$

$$\frac{D^2}{A'} \leq f_2(M), \quad (9.2)$$

$$\frac{D}{D'} \leq f_3(M), \quad (9.3)$$

$$\frac{D}{B} \leq f_4(M) \quad (9.4)$$

with explicit expressions for the functions f_k . Inequality (9.1) is a classical result of Carleman (1918) (cf. [36, p. 503]), for (9.2) cf. [20, 57], for (9.3) and (9.4) cf. [57] (there further references).

Because the $f_k(M)$ are decreasing functions of M the from (9.1)–(9.4) resulting estimates of M are always of the form $M \leq \dots$. These estimates are naturally “good” if G is close to the extremal configuration.

For example, in the simplest case (9.1) the extremal configuration is an annulus G . This yields $f_1(M) = e^{-4\pi M}$, therefore

$$\frac{A}{A'} \leq e^{-4\pi M}, \quad M \leq \frac{1}{4\pi} \log \frac{A'}{A} \quad (9.1')$$

with equality only in the case of an annulus G .

A related type of extremal problems was started by Acker [1]. Here it was asked for quadrilaterals of minimal module, when the area and some side conditions were prescribed. The solutions of these extremal problems satisfy at the free part of the boundary the condition that the derivative of a corresponding conformal mapping has a constant absolute value. This condition is similar to the case of free boundary value problems in the theory of wakes (in German “Totwasserströmungen”) in hydrodynamics: Along the line of discontinuity the pressure and therefore the absolute value of the velocity (= absolute value of the derivative of the complex potential) has a constant value. Therefore classical methods (hodograph method) of von Helmholtz, Kirchhoff etc. are applicable; cf. [11, p. 327], [44, pp. 489, 553], [71, p. 293]. This yields the solution also in explicit form in some interesting special cases: [26].

In [19] the problem of maximal module among all ring domains with a fixed outer boundary component C and a variable inner boundary component surrounding a fixed area was considered. For small values of this area the solutions are in some sense nearly circles close to the points of maximal conformal radius of the interior of C .

Another nice estimate for the module arises from the following fact [96] (cf. also [3, p. 74]): Every ring domain with module $M > \frac{1}{2}$ contains a circle which separates the two boundary components.

Finally we mention an extremal property in [42, Theorem 3], although not with an usual geometric quantity, namely with the Fredholm eigenvalue λ_2 of a ring domain with conformal module M (in the sense of (1.1)):

$$\lambda_2 \leq \exp\{2\pi M\}, \quad (9.5)$$

with equality only in the case of a circular domain.

10. Extremal decomposition problems

If in the unit disk n points are fixed, we can ask for those rings which separate these points and the unit circle, and for which the module is maximal.

In the simplest nontrivial case $n = 2$ we can assume the two points as 0 and r ($0 < r < 1$). Then the solution of our extremal problem is given by the Grötzsch ring (cf. [72]). For $n \geq 3$, the solution is much more complicated and was given by Grötzsch [33]; in [68] additionally explicit formulas are given.

Indirectly this type of extremal problem yields estimates for the module of rings with the desired separating property.

A much more general and fruitful question in the history of Geometric Function Theory was the following. If there is given a fixed multiply-connected domain D on the Riemann sphere (or on a Riemann surface), then we can consider finite sets of ring domains $\subset D$ in given homotopy classes. If we denote by M_k the corresponding modules and define a set of weights $\alpha_k > 0$, we can consider the extremal problem of the type

$$\sum \alpha_k M_k \rightarrow \max. \quad (10.1)$$

This gives rise to important connections to the theory of quadratic differentials. The first example for (10.1) was given by Grötzsch [34, pp. 10–11, especially footnote on p. 11]. General results were obtained later by Jenkins (cf. [45,46]) and Pirl [80,81]; cf. also the literature in [68,85], [93, Chapter VI]; in [91] some newer results and generalizations beside [80,81].

In [11] concrete estimates for the module of ring domains which separate pairwise 4 fixed points are given.

11. Method of extremal length

(a) Now it is time to consider this extremally fruitful idea of Beurling and Ahlfors; cf. [3,45,77] for more details. This gave rise for a very simple new definition of conformal module, surprisingly without conformal mappings. Remarkable is also that the integrals in the following variational problem do not contain derivatives.

In the simplest case of a quadrilateral $\mathfrak{V}$ in the plane $z = x + iy$ (cf. Section 1) we have

$$M = \inf_{\rho} \iint_{\mathfrak{V}} \rho^2 dx dy \quad (11.1)$$

where the side condition for the admissible and, e.g., continuous functions $\rho = \rho(x, y) \geq 0$ is

$$\int_{\gamma} \rho |dz| \geq 1 \quad (11.2)$$

for all (local rectifiable) Jordan arcs γ which join the opposite sides Γ_0 and Γ_1 . Historically the reciprocal value of (11.1) is called “extremal length” of the family of the curves γ with (11.2). But in our context it is of course more convenient to consider (11.1) itself as the “module of a curve family”.

The unique solution $\rho = \rho_0$ of the infimum-problem (11.1) is the “extremal metric”

$$\rho_0(x, y) = |f'(z)|, \quad (11.3)$$

where $w = f(z)$ is the conformal rectangle-mapping of $\mathfrak{V}$, prescribed in Section 1. This means that this solution ρ_0 is connected with the system of the Cauchy–Riemann equations.

(b) For some purposes it is very useful to consider also in (11.1) a fixed weight-function $p = p(x, y) > 0$, defined in $\mathfrak{V}$:

$$M_p = \inf_{\rho} \iint_{\mathfrak{V}} p \rho^2 dx dy \quad (11.4)$$

with the same side condition (11.2) for the ρ . The resulting “ p -module” of $\mathfrak{V}$ appears if we replace the Cauchy–Riemann system for the quadrilateral mapping of $\mathfrak{V}$ in Section 1 by the more general elliptic system

$$u_x = p v_y, \quad u_y = -p v_x. \quad (11.5)$$

This system is very important in mathematical physics; cf. [52]. For example, the p -module represents in electrostatics the capacity of a condenser where the dielectric is not homogeneous but depends in defined manner on the point $z = x + iy$; cf. [49, p. 151].

It also turns out that the system (11.5) and the p -module are essentially connected with the theory of extremal problems for quasiconformal mappings where the dilatation bound is not a constant but depends on the place z ; cf. [8,9,50,51], [49, p. 85 especially pp. 92–93], (there further references, especially to papers of Schiffer and Schober).

The definition (11.4) of the p -module formally goes back to Ohtsuka, but it seems that the connection with the system (11.5) appeared at first in [16].

(c) Now we are going a further step forward to a nonlinear situation. If we do not fix the weight-function $p(x, y)$ but instead write down the side condition

$$\iint_{\mathfrak{V}} \Phi(p(x, y)) dx dy \leq C \quad (11.6)$$

with a (sufficiently great) constant C and a fixed convex function $\Phi = \Phi(p)$, then we get a new generalized “module”

$$M' = \sup_p \left\{ \inf_{\rho} \iint_{\mathfrak{V}} p \rho^2 dx dy \right\}. \quad (11.7)$$

Here we require for the functions ρ still the side condition (11.2). Now we obtain for the solution of (11.7) again a connection with the system (11.5), but now p is a defined function of $u_x^2 + u_y^2$. This means that we have now a nonlinear system (11.5). Surprisingly this is of the same type as in gas dynamics (subsonic motion of a compressible fluid), and the resulting M' of (11.7) represents some flux through $\mathfrak{V}$ from Γ_0 to Γ_1 ; cf. [58,59].

In particular, for the prescribed function

$$\Phi(p) = \frac{1}{p} + \frac{p}{\alpha}, \quad \alpha = \text{const}, \quad (11.8)$$

$U = cu$ fulfills with a suitable constant c the minimal surface equation (cf. [58,59])

$$(1 + U_y^2)U_{xx} - 2U_x U_y U_{xy} + (1 + U_x^2)U_{yy} = 0. \quad (11.9)$$

The significance for the theory of minimal surfaces is a *cura posterior*.

12. Module of one-parameter curve families

Because of the “inf” in the formula (11.1) this characterization of the module M is of course in the first instance of very theoretical nature. But, on the other hand, (11.1) yields immediately an estimate for M if we use in (11.1) any concrete fixed admissible ρ . For practice it is important to have a feeling for a good ρ , that means a ρ which is close to the extremal metric (11.3).

Another more geometric possibility to get estimations for M is to restrict the curves γ . Namely, let us now use a fixed family $\mathfrak{S}$ of curves γ which depend only on one parameter t : $\gamma = \gamma(t)$. All the $\gamma(t)$ again have to join Γ_0 and Γ_1 inside $\mathfrak{V}$, and through all points of $\mathfrak{V}$ there is exactly one $\gamma(t)$. Let the $\gamma(t)$ be piecewise continuously differentiable and let also the dependence on the parameter t be continuously differentiable. Further let $a(z) dt$ be the infinitesimal distance between the curves $\gamma(t)$ and $\gamma(t + dt)$ in the corresponding point $z \in \mathfrak{V}$. As usual let s be the arclength. Surprisingly we can now explicitly calculate the infimum in (11.1) by using the Cauchy–Schwarz inequality. Let us denote this as module $M(\mathfrak{S})$ of the family $\mathfrak{S}$. Then we have

$$M(\mathfrak{S}) = \int \frac{dt}{\int_{\gamma(t)} \frac{ds}{a}}; \quad (12.1)$$

cf. [50] and several references in [9] (also in [14,86]).

Because now more functions ρ are admissible in (11.1) we have

$$M(\mathfrak{S}) \leq M \quad (12.2)$$

with equality if and only if $\mathfrak{S}$ is the family of the module-lines of $\mathfrak{V}$ (cf. Section 1).

Again (12.2) yields concrete estimations for M , if we use a concrete family $\mathfrak{S}$. To get a “good” inequality we have to use “by feeling” a “good” family, that means a family close to the module-lines. For the practical calculation it is also important to use a family $\mathfrak{S}$ for which the calculation of the function $a(z)$ and the integrals in (12.1) are not too complicated. Possibly we have to estimate this $a(z)$ to receive simpler expressions.

If we have, for example, to determine an asymptotic formula for the conformal module M of a quadrilateral (resp. ring) with some extreme geometric shape, then in principle the asymptotic expansion for M can be obtained by means of (12.1), this because we have in (12.2) equality for the module lines. The “only” problem is to “seek” a family $\mathfrak{S}$ of curves close to the module lines. Compare as examples [62,66] or Section 15. It would be helpful for practical reasons to develop an algorithm for calculation (instead of “seeking” with try) of curve families $\mathfrak{V}$ with the result that the asymptotic expansion will be better and better.

It should be remarked that inequality (12.2) with the right-hand side of (12.1) also appears with the Dirichlet (Gauß–Thomson) principle after some additional calculations –

cf. [82, 2.5 and 2.7 in the analogue for the three-space]. This means that by using of one-parameter curve families the method of extremal length and Dirichlet's principle yields the same result.

Without the context of extremal length some formulas of the type on the right-hand side of (12.1) appears also in the important papers of Volkovyskii (cf. [102], there "continuous form of Grötzsch's principle"). The reason is that the integral in (12.1) can be considered as the limit case of the sum of infinitesimally small quadrilaterals which are defined by the curves of $\mathfrak{S}$. It should be mentioned that integrals of the form (12.1) appear also in many papers of Andreian Cazacu; cf. [9], [56, p. 100]; special cases, e.g., in [4].

13. Small changes of a quadrilateral

Because the calculation of the exact value of the conformal module of a quadrilateral or of a ring domain is generally very difficult, asymptotic formulas often are very useful.

Let us consider in the following the "first variation" of the conformal module of a quadrilateral in the case one side is changed in a special way. For this reason we study the configuration of Figure 5.

Here 3 sides of the quadrilateral $\mathfrak{V}$ are segments, only Γ_1 is a Jordan arc in the right half plane. Γ_1 connects the real axis with $y = 1$ in the strip $0 < y < 1$ whereas Γ_1 has at least one point on the imaginary axis. If we replace (with Γ_1 fixed) h by $h + \delta$ (δ not necessarily positive), then we have to replace the module M by a new value M^* . Here we have for $\delta \rightarrow 0$ the asymptotic formula

$$M^* = M - \delta M^2 \int_0^1 f'^2(-h + iy) dy + \delta^2 O(1), \quad (13.1)$$

where $O(1)$ is a function of δ , bounded by a constant independent of Γ_1 and h for sufficiently small values of M and $|\delta|$. The function $w = f(z)$ denotes the conformal mapping of $\mathfrak{V}$ onto the rectangle $0 < \Im w < 1$, $-\frac{1}{M} < \Re w < 0$ with $f(-h) = -\frac{1}{M}$, $f(-h + i) = -\frac{1}{M} + i$, whereas the endpoints of Γ_1 transform into 0 and i .

For the proof of (13.1) in [61] an estimation of M^* in both directions with formula (12.1) was applied. For the estimation in one direction the family of the module lines which join Γ_0 and Γ_1 with additionally segments of length δ (in the case $\delta > 0$), parallel to the real axis was used.

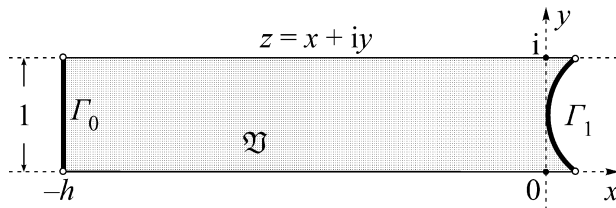


Fig. 5.

Especially (13.1) shows that M^* is a differentiable function of h with a simple and explicit expression for the first derivative. This was already shown in [90] with the classical Hadamard variational formula.

A much more refined question is: Is M^* a real analytic function of h ? More generally we can ask: *Is the conformal module of a quadrilateral or a ring a real analytic function of prescribed geometric parameters?* It seems that this question (in another configuration) was at first attacked by Teichmüller [98]. In a quite new paper [18] this general question was studied again; cf. also [69].

14. Long quadrilaterals

We obtain another asymptotic formula for the conformal module M of the quadrilateral $\mathfrak{V}$ of Figure 5 if we ask for M for great values of h (again for fixed Γ_1). Then there holds for $h \geq 1$ after [29] the inequality

$$-0.22e^{-2\pi h} \leq \frac{1}{M} - h - \frac{1}{\pi} \log R \leq 0. \quad (14.1)$$

Here R denotes the conformal radius (in the point 0) of that simply-connected domain which occurs after the mapping $e^{\pi z}$ from that part of the strip $|y| \leq 1$ which lies to the left-hand side of $\Gamma_1 \cup$ (reflection of Γ_1 at the real axis). The geometry of Γ_1 is contained in R .

We can find results of similar type in [30]; further extensive studies in this direction in [78,79] (with references there). Beside the conformal module of long quadrilaterals the corresponding mapping function was considered in [70].

15. Module of a thin worm

An attractive problem arises also if we consider quadrilaterals $\mathfrak{V}$ in form of a thin worm. For this purpose let a Jordan arc $\mathfrak{C}$ of the class C^3 , $Z = Z(S)$ with arclength S , $0 \leq S \leq L$, be given. We denote by $k(S)$ the curvature with such a sign that $iZ'' = kZ'$. By the expression

$$z(S, t) = Z(S) + itZ'(S), \quad (15.1)$$

we obtain for fixed real t parallel curves with respect to $\mathfrak{C}$ with the distance $|t|$. These parallel curves for $|t| \leq \varepsilon$ (sufficiently small) paint over a quadrilateral $\mathfrak{V}$ in form of a worm. The opposite sides Γ_0 and Γ_1 are the two segments of length 2ε arising for $S = 0$ and $S = L$. For the corresponding module M , we have according to [62]

$$M = \frac{2\varepsilon}{L} + \frac{2}{3} \frac{\varepsilon^3}{L^2} \int_0^L k^2(S) dS + O(\varepsilon^5). \quad (15.2)$$

Here it is possible to give a concrete estimate for $O(\varepsilon^5)$.

In the proof of (15.2) in the first part we use in (12.1) the family of segments in $\mathfrak{V}$ orthogonal to $\mathfrak{C}$. This yields an estimate of M in one direction. To estimate M also in the other direction the first idea is of course to use the family of parallel curves for $\mathfrak{C}$ in $\mathfrak{V}$. But this yields only

$$M \geq \frac{1}{k^*} \log \frac{1 + \frac{k^* \varepsilon}{L}}{1 - \frac{k^* \varepsilon}{L}} \quad \text{with } k^* = \int_0^L k(S) dS. \quad (15.3)$$

But this is for small ε weaker than (15.1), because then (15.3) leaves us only with

$$M \geq \frac{2\varepsilon}{L} + \frac{2}{3} \frac{k^{*2}}{L^3} \varepsilon^2 + \dots \quad \text{with } k^{*2} \leq L \int_0^L k^2(S) dS. \quad (15.4)$$

To get the sharper inequality $M \geq \dots$ contained in (15.2) we use a suitable deformation of the family of parallel curves:

$$z^*(S, t) = Z(S) + i \left[t - \frac{1}{2} k(S) (\varepsilon^2 - t^2) \right] Z'(S). \quad (15.5)$$

By the way we get from (15.3): *In the class of all quadrilaterals $\mathfrak{V}$ in form of a worm (15.1) with length L and width 2ε the corresponding rectangle (with $k(S) \equiv 0$) has the smallest modulus.*

There is of course an analogous formula (15.2) for rings if we start with a closed curve $\mathfrak{C}$.

Similar considerations with parallel curves are already studied, e.g., in [94]. We mention here only as a typical example the following result [94, p. 340].

Let l_1 be a closed convex curve and l_0 an arbitrary outer curve parallel to l_1 . If L_0 is the length of l_0 and L_1 the length of l_1 , we have for the module (in the sense of (1.1)) of the ring domain bounded by l_0 and l_1

$$\frac{1}{2\pi} \log \frac{L_0}{L_1} < M < \frac{1}{2\pi} \left(\frac{L_0}{L_1} - 1 \right) \quad (15.6)$$

unless l_1 is a circle.

It seems extremely difficult to get more members in the expansion (15.2) for M . Another question is: Is M a real analytic function of ε ? (Cf. for this question also Section 13.)

Finally, let us formulate as a desideratum the problem to determine the analogue for the formula (15.2) in the case, the curve $\mathfrak{C}$ is only piecewise of class C^3 , that means $\mathfrak{C}$ contains corners. Of special interest is the case of a polygonal line $\mathfrak{C}$. This problem needs the consideration of new curve families $\mathfrak{S}$ in (15.5).

Finally, we mention that a formula analogous to (15.2) was given in [66] for the case the complete complement of the two “plates” Γ_0 and Γ_1 is considered as a condenser (ring domain).

16. Module and hyperbolic/elliptic transfinite diameter

Of special interest there are also those rings which arise by deleting a continuum E inside the unit disk. Then we get a surprisingly new definition of the module of the ring with hyperbolic geometry. For this purpose we define at first the m th hyperbolic diameter of E by

$$d_m^{m(m-1)/2} = \max_{z_i, z_k \in E} \prod_{1 \leq i < k \leq m} [z_i, z_k] \quad (16.1)$$

with the hyperbolic pseudodistance

$$[a, b] = \left| \frac{a - b}{1 - \bar{a}b} \right| \quad (16.2)$$

for any two points a, b in the unit disk. Then we can show that the sequence d_m is nonincreasing. Therefore the limit

$$d(E) = \lim_{m \rightarrow \infty} d_m(E) \quad (16.3)$$

exists. Following Tsuji [100] this limit is called the *hyperbolic transfinite diameter* of E ; cf. also [53,56] (there in (8.1) is misprint).

Now we consider the ring domain on the Riemann sphere between E and the reflected set E^* at the unit circle. Let us denote the corresponding module (defined by (1.1)) by M (the ring between E and the unit circle then has the module $\frac{1}{2}M$). Then the surprisingly simple connection between M and d is given by [53,56,100]

$$M = \frac{1}{\pi} \log \frac{1}{d}. \quad (16.4)$$

There is also an elliptic analogue of the formula (16.4). Namely, let there be given the ring domain on the Riemann sphere between a continuum E and the set E^* which is antipodal to E on the sphere and has to satisfy $E \cap E^* = \emptyset$. Then we take instead of (16.2) the elliptic pseudodistance

$$[a, b] = \left| \frac{a - b}{1 + \bar{a}b} \right|, \quad (16.5)$$

and formally define again with (16.3) and (16.4) now the *elliptic transfinite diameter* d of E . Now we have again for the module M of the ring between E and E^* the formula (16.4) where d means the elliptic transfinite diameter of E [53,56]. (It should be mentioned that Tsuji [100] defined another elliptic transfinite diameter with another pseudometric which turns out to have been an unfortunate choice.)

By the way: There is the possibility to estimate M with the hyperbolic/elliptic area (resp. perimeter of E) similar to the considerations in Section 9; cf. [56, p. 96].

This hyperbolic/elliptic transfinite diameter is in some sense the discrete analogue of a hyperbolic/elliptic capacity defined as

$$\exp \left\{ \sup_{\mu} \int_E \int_E \log[a, b] d\mu(a) d\mu(b) \right\}, \quad (16.6)$$

where $\mu \geq 0$ is a positive mass distribution on E of total mass 1, $\mu(E) = 1$; this turns out as the same as the transfinite diameter (cf. [100]; in the elliptic case the same remark as above).

Finally it should be mentioned that there is also the possibility to define the module between more general sets E and E^* with a more general limit procedure than in (16.1)–(16.3): see [10]. This is the discrete analogue of the classical Gauß–Thomson principle of minimal energy. Compare for example [104, p. 80], where this is used with a spline approximation for a numerical procedure to calculate the conformal module.

17. Higher dimensions

There are several generalizations of the conformal module in higher dimensions. The situation here is much more complicated than in the complex plane. Especially we have to distinguish between generalizations as a tool for quasiconformal mappings and generalizations for physical reasons. Also the possibility to consider families of curves or families of surfaces in the definition of the extremal length gives rise for more possibilities. We will restrict ourselves here to the following remarks.

In three-space we can consider curve families and admissible metrics ρ with the side condition (11.2) and then define a module in form of

$$M = \inf_{\rho} \iint \rho^m dx dy \quad \text{with } m = 2 \text{ or } m = 3. \quad (17.1)$$

The case $m = 3$ is very fruitful in the theory of quasiconformal mappings, while the case $m = 2$ is “natural” to obtain a characterization of the physical capacity of a condenser. The latter case, $m = 2$, was discovered by Hersch [39] and has remained almost unknown up to now; cf. also [54] for condensers with nonhomogeneous dielectrics; in [63] results similar to (15.2).

For the case $m = 3$ we can refer to the comprehensive book [6] with an extensive bibliography, also about such things as Grötzsch’s ring, etc. in higher dimensions.

18. Limit cases: Reduced modules

Let G be a simply-connected domain in the finite z -plane with the interior point $z = 0$. We denote by r the conformal radius of G with respect to $z = 0$ and by G_{ρ} the ring domain

which is that part of G outside $|z| = \rho$ (small). Then Teichmüller [96] proved for the corresponding module m_ρ (in the sense of (1.1))

$$|2\pi m_\rho + \log \rho - \log r| < \frac{2\rho}{r - 4\rho}. \quad (18.1)$$

He defined

$$\log r = \lim_{\rho \rightarrow 0} (2\pi m_\rho + \log \rho)$$

as the “reduced module” of G relative to the point $z = 0$; cf. [93, p. 10].

In the special case G is symmetric with respect to the real axis the question was considered again and independently in more detail in [29]. Another approach to a result of this type is possible with (12.1).

In this connection we have to mention also the limit case in form of the *reduced module of triangles and bigons* [91] (cf. also [83, p. 89], with some more references and the proposal to replace the word “bigon” by “biangle”).

For example in the case of a triangle we have to consider simply-connected domains G with 3 marked boundary points z_1, z_2, z_3 and with some smoothness conditions for the 3 boundary arcs that arise. Then we delete from G the part inside $|z - z_1| < \varepsilon$. For the remaining part of G as a quadrilateral we have then to consider the limit case $\varepsilon \rightarrow 0$.

There arise of course new complications because of the geometric nature of boundary arcs of G .

19. Symmetrization and other geometric transformations

Beside the change of conformal module under quasiconformal mappings (cf. Section 22) of the quadrilateral or ring domain it is very interesting and useful to study the change under geometric transformations of other type such as symmetrization of Schwarz or Steiner type, circular symmetrization, dissymmetrization, polarization, etc. For these considerations the change of the Dirichlet integral in (2.1) is essential. This gives rise to a lot of useful results.

We mention here only some references for this topic: [35, p. 109], [45, Chapter VIII], [15, 75, 82] with many other references there.

A nice piece of work is [84], namely a very special monotonicity and convexity property of the conformal module of parallelograms is discussed, without discussing the corresponding Schwarz–Christoffel formula (cf. for parallelograms also [29, p. 463]).

20. Examples of ring domains and quadrilaterals

Some special ring domains and quadrilaterals allow the determination of the conformal module in a simple way. Some of them are very important and useful in the theory of conformal and quasiconformal mappings.

(a) *Eccentric annulus*. If both boundary components of the ring domain are circles on the Riemann sphere then we can assume without loss of generality that the configuration is symmetric with respect to the real axis. Equality between the cross-ratio of the four points of intersection with the real axis and the four points of intersection of the boundary circles of a concentric annulus with center 0 gives rise to a simple equation for the conformal module.

(b) *Grötzsch's extremal domain*. In this case the boundary components of the ring domain are the unit circle and the segment with endpoints 0 and r ($0 < r < 1$). Then the conformal module in the sense of (1.1) is

$$\frac{1}{2\pi}\mu(r) \quad \text{with } \mu(r) = \frac{\pi}{2} \frac{K(\sqrt{1-r^2})}{K(r)}, \quad (20.1)$$

where $K(\cdots)$ denotes the elliptic integral of the first kind; cf. [72, p. 53] [6, pp. 124, 158] (in [67, p. 6], after reflection at the unit circle). Interestingly in this case the module-lines are ellipses in the hyperbolic (non-Euclidean) geometry of the unit disk, with foci 0 and r ; cf. [56, p. 24], [6, p. 124].

(c) *Teichmüller's extremal domain*. In this case the boundary components of the ring domain are the segment $-r_1 \cdots 0$ and the ray $r_2 \cdots +\infty$ on the real axis ($r_1 > 0, r_2 > 0$). Here we have for the conformal module in the sense of (1.1)

$$\frac{1}{\pi}\mu\left(\sqrt{\frac{r_1}{r_1+r_2}}\right) \quad \text{with } \mu(\cdot) \text{ as in (20.1);} \quad (20.2)$$

cf. [6, p. 158], [67, p. 8], [72, p. 57].

(d) *Mori's extremal domain*. Here the boundary components are the ray $-\infty \cdots 0$ on the real axis and the circular arc $|z| = 1$, $-\alpha \leq \arg z \leq \alpha$, with a fixed positive $\alpha \leq \pi/2$. For the conformal module there exists again an explicit expression with the function $\mu(\cdot)$; cf. [6, p. 305], [67, p. 9], [72, p. 61].

There is an interesting interpretation of the module-lines of this ring domain in the hyperbolic (non-Euclidean) geometry of the unit disk, analogously to the Grötzsch extremal domain. Namely the parts of these lines inside the unit circle are semihyperbolas of hyperbolic geometry; cf. [56, p. 26].

(e) *A condenser whose plates are two parallel segments*. This example is of great interest in mathematical physics. Here the boundary of the ring domain consists in two parallel segments. In the most interesting case the segments are situated symmetrically with respect to a parallel line. We can find the complicated calculations for the module in [48, p. 340]. It should be remarked that we can already get some useful informations and approximations for the limit case in which the plates are parallel semiinfinite lines; cf. [71, p. 300].

Interestingly this limit case was already known to von Helmholtz, Clausius and Kirchhoff. In the other limit case in which the plates have a small distance ε , the behavior of the capacity as a function of ε is surprisingly complicated; cf. [88].

(f) *Unit disk as a quadrilateral with corners z_1, z_2, z_3, z_4 and the two opposite sides Γ_0 and Γ_1 as the arcs $z_1 \cdots z_2$ and $z_3 \cdots z_4$ on the unit circle (as in Section 1)*. After a

Möbius transformation $w(z)$ with $z_1 \rightarrow 1$, $z_2 \rightarrow e^{i\theta}$, $z_3 \rightarrow -1$, $z_4 \rightarrow -e^{i\theta}$ and with a suitable θ ($0 < \theta < \pi$) we have because of equality between the cross-ratios:

$$\cos^2 \frac{\theta}{2} = \frac{z_1 - z_4}{z_2 - z_4} : \frac{z_1 - z_3}{z_2 - z_3}. \quad (20.3)$$

For the conformal module M of this quadrilateral we have

$$M = \frac{\pi}{2\mu(\sin \frac{\theta}{2})} = \frac{K(\sin \frac{\theta}{2})}{K(\cos \frac{\theta}{2})} \quad (20.4)$$

with the functions $\mu(\cdot)$ as in (20.1). For the proof of (20.4) we consider the Möbius transformation

$$\zeta = i \frac{w - 1}{w + 1}$$

of the unit disk onto the lower half-plane with $1 \rightarrow 0$, $-1 \rightarrow \infty$, $e^{i\theta} \rightarrow -\operatorname{tg} \frac{\theta}{2}$, $-e^{i\theta} \rightarrow \operatorname{ctg} \frac{\theta}{2}$.

If we see the ζ -plane with cuts along the segment $-\operatorname{tg} \frac{\theta}{2} \cdots 0$ and the ray $\operatorname{ctg} \frac{\theta}{2} \cdots +\infty$ as a Teichmüller extremal domain we obtain (20.4) with (20.3).

There is again an interesting interpretation of the module-lines of this quadrilateral in the hyperbolic (non-Euclidean) geometry in the unit disk. Namely these lines are convex hyperbolas of hyperbolic geometry; cf. [56, p. 27].

Clearly, for an arbitrary simply-connected domain with 4 marked “corners” the problem of calculation the conformal module is immediately solved with (20.4) after the determination of the Riemann mapping function.

The case of a half-plane as a quadrilateral with 4 marked boundary points as corners is of course similar.

(g) Some other special examples of quadrilaterals and doubly-connected domains with the corresponding conformal module are posed in [36, pp. 438, 456]. Additionally we remark that a useful test example for numerical procedures to calculate the conformal module is the case of a square frame. This ring domain is defined by

$$\{(X, Y): |X| < 1, |Y| < 1\} \cap \{(X, Y): |X| > a, |Y| > a\},$$

where $0 < a < 1$,

(20.5)

and the module was calculated in closed form in [13]; cf. also [27] (curiously enough: in a special case this was already known to Burnside (1893); cf. [21]).

21. Harmonic measure and conformal module

(a) In the following we prescribe a simple relation between the harmonic measure of a boundary arc [3] and the conformal module of a related quadrilateral.

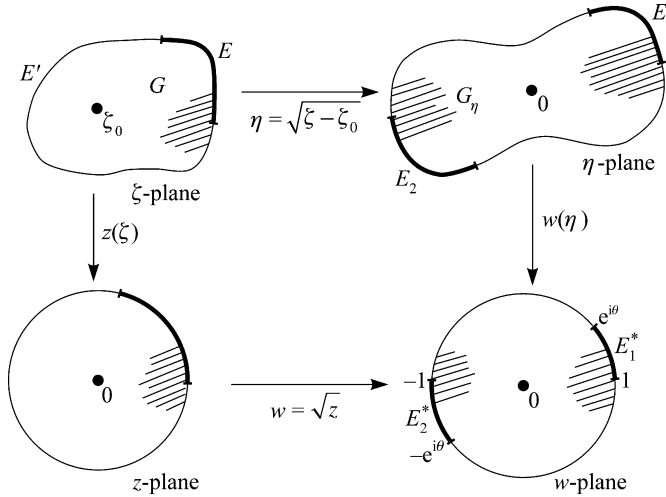


Fig. 6.

Let G be a simply-connected region in the extended complex plane ζ whose boundary ∂G consists in a Jordan curve. Suppose that ∂G is divided into two parts E and E' , each consisting in a finite number of arcs. There exists a unique bounded harmonic function $\omega(\zeta)$ in G such that $\omega(\zeta) \rightarrow 1$ when ζ tends to an interior point of E and $\omega(\zeta) \rightarrow 0$ when ζ tends to an interior point of E' . The number $\omega(\zeta)$ is called the harmonic measure of E at the point ζ with respect to the region G . It is denoted by $\omega(\zeta, G, E)$.

(b) At first we restrict ourselves to the case when E consists in only one simple arc; cf. Figure 6. Let ζ_0 be a fixed interior point of G . We consider now a Riemann mapping function $z = z(\zeta)$ of G with $z(\zeta_0) = 0$ onto $|z| < 1$.

On the other hand, we consider the mapping $\eta = \sqrt{\zeta - \zeta_0}$ of the two-sheeted region G onto a region G_η which is symmetric with respect to $\eta = 0$ and possesses the two symmetric boundary arcs E_1 and E_2 as images of E . We transform now also G_η with a corresponding Riemann mapping function $w = w(\eta)$ with $w(0) = 0$ onto $|w| < 1$. Then there is induced also a mapping $w = \sqrt{z}$ of the two sheeted disk $|z| < 1$ onto $|w| < 1$.

We now interpret the region G_η as a quadrilateral with opposite sides E_1 and E_2 (corresponding to Γ_0 and Γ_1 at the beginning of the article).

THEOREM 1. *We have with the notation $\mu(\cdot)$ for the module of the Grötzsch ring [6,72] the following relation between the harmonic measure $h = \omega(\zeta_0, G, E)$ of E at ζ_0 and the conformal module of G_η :*

$$M(G_\eta) = \frac{\pi}{2\mu(\sin \frac{\pi h}{2})} = \frac{K(\sin \frac{\pi h}{2})}{K(\cos \frac{\pi h}{2})} \quad (21.1)$$

(K = complete elliptic integral of the first kind).

We can consider the situation in the planes z and w for the *proof* because of the conformal invariance of the harmonic measure and of the module. Let E_1^* and $E_2^*(= -E_1^*)$ be the images of E_1 and E_2 in the plane w . Without loss of generality we can assume that the endpoints of E_1^* are $w = 1$ and $w = e^{i\theta}$ ($0 < \theta < \pi$). Then obviously $h = 2\theta/(2\pi)$, because this is the harmonic measure of the image of E (an arc of the unit circle) in the z -plane at $z = 0$. On the other hand, if we use the Möbius transformation

$$W = -i \frac{1-w}{1+w},$$

we obtain as the image of the unit disk the lower half-plane W . Furthermore it transforms $1 \rightarrow 0$, $e^{i\theta} \rightarrow -\operatorname{tg} \frac{\theta}{2}$, $-1 \rightarrow \infty$, $-e^{i\theta} \rightarrow \operatorname{ctg} \frac{\theta}{2}$. The plane W with slits along the segments $-\operatorname{tg} \frac{\theta}{2} \cdots 0$ and $\operatorname{ctg} \frac{\theta}{2} \cdots \infty$ is a Teichmüller domain [72]. The module of this special ring in the sense of (1.1) is easily calculated as

$$\frac{1}{2\pi} 2\mu \left(\sqrt{\frac{\operatorname{tg} \frac{\theta}{2}}{\operatorname{tg} \frac{\theta}{2} + \operatorname{ctg} \frac{\theta}{2}}} \right) = \frac{1}{\pi} \mu \left(\sin \frac{\theta}{2} \right).$$

This yields (21.1) because a twofold of this quantity is the reciprocal module of the quadrilateral consisting in the lower half-plane, with the mentioned segments as opposite sides. And this module is the same as the module of G_η .

For the relation between the harmonic measure and the module in a more general situation cf. [38].

(c) With (21.1) the methods of calculation for the module provide a way of calculating the harmonic measure. Furthermore because there are several good estimates of the function $\mu(\cdot)$ (cf. [72,6]), we can obtain estimates of the harmonic measure by means of estimates of the conformal module.

(d) Because of (21.1) there are several relations between qualitative rules for the harmonic measure and those for the conformal module. For example, the harmonic measure increases if we replace E by a larger arc, and this follows here from a related rule for the conformal module of a quadrilateral.

(e) Because of the extremal length characterization of the conformal module (cf. Section 11) we get immediately from (21.1) also an extremal length characterization of the harmonic measure $h = \omega(\zeta_0, G, E)$ of an arc E on the boundary ∂G of the simply-connected region G (cf. [77, Theorem 2.75]):

$$\inf_{\rho} \iint_G \rho^2 d\sigma = \frac{\pi}{4\mu(\sin \frac{\pi h}{2})}, \quad (21.2)$$

where the admissible ρ are characterized by

$$\int_{\gamma} \rho |d\zeta| \geq 1 \quad (21.2')$$

for all $\gamma \subset G$ which start from E and come back to E after surrounding ζ_0 .

Let us denote by $\mathfrak{S}$ this family of curves γ , the corresponding modules (= left-hand side of (11.1)) by $M(\mathfrak{S})$. Then we can write also (21.1) in the form

$$h = \frac{2}{\pi} \arcsin \left(\mu^{-1} \left(\frac{\pi}{4M(\mathfrak{S})} \right) \right) \quad (21.3)$$

(μ^{-1} = inverse function of μ).

(f) Now we go over to the situation where at the beginning of (a) the boundary part E consists in a finite number of disjoint (open) arcs E_v . Then we only have to add the measures of the individual E_v :

$$\omega(\zeta_0, G, E) = \frac{2}{\pi} \sum_v \arcsin \left(\mu^{-1} \left(\frac{\pi}{4M(\mathfrak{S}_v)} \right) \right). \quad (21.4)$$

Here the family $\mathfrak{S}_v$ corresponds to E_v (as in (21.3) the family $\mathfrak{S}$ corresponds to E).

The same idea yields a curious extremal length representation for the solution of the Dirichlet problem for our simply-connected region G . We restrict ourselves here to the case of piecewise constant boundary values (the general case of course needs a suitable limit process). Namely, our aim is to find the bounded harmonic function $U(\zeta)$ with the boundary value U_v at the arc E_v , where the finite number of disjoint (open) arcs E_v satisfies $\bigcup_v \overline{E}_v = \partial G$. Then we have

$$U(\zeta_0) = \frac{2}{\pi} \sum_v U_v \arcsin \left(\mu^{-1} \left(\frac{\pi}{4M(\mathfrak{S}_v)} \right) \right). \quad (21.5)$$

22. Conformal module and quasiconformal mappings

If we consider mappings, more general than conformal ones, obviously the conformal module will be changed in general. It was the simple but pioneering observation of Grötzsch that the change of the module is bounded in the case of mappings which are today called Q -quasiconformal. Then we have the following fundamental inequality which was in some sense the starting point of the theory of quasiconformal mappings:

$$\frac{1}{Q} M(\mathfrak{V}) \leq M(\mathfrak{V}^*) \leq Q M(Q). \quad (22.1)$$

Here $M(\mathfrak{V})$ denotes as before the conformal module of the given quadrilateral $\mathfrak{V}$ in the z -plane, $M(\mathfrak{V}^*)$ the module of the image $\mathfrak{V}^*$ under the quasiconformal mapping of $\mathfrak{V}$, if the dilatation $p(z)$ satisfies $p(z) \leq Q$ for all $z \in \mathfrak{V}$. Already Teichmüller asked in a general remark [97, p. 15], for the analogous sharp inequality for $M(\mathfrak{V}^*)$ if we have the more general dilatation bound

$$p(z) \leq p_0(z) \quad (22.2)$$

with a fixed and bounded function $p_0(z)$. Later independently also Volkovyskii [103] formulated this problem. The solution of this problem needs a suitable solution of the elliptic system (11.5) and for the extremality a refined application of the strip method of Grötzsch resp. of the method of extremal length; cf. [50], [49, p. 92], and for the proof of extremality also [8,9] and references there.

The extremal values of $M(\mathfrak{V}^*)$ with the side condition (22.2) are called p_0 -module of $\mathfrak{V}$; cf., e.g., [28,41,52]. In physics this value appears in the case of condensers with a nonhomogeneous dielectric. There is also a characterization with a modified Dirichlet integral. (The word “ p -module” is sometimes also used in another sense.)

In the case of this general restriction (22.2) there is no simple inequality for $M(\mathfrak{V}^*)$ analogously to (22.1). In [41,55] a reduction to an infinite system of linear equations using Fourier series was given. There are also inequalities for $M(\mathfrak{V}^*)$ which are not sharp; cf. references in [50].

Extremal problems for quasiconformal mappings with the side condition (22.2) gave rise to the theory of conformal mappings with a quasiconformal extension.

Another type of problem arises with side conditions for the orientation of the ellipses which transform into infinitesimal circles. This topic looks much more difficult. We mention here only the early paper [7]. This is a pearl, up to now almost unknown. Some remarks also in [51].

Further we mention that there is also the possibility to replace the side condition (22.2) by a restriction for the dilatation in the mean; cf. [49, p. 146].

23. Harmonic mappings

Contrary to the case of quasiconformal mappings the behavior of the conformal module in the case of harmonic mappings is not so clear till now. If we consider univalent harmonic mappings $f(z) = u(z) + iv(z)$ ($u(z)$ and $v(z)$ are harmonic functions) of a ring domain, then generally the conformal module is not invariant. If for example the ring domain is the annulus $(0 <) r < |z| < 1$, then the harmonic mapping

$$w = f(z) = \frac{1}{1-r^2} \left(z - \frac{r^2}{\bar{z}} \right)$$

transforms this annulus onto the punctured disk $0 < |w| < 1$. This shows also that in the class of all harmonic mappings of $(0 <) r < |z| < 1$ onto another annulus $r' < |w| < 1$ the problem $r' \rightarrow \min$ has no sense. But the max-problem is of another type as was shown by Nitsche. Now in [73] it was shown that always

$$r' < s,$$

where s is defined with the Grötzsch ring (cf. Section 20 or [72,74]) which is conformally

equivalent to $r < |z| < 1$: The boundary components of this Grötzsch ring are the unit circle and the segment $0 \cdots s$. This yields the equation

$$\log \frac{1}{r} = \mu(s) = \frac{\pi}{2} \frac{K(\sqrt{1-s^2})}{K(s)} \quad (23.1)$$

(K = complete elliptic integral of the first kind) for the calculation of s as a function of r .

The exact value of $\sup r'$ in this class of harmonic mappings is not known. Recently in [105] another unsharp estimate of r' was given:

$$r' < \frac{1}{\frac{r^2}{2}(\log r)^2 + 1}. \quad (23.2)$$

24. Inner and outer domain of a Jordan curve $\mathfrak{C}$

Let a closed Jordan curve $\mathfrak{C}$ with 4 marked points and with 2 marked opposite sides be given (Figure 7). What can we say about the modules M_1 and M_2 of the quadrilaterals corresponding to the domain, respectively, inside and outside of $\mathfrak{C}$?

The example of a rectangular line $\mathfrak{C}$ (with the corners as the marked 4 points) with lengths a and $b < a$ shows that there is not always a simple relation between $M_1 = \frac{a}{b}$ and M_2 , because in this case the calculation of M_2 needs elliptic integrals [48, p. 245].

Moreover, it is generally impossible to compute M_2 only from the knowledge of M_1 . For this reason let us consider the curve $\mathfrak{C}$ of Figure 8. Here we have

$$M_1 = \frac{\log R}{2\pi - \alpha}.$$

On the other side, we have the simple estimation

$$M_2 > \frac{\log R}{\alpha}$$

(comparison of M_2 with the modulus of the quadrilateral $1 < |z| < R$, $0 < \arg z < \alpha$, using inequality (7.1)). The desired assertion follows with $\alpha \rightarrow 0$.

But it is easy to obtain an inequality between M_1 and M_2 , for example, if there exists a Q -quasiconformal reflection at $\mathfrak{C}$. Then we have immediately

$$\frac{1}{Q} \leq \frac{M_2}{M_1} \leq Q. \quad (24.1)$$

For the nontrivial question about equality in (24.1) compare the references in [64].

It is possible to give a sharper inequality for M_2/M_1 by using the Fredholm eigenvalue $\lambda_{\mathfrak{C}} \geq 1$ of $\mathfrak{C}$ [60]. For this purpose let us define

$$M_{\mathfrak{C}} = \sup \frac{M_2}{M_1}, \quad (24.2)$$

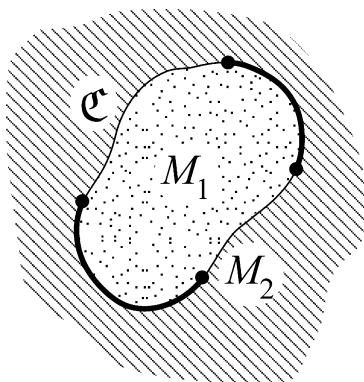


Fig. 7.

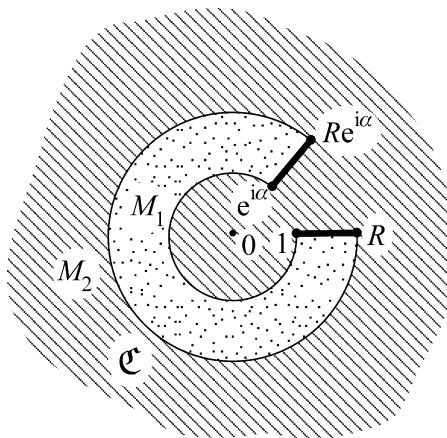


Fig. 8.

where we consider for the fixed Jordan curve $\mathcal{C}$ all such possible pairs of corresponding quadrilaterals; this means now that the 4 marked points on $\mathcal{C}$ are variable. We have $M_{\mathcal{C}} \geq 1$ because for all values M_2/M_1 we have as another possibility also the reciprocal value (by considering the other orientation of the quadrilateral). Because of continuity for M_2/M_1 all values between $M_{\mathcal{C}}$ and $1/M_{\mathcal{C}}$ are possible, also the value 1.

Additionally let us define by

$$Q_{\mathcal{C}} = \inf Q \geq 1 \quad (24.3)$$

the reflection coefficient of $\mathcal{C}$, that means the infimum of the dilatation bounds for all reflections at $\mathcal{C}$ [60]. We have $Q_{\mathcal{C}} < \infty$ if and only if $\mathcal{C}$ is a quasicircle, and $Q_{\mathcal{C}} = 1$ if and only if $\mathcal{C}$ is a circle. Also $M_{\mathcal{C}} = 1$ and $\lambda_{\mathcal{C}} = \infty$ only in the case of a circle [65].

Then we have [64] with $\Lambda_{\mathcal{C}} = \frac{\lambda_{\mathcal{C}} + 1}{\lambda_{\mathcal{C}} - 1}$ the following inequality

$$M_{\mathcal{C}} \leq \Lambda_{\mathcal{C}} \leq Q_{\mathcal{C}}. \quad (24.4)$$

This yields

$$\frac{1}{\Lambda_{\mathcal{C}}} \leq \frac{M_2}{M_1} \leq \Lambda_{\mathcal{C}}. \quad (24.5)$$

This means: With the knowledge of $\Lambda_{\mathcal{C}}$ we can estimate M_2 by M_1 and vice versa.

In the other direction, we have [65]

$$Q_{\mathcal{C}} \leq \max(\lambda^{3/2}(M_{\mathcal{C}}), 2\lambda(M_{\mathcal{C}}) - 1). \quad (24.6)$$

Here λ denotes an analytic expression in elliptic integrals. Inequality (24.6) is not sharp and goes back to Lehtinen; there exist also several newer inequalities of this type which are a little bit better [65].

From another point of view we can find something about the module of the quadrilateral over the outside of a Jordan curve in [17].

25. Miscellaneous. Problems

(a) A very *special numerical procedure* for the numerical calculation of a conformal module in [101].

(b) A *variational formula* of Schiffer type for the module of doubly-connected domains is derived in [87]; also some applications for extremal problems.

(c) An interesting *problem* was posed by Solynin at the Oberwolfach-Conference “Funktionentheorie” in February 2001: Let a continuum E be given in the complex plane, and let a suitable sequence of circles $|z| = R_n$, R_n monotonous and such that $|z| < R_n$ always contains E be given. Is it possible to determine E from the values M_n of the modules of the doubly-connected domains between E and $|z| = R_n$?

Remark from Solynin. If E is a disk $|z| = R$, then any E^* with the same values M_1 and M_2 (two radii R_1 and R_2 are enough in this case) must be identical with E . Indeed, if we transform with a conformal mapping $w(z)$ the ring domain between E^* and $|z| = R_2$ onto the annulus $R < |w| < R_2$ such that $|z| = R_2$ transforms onto $|w| = R_2$, then $|w| = R_1$ transforms onto a Jordan curve L . This L gives rise to a decomposition of the annulus $R < |w| < R_2$ into two ring domains with the same modules as the annuli $R < |z| < R_1$ and $R_1 < |z| < R_2$ have. Then after a classical result of Grötzsch L must be a circle with center 0; therefore $E^* = E$ because $w(z)$ must be a rotation about 0.

(d) Determination of the *conformal module with Monte Carlo methods*. Because there exists a random walk characterization of harmonic measure (Kakutani, Levy) such a characterization is also possible for the conformal module. A by chance chosen path in the unit disk, starting in 0, arrives an arc on the unit circle of length α with probability $\frac{\alpha}{2\pi}$ (= harmonic measure of this arc with respect to 0). Therefore a path in a simply-connected domain, starting in a fixed inner point z_0 , arrives a boundary arc with a probability which equals the corresponding harmonic measure with respect to z_0 . If we now have a quadrilateral $\mathfrak{V}$ then we fix an arbitrary inner point z_0 and consider paths in the quadrilateral, starting in z_0 and ending on the boundary. With the obtained probabilities for the 4 boundary arcs we obtain for the Riemann mapping function with the side condition $z_0 \rightarrow 0$ the length of the corresponding arcs of the unit circle, and therefore the boundary correspondence for the 4 corners, up to a rotation. This leaves us finally, with formulas (20.3) and (20.4), with the desired conformal module.

The practical construction of such paths works for example with points on a fixed lattice. Another possibility is the following. We start with the greatest circle with center z_0 inside of $\mathfrak{V}$ and choose by chance a point on the boundary of the greatest circle with center z_1 , lying in $\mathfrak{V}$, etc. Practically the process comes to an end if the radius is “sufficiently” smooth.

Unfortunately the convergence of Monte Carlo methods is very slow.

Another related topic was given by the so-called critical percolation [88].

(e) An interesting open problem is the determination of the conformal module of quadrilaterals or ring domains which lie on a surface in three-dimensional space. What is the influence of the curvature? Are there asymptotic formulas in the case of “small” curvature?

(f) It is also possible to define a ring domain by a conformal welding procedure. Again the corresponding conformal modulus is defined by a conformal mapping onto an annulus. For example it is possible to consider a simply-connected domain with a suitable welding of two boundary arcs; a special example in [75]. The case of a Möbius band is in some sense related to the case of a ring domain; cf., e.g., [95].

(g) Of special charm is the question for an discrete analogue of the conformal modulus, especially in connection with the theory of circle packing, founded by Koebe. The reader can find something about this “combinatorial quadrilateral” in [92].

(h) Finally, it should be mentioned that the theory of the conformal modules of quadrilaterals or ring domains is of course a special case of the theory of modules of multiply-connected domains or, more generally, of Riemann surfaces. But here the theory is so far not so rich and easily visible as in our “simple case” of quadrilaterals or ring domains.

References

- [1] A. Acker, *A free boundary optimization problem involving weighted areas*, Z. Angew. Math. Phys. **29** (1978), 395–408.
- [2] L.V. Ahlfors, *Extremalprobleme in der Funktionentheorie*, Ann. Acad. Sci. Fenn. Ser. A I Math. **249** (1) (1958), 9.
- [3] L.V. Ahlfors, *Conformal Invariants – Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [4] T. Akaza and T. Kuroda, *Module of annulus*, Nagoya Math. J. **18** (1961), 37–41.
- [5] G.D. Anderson, S.-L. Qiu, M.K. Vamanamurthy and M. Vuorinen, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000), 1–37.
- [6] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen, *Conformal Invariants, Inequalities and Quasi-conformal Maps*, Wiley, New York (1997).
- [7] C. Andreian Cazacu, *Sur les transformations pseudo-analytiques*, Rev. Math. Pures Appl. **2** (1957), 383–397.
- [8] C. Andreian Cazacu, *Sur un problème de L.I. Volkovyski*, Rev. Roumaine Math. Pures Appl. **10** (1965), 43–63.
- [9] C. Andreian Cazacu, *Influence of the orientation of the characteristic ellipses on the properties of the quasiconformal mappings and Some formulae on the extremal length in n-dimensional case*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Braşov (Kronstadt) 1969, Publ. House of the Acad. R.S.R., Bucurest (1971), 65–85, 87–102.
- [10] T. Bagby, *The modulus of a plane condenser*, J. Math. Mech. **17** (1967), 315–329.
- [11] D. Betsakos and M. Vuorinen, *Estimates for conformal capacity*, Constr. Approx. **16** (2000), 589–602.
- [12] A. Betz, *Konforme Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1948).
- [13] F. Bowman, *Notes on two-dimensional electric field problems*, Proc. London Math. Soc. **39** (1935), 205–215.
- [14] M.A. Chinak, *Isoperimetric properties of the module and quasiconformal mappings*, Siberian Math. J. **27** (1986), 591–598.
- [15] V.N. Dubinin, *Symmetrization in the geometric theory of functions of a complex variable*, Russian Math. Surveys **49** (1) (1994), 1–79; Transl. of: Uspekhi Mat. Nauk **49** (1) (1994), 3–76.
- [16] R.J. Duffin, *The extremal length of a network*, J. Math. Anal. Appl. **5** (1962), 200–215.
- [17] P. Duren, *Robin capacity*, Computational Methods and Function Theory (CMFT’97), World Scientific, Singapore (1999), 177–190.
- [18] C.J. Earle and S. Mitra, *Variation of moduli under holomorphic motions*, Contemp. Math. **256** (2000), 39–67.
- [19] M. Flucher, *An asymptotic formula for the minimal capacity among sets of equal area*, Calc. Var. Partial Differential Equations **1** (1993), 71–86.

- [20] D. Gaier, *Über ein Extremalproblem der konformen Abbildung*, Math. Z. **71** (1959), 83–88.
- [21] D. Gaier, *Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden*, Numer. Math. **19** (1972), 179–194.
- [22] D. Gaier, *Determination of conformal modules of ring domains and quadrilaterals*, Lectures Notes in Math., Vol. 399 (1974), 180–188.
- [23] D. Gaier, *Capacitance and the conformal module of quadrilaterals*, J. Math. Anal. Appl. **70** (1979), 236–239.
- [24] D. Gaier, *Das logarithmische Potential und die konforme Abbildung mehrfach zusammenhängender Gebiete*, E.B. Christoffel, P.L. Butzer and F. Fehér, eds, Birkhäuser, Basel (1981), 290–303.
- [25] D. Gaier, *Numerical methods in conformal mapping*, Proc. Comput. Aspects of Complex Analysis, Braunschweig, H. Werner, L. Wuytack, E. Ng and H.J. Bünger, eds, Reidel, Dordrecht–Boston–Lancaster (1983), 51–78.
- [26] D. Gaier, *On an area problem in conformal mapping*, Results Math. **10** (1986), 66–81.
- [27] D. Gaier, *On the comparison of two numerical methods for conformal mapping*, IMA J. Numer. Anal. **7** (1987), 261–282.
- [28] D. Gaier, *Conformal modules and their computation*, Computational Methods and Function Theory, World Scientific, Singapore (1995), 159–171.
- [29] D. Gaier and W. Hayman, *On the computation of modules of long quadrilaterals*, Constr. Approx. **7** (1991), 453–467.
- [30] D. Gaier and W.K. Hayman, *Moduli of long quadrilaterals and thick ring domains*, Rend. Mat. Ser. VII **10** (1990), 809–834.
- [31] G.M. Golusin, *Geometrische Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1957); Transl. of the 2nd Russian edn: *Geometric Theory of Functions of a Complex Variable*, Amer. Math. Soc., Providence, RI (1969).
- [32] H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung*, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig **80** (1928), 367–376.
- [33] H. Grötzsch, *Über ein Variationsproblem der konformen Abbildung*, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig **82** (1930), 251–263.
- [34] H. Grötzsch, *Über die Geometrie der schlichten konformen Abbildung*, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. (1933), 654–671.
- [35] W.K. Hayman, *Multivalent Functions*, 2nd edn, Cambridge Univ. Press, Cambridge (1994).
- [36] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, Wiley, New York–London–Sydney–Toronto (1986).
- [37] J. Hersch, (a) *Sur une forme générale du théorème de Phragmén–Lindelöf*; (b) *“Longueurs extrémales” dans l’espace, résistance électrique et capacité*, C. R. Acad. Sci. Paris **237** (1953), 641–643; **238** (1954), 1639–1641.
- [38] J. Hersch, *A propos d’un problème de variation de R. Nevanlinna*, Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys. **168** (1954), 1–7.
- [39] J. Hersch, *Représentation conforme et symétries: une détermination élémentaire du module d’un quadrilatère en forme de L*, Elem. Math. **37** (1982), 1–5.
- [40] J. Hersch, *On harmonic measures, conformal moduli, and elementary symmetry methods*, J. Anal. Math. **42** (1982/83), 211–228.
- [41] E. Hoy, *Über die Approximation einiger extremaler quasikonformer Abbildungen*, Ann. Univ. Mariae Curie-Skłodowska Lublin Sect. A, **40** (1986), 45–61.
- [42] E. Hoy, *Area theorems and Fredholm eigenvalues*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 137–148.
- [43] C. Hu, *A Software Package for Computing Schwarz–Christoffel Conformal Transformation for Doubly Connected Polygonal Regions*, ACM Trans. Math. Software **24** (1998).
- [44] C. Jacob, *Introduction mathématique à la mécanique des fluides*, Ed. Acad. R. P. R., Bucharest/ Gauthier-Villars, Paris (1959).
- [45] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1958).
- [46] J.A. Jenkins, *The method of the extremal metric*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier, Amsterdam (2002), 393–456.

- [47] Y. Juve, *Über gewisse Verzerrungseigenschaften konformer und quasikonformer Abbildungen*, Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys. **174** (1954).
- [48] W. von Koppenfels und F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1959).
- [49] S.L. Krushkal und R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner, Leipzig (1983); Nauka, Novosibirsk (1984) (in Russian).
- [50] R. Kühnau, *Über gewisse Extremalprobleme der quasikonformen Abbildung*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **13** (1964), 35–40.
- [51] R. Kühnau, *Einige Extremalprobleme bei differentialgeometrischen und quasikonformen Abbildungen*, Math. Z. **94** (1966), 178–192.
- [52] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien*, J. Reine Angew. Math. **231** (1968), 101–113.
- [53] R. Kühnau, *Transfiniten Durchmesser, Kapazität und Tschebyschewsche Konstante in der Euklidischen, hyperbolischen und elliptischen Geometrie*, J. Reine Angew. Math. **234** (1969), 216–220.
- [54] R. Kühnau, *Der Modul von Kurven- und Flächenscharen und räumliche Felder in inhomogenen Medien*, J. Reine Angew. Math. **243** (1970), 184–191.
- [55] R. Kühnau, *Darstellung quasikonformer Abbildungen durch Fouriersche Reihen*, Publ. Math. Debrecen **18** (1971), 119–127.
- [56] R. Kühnau, *Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag Wiss., Berlin (1974).
- [57] R. Kühnau, *Einige Extremalprobleme bei konformen Abbildungen mit geometrischen Funktionalen*, Math. Z. **181** (1982), 287–292.
- [58] R. Kühnau, *Eine Extremalcharakterisierung von Unterschallgasströmungen durch quasikonforme Abbildungen*, Banach Center Publ. **11** (1983), 199–210.
- [59] R. Kühnau, *Über Extremalprobleme bei im Mittel quasikonformen Abbildungen*, Complex Analysis – 5, Romanian–Finnish Seminar, Bucharest 1981, Lecture Notes in Math., Vol. 1013 (1983), 113–124.
- [60] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jahresber. Deutsch. Math.-Verein. **90** (1988), 90–109.
- [61] R. Kühnau, *Zum konformen Modul eines Vierecks*, Mitt. Math. Seminar Gießen, Heft **211** (1992), 61–67.
- [62] R. Kühnau, *Der konforme Modul schmaler Vierecke*, Math. Nachr. **175** (1995), 193–198.
- [63] R. Kühnau, *Die Kapazität dünner Kondensatoren*, Math. Nachr. **203** (1999), 125–130.
- [64] R. Kühnau, *Drei Funktionale eines Quasikreises*, Ann. Acad. Sci. Fenn. Math. **25** (2000), 413–415.
- [65] R. Kühnau, *Zu den Grunskyschen Koeffizientenbedingungen*, Ann. Univ. Mariae Curie-Skłodowska Lublin Sect. A **54** (2000), 53–60.
- [66] R. Kühnau, *Boundary effects for an electrostatic condenser*, J. Math. Sci. **105** (2001), 2210–2219.
- [67] H.P. Künzi, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
- [68] G.V. Kuz'mina, *Modules of Families of Curves and Quadratic Differentials*, Proc. Steklov Inst. Math., no. 139 (1982); Transl. from Russian.
- [69] M. Lanza de Cristoforis, *Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces*, Comput. Methods Funct. Theory **2** (2002), 1–27.
- [70] R. Laugesen, *Conformal mapping of long quadrilaterals and thick doubly connected domains*, Constr. Approx. **10** (1994), 523–554.
- [71] M.A. Lawrentjew und B.W. Schabat, *Methoden der komplexen Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1967); Transl. from Russian.
- [72] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edn, Springer-Verlag, New York–Heidelberg–Berlin (1973).
- [73] A. Lyzzaik, *Univalent harmonic mappings and a conjecture of J.C.C. Nitsche*, Ann. Univ. Mariae Curie-Skłodowska Lublin Sect. A. **53** (1999), 147–150.
- [74] A. Lyzzaik, *The modulus of the image annuli under univalent harmonic mappings and a conjecture of Nitsche*, J. London Math. Soc. **64** (2001), 369–384.
- [75] M. Marcus, *Radial averaging of domains, estimates for Dirichlet integrals and applications*, J. Anal. Math. **27** (1974), 47–78.

- [76] K. Nishikawa and F. Maitani, *Moduli of ring domains obtained by a conformal welding*, Kodai Math. J. **20** (1997), 161–171.
- [77] M. Ohtsuka, *Dirichlet Problem, Extremal Length and Prime Ends*, Van Nostrand, New York (1970).
- [78] N. Papamichael and S.N. Stylianopoulos, *On a domain decomposition method for the computation of conformal modules*, Appl. Math. Lett. **1** (1988), 277–280.
- [79] N. Papamichael and S.N. Stylianopoulos, *The asymptotic behaviour of conformal modules of quadrilaterals with applications to the estimation of resistance values*, Constr. Approx. **15** (1999), 109–134.
- [80] U. Pirl, *Isotherme Kurvenscharen und zugehörige Extremalprobleme der konformen Abbildung*, Wiss. Zeitschr. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **4** (1955), 1225–1251.
- [81] U. Pirl, *Über isotherme Kurvenscharen vorgegebenen topologischen Verlaufes und ein zugehöriges Extremalproblem der konformen Abbildung*, Math. Ann. **133** (1957), 91–117.
- [82] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton, Princeton Univ. Press (1951); Moskva (1962) (in Russian).
- [83] C. Pommerenke and A. Vasil'ev, *On bounded univalent functions and the angular derivative*, Ann. Univ. Mariae Curie-Skłodowska **54** (2000), 79–106.
- [84] E. Reich, *Steiner symmetrization and the conformal moduli of parallelograms*, Analysis and Topology, World Scientific, Singapore (1998), 615–620.
- [85] H. Renelt, *Konstruktion gewisser quadratischer Differentiale mit Hilfe von Dirichletintegralen*, Math. Nachr. **73** (1976), 125–142.
- [86] B. Rodin, *The method of extremal length*, Bull. Amer. Math. Soc. **80** (1974), 587–606.
- [87] M. Schiffer, *On the modulus of doubly-connected domains*, Quart. J. Math. Oxford Ser. **17** (1946), 197–213.
- [88] I.B. Simonenko and A.A. Chekulaeva, *On the capacity of a condenser consisting of infinite strips*, Izv. Vuzov. Elektromekh. **4** (1972), 362–370 (in Russian).
- [89] S. Smirnov, *Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I. Math. **333** (2001), 239–244.
- [90] A.Yu. Solynin, *Boundary distortions and change of module under extension of a doubly connected domain*, Zapiski Naučn. Sem. POMI **201** (1992), 157–164 (in Russian).
- [91] A.Yu. Solynin, *Modules and extremal metric problems*, St. Petersburg Math. J. **11** (1) (2000), 1–65.
- [92] K. Stephenson, *Circle packing and discrete analytic function theory*, Handbook of Complex Analysis, Geometric Function Theory, Vol. 1, Elsevier, Amsterdam (2002), 333–370.
- [93] K. Strebel, *Quadratic Differentials*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo (1984).
- [94] G. Szegő, *On the capacity of a condenser*, Bull. Amer. Math. Soc. **51** (1945), 325–350.
- [95] P.M. Tamrazov, *Moduli and extremal metrics in nonorientable and twisted Riemannian manifolds*, Ukrain. Math. J. **50** (1998), 1586–1598; Transl. of Ukrain. Mat. Zh. **50** (1998), 1388–1398.
- [96] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. **3** (1938), 621–678; cf. also Gesammelte Abhandlungen, Springer-Verlag, Berlin–Heidelberg–New York (1982).
- [97] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuß. Akad. Wiss. Math.-Nat. Kl. **22** (1939); cf. also Gesammelte Abhandlungen.
- [98] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern*, Deutsche Math. **7** (1944), 309–336; cf. also Gesammelte Abhandlungen.
- [99] L.N. Trefethen, *Analysis and design of polygonal resistors by conformal mapping*, Z. Angew. Math. Phys. **35** (1984), 692–704.
- [100] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959); 2nd edn: Chelsea Publ. Co., New York (1975).
- [101] E.A. Volkov and A.K. Kornoukhov, *An approximate conformal mapping of a trapezoid onto a rectangle by the block method, and its inversion*, Zh. Vychisl. Mat. Mat. Fiz. **39** (1999), 1142–1150 (in Russian); Transl.: Comput. Math. Math. Phys. **39** (1999), 1100–1108.
- [102] L.I. Volkovskii, *Investigation of the type problem for simply connected Riemann surfaces*, Trudy Mat. Inst. Steklov. **34**, Izdat. Akad. Nauk SSSR, Moskva–Leningrad (1950) (in Russian).

- [103] L.I. Volkovskii, *On the conformal moduli and quasiconformal mappings*, Some Problems of Mathematics and Mechanics, Izdat. Sibirsk. Otd. AN SSSR, Novosibirsk (1961), 65–68 (in Russian).
- [104] J. Weisel, *Lösung singulärer Variationsprobleme durch Verfahren von Ritz und Galerkin mit finiten Elementen – Anwendungen in der konformen Abbildung*, Mitt. Math. Seminar Giessen, Heft 138 (1979).
- [105] A. Weitsman, *Univalent harmonic mappings of annuli and a conjecture of J.C.C. Nitsche*, Israel J. Math. **124** (2001), 327–331.

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CHAPTER 4

Canonical Conformal and Quasiconformal Mappings. Identities. Kernel Functions

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1. Introduction

Let a fixed domain G be given in the complex z -plane or on the Riemann sphere. With the exception of Section 7 “domain” always means a finitely-connected domain. A fundamental question is then: Are there in the class of all schlicht conformal mappings of G special mappings which are in some sense distinguished? That means the question for mappings with some special analytic or mainly geometric properties. In the latter case we seek mappings which produce image domains with some prescribed geometric shape, so called “canonical” or “representative” domains. The question for canonical conformal mappings consists in the question of the existence and in the question of uniqueness by setting of side conditions, so called normalizations. Such normalizations are necessary for uniqueness because we have otherwise some freedom with the use of Möbius transformations. Because Möbius transformations depend on 3 complex numbers, also these normalizations consist in the simplest case in 3 side conditions.

If we set more than 3 normalizations (“höhere Normierungen” in the terminology of Grötzsch [26]), the things become more complicated, also in the case we consider G and its images as a part of a fixed Riemann surface.

We will see that there is a very rich theory of canonical conformal mappings with many aspects, and connections to many other fields and questions: Identities between these mappings and connections with fundamental solutions and kernel functions, extremal problems to characterize these canonical mappings, etc.

In the (mainly considered) conformal case there exists a rich literature about canonical mappings. We mention mainly the books [9,17,23,29,32,35,82,97,99,110] and the article [18]. Therefore we will give here only some typical examples and will then restrict us to sketch some additional aspects and especially to the generalization of the theory to conformal mappings with quasiconformal extension.

What concerns the main methods for proving the existence of canonical conformal mappings we mention: methods for solving boundary value problems, extremal problems combined with normal family arguments (the most elegant method, e.g., in the case of the parallel slit mapping), Koebe’s method of continuity, functional-analytic fix-point methods, orthonormal expansions, integral equations.

To begin with let us start with the simplest case of a simply-connected domain G on the Riemann sphere. If G has no or only one boundary point then the situation is trivial because the Möbius transformations are the only schlicht conformal mappings of G . If G has more than one boundary point we have the fundamental *Riemann mapping theorem*: *For every such G there exists a schlicht conformal mapping onto the unit disk. The mapping is unique up to a following Möbius transformation of the unit disk onto itself. These Möbius transformations contain 3 real parameters.*

We can find the Riemann mapping theorem in almost all textbooks on Complex Analysis (cf. also references in [70]), with all corresponding aspects as the connection with the Green’s function of G , construction of the Riemann mapping for polygonal domains G (Schwarz–Christoffel formula), construction with orthogonal expansions etc. We can find the boundary behavior of the Riemann mapping, e.g., in [90,91]. For numerical procedures cf. [17] and [112] with many references.

From the Riemann mapping theorem it follows that all simply-connected domains with more than one boundary point are conformally equivalent. If we pass now over to doubly-connected domains G we have a new situation: Two such domains are conformally equivalent if and only if the so-called conformal module is the same. This means in particular that every doubly-connected domain G is conformally equivalent to an annulus with the same conformal module. (This annulus can degenerate.) One can find a detailed discussion of the corresponding questions in [69]. We will restrict ourselves here therefore to the general case of conformal mappings of multiply-connected domains, including the case of connectivity greater than 2.

2. Some simple canonical conformal mappings

(a) We start with the following representative and classical example of the so-called parallel slit mapping (cf. Figure 1).

THEOREM 2.1. *For every G of finite connectivity on the Riemann sphere z , which contains $z = \infty$ as an inner point, there exists a unique schlicht conformal mapping $w = g_\Theta(z)$ with the hydrodynamical normalization*

$$z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \quad (2.1)$$

at $z = \infty$, such that the image domain on the Riemann sphere w is bounded by segments of the prescribed inclination Θ .

Such a segment degenerates to a point only in the case, the corresponding boundary component of G as the preimage is a single point, therefore removable for the mapping function. The middle point and the length of the segments are uniquely determined by G . This is typical for canonical conformal mappings: One can prescribe only the “shape” (here the shape as a segment) of the images of the boundary components of G .

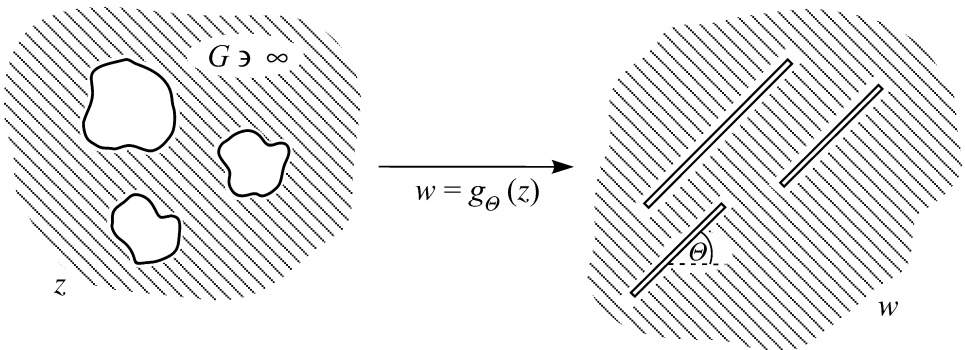


Fig. 1.

The normalization (2.1) means in particular that $g_\Theta(\infty) = \infty$. This and the conditions

$$\lim_{z \rightarrow \infty} g'_\Theta(z) = 1, \quad \lim_{z \rightarrow \infty} (g_\Theta(z) - z) = 0,$$

equivalent to (2.1), correspond to the 3 at the beginning of the introduction mentioned complex parameters in the mapping function.

The reason for the terminology “hydrodynamical normalization” in (2.1) is the following. We can restrict ourselves to the case $\Theta = 0$. Then the mapping function $g_0(z)$ is the complex potential to prescribe the planar potential flow (without circulation) of an ideal fluid around the boundary components of G , when the vector of velocity at $z = \infty$ has the length 1 and is in the direction of the real axis. “Complex potential” means that $\Re g_0(z)$ is the usual real potential, $\Im g_0(z)$ is the stream function, the streamlines are defined by $\Im g_0(z) = \text{const.}$ And the complex number $\overline{g'_0(z)}$ represents the vector of velocity. This interpretation of $g_0(z)$ is classical at least since the days of Klein. Concerning the hydrodynamical interpretation one has also to consult the great papers [44,46] of Koebe, Courant [9] and others, today not always known [5].

For the proof of Theorem 2.1 it is of course enough to consider the case $\Theta = 0$. There exist today many proofs of Theorem 2.1, such as methods of potential theory (because of the simple boundary condition for $g_0(z)$), integral equations (Section 10), iterative procedures (Section 12), another idea in [9, p. 45]. But the simplest proof uses the extremal property with respect to $\Re a_1$ in the development (2.1); cf., e.g., [17, p. 235], [23, p. 179], [29, p. 82], [82, p. 346] and many other books [70].

It is useful also to introduce for a domain G (not necessarily $\ni \infty$) the function $g_\Theta(z, a)$ which maps again G onto a domain bounded by segments of inclination Θ , but now with the side condition

$$g_\Theta(z, a) = \frac{1}{z-a} + A_1(z-a) + A_2(z-a)^2 + \dots \quad (2.2)$$

for any given and fixed $a \in G$. Furthermore it is useful to introduce the two new functions

$$\begin{aligned} M(z, a) &= \frac{1}{2} (g_0(z, a) - g_{\pi/2}(z, a)), \\ N(z, a) &= \frac{1}{2} (g_0(z, a) + g_{\pi/2}(z, a)). \end{aligned} \quad (2.3)$$

The function N represents a schlicht conformal mapping with an interesting extremal property with respect to an area. But this is not our theme here – cf., for example, [80, Satz IX 2], [82, p. 362], [99].

(b) Another important type of canonical mapping consists in the *spiral slit mapping* of G , namely $j_\Theta(z, a, b)$ as the schlicht conformal mapping of G with the normalization

$$\begin{aligned} j_\Theta(a, a, b) &= 0, \\ j_\Theta(z, a, b) &= \frac{1}{z-b} + \alpha_0 + \alpha_1(z-b) + \alpha_2(z-b)^2 + \dots \end{aligned} \quad (2.4)$$

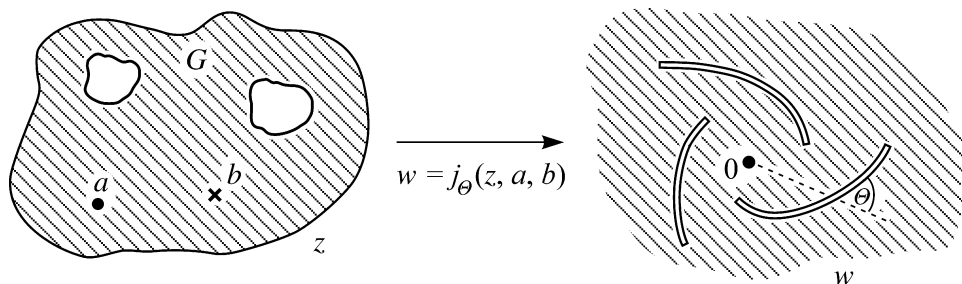


Fig. 2.

in the different points $a \in G$ and $b \in G$ (in particular $j_\Theta(b, a, b) = \infty$), where the boundary components of G transform onto slits on the family of the spirals with inclination Θ (cf. Figure 2).

We obtain, in particular for $\Theta = 0$, the normalized *radial slit mapping*, for $\Theta = \pi/2$, the normalized *circular slit mapping*. All these mappings have again a simple extremal property. The situation is in many aspects similar to the case of the parallel slit mappings; cf. again [23,29,82]. Also it is useful to introduce two further functions:

$$\begin{aligned} P(z, a, b) &= \frac{1}{2}(\log j_{\pi/2}(z, a, b) - \log j_0(z, a, b)), \\ Q(z, a, b) &= \frac{1}{2}(\log j_{\pi/2}(z, a, b) + \log j_0(z, a, b)). \end{aligned} \quad (2.5)$$

(c) There are several other important slit mappings which are connected to an extremal problem. We will introduce only one more, namely the so-called *parabola slit mapping* of G . Let a and b again be different interior points of G , Θ is a prescribed real value. Then Grötzsch proved in connection with an extremal property (cf. [35, Theorem 5.9]) that there exists exactly one schlicht conformal mapping $w = p_\Theta(z, a, b)$ of G with the normalization

$$p_\Theta(z, a, b) = \frac{1}{z - b} + \alpha_1(z - b) + \alpha_2(z - b)^2 + \dots$$

(in the case of a finite b), where the boundary components of G transform onto slits on the family of the parabolas with the focus axis which starts in the image of a and has the inclination Θ (cf. Figure 3).

(d) There are many other mapping theorems of a geometric nature beside the classical results of Koebe (e.g., [43]). Very general slit theorems were given by Grötzsch [27]. He fixed n families of curves which cover the plane in some sense completely. Then for every domain $G \ni \infty$ of connectivity n there exists exactly one hydrodynamically normalized schlicht conformal mapping with the property that the k th boundary component transforms onto a slit on the k th family of curves ($k = 1, 2, \dots, n$). His proof uses Koebe's method of continuity, where the corresponding uniqueness result was proved with an older

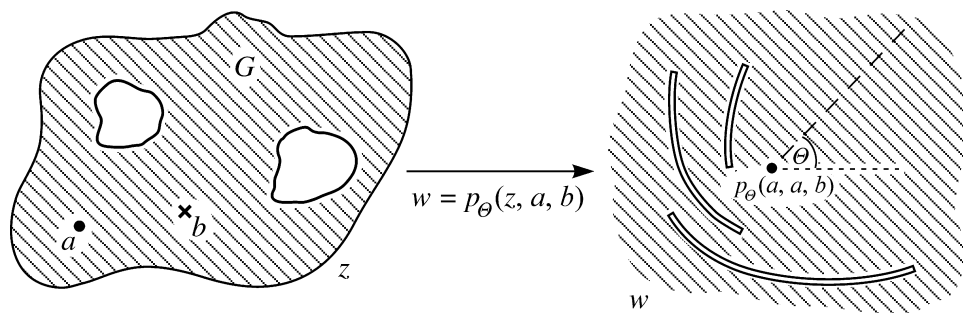


Fig. 3.

idea of Carleman¹ (argument principle). If the n prescribed families of curves are simply parallel lines, then we obtain in particular the classical Koebe *Geradenschlitztheorem*. This means that every $G \ni \infty$ of connectivity n can be mapped in such a way, that the image of the k th boundary component is a segment of inclination Θ_k , where Θ_k is prescribed for $k = 1, \dots, n$. If all Θ_k are the same, then we have again the parallel slit mapping. Only in this special case a simple extremal property is known. This failure of a suitable extremal property in the general case of the *Geradenschlitztheorem* is a great impediment for proving the existence with an extremal property as for example in the case of the parallel slit mapping, or for proving the convergence of the correspondent iterative process – cf. Section 12.

In [28] one can find also similar results for another normalization. Later very general interesting results of this type (slit mappings) are derived in [7] with other ideas (fixed point consideration); cf. also [30,105].

Grötzsch [26,28] gave also a very general mapping theorem of a geometric nature where the boundary components of the image domain are not slits but for example complete Jordan curves of prescribed geometric type. These Jordan curves are special curves of a given fixed family S_k which depends in some sense continuously on 3 real parameters ($k = 1, \dots, n$, where n is again the number of connectivity), and satisfy some simple geometric conditions, as in the case of the family of all circles. A very simple example is given by the family of all ellipses which are homothetic to a fixed ellipse.

(e) Beside the canonical conformal mappings of a geometric nature there are also several mapping theorems which yield canonical conformal mappings where the image domain is defined in a more analytic manner. We will restrict ourselves here to formulate the following general theorem.

THEOREM 2.2 [53]. *Let G be a domain in the z -plane with the distinct interior points $z = \infty$ and $z_1, z_2, \dots, z_m$. Let $R_{m+1}, \dots, R_N$ ($0 \leq m < N$, for $m = 0$ the z_k are absent) be*

¹Now I found in the inheritance of Grötzsch a hand-written copy from him (probably from the early fifties) of a paper of Shiffman from 1941 (cited in [9, footnote at p. 187], concerning uniqueness proofs with this idea of Carleman.

The margine note “Betreffend zu schreibende Notiz” shows that Grötzsch planned to write something, probably about generalizations and relations to his own paper [27]. But he never wrote such a note.

the boundary components. These are assumed as closed analytic Jordan curves. Further let there be given a matrix $\{\gamma_{v\mu}\}$ of complex numbers $\gamma_{v\mu}$ ($v, \mu = 1, \dots, n$; $m \leq n \leq N$), where not all of them vanish, and $\gamma_{vv} \neq 0$ for $v = m+1, \dots, n$. On each R_v with $m < v \leq n$ and with a nonpositive γ_{vv} let there be fixed a point z_v (for $n = m$ this condition is vacuous). Then there exists a hydrodynamically normalized schlicht conformal mapping $w = f(z)$ of G , such that on the images $f(R_{m+1}), \dots, f(R_N)$

$$-\sum_{v,\mu=1}^n \frac{\gamma_{v\mu}}{(w - w_v)(w - w_\mu)} dw^2 \geq 0, \quad (2.6)$$

with $w_v = f(z_v)$ for $v = 1, \dots, m$ and some further points $w_{m+1}, \dots, w_n$ (which we cannot prescribe). Such a point w_v with $v > m$ is an interior point of $f(G)$ in the case $\gamma_{vv} > 0$ (whereby the corresponding $f(R_v)$ surrounds this w_v , not necessarily as a Jordan curve – some part of $f(R_v)$ can be a slit). And for complex γ_{vv} ($v \leq n$) which are not positive (for short $\gamma_{vv} \not> 0$) the corresponding $f(R_v)$ is a slit with endpoint w_v . In the case $\gamma_{vv} < 0$ this slit is analytic also in w_v , but in the other cases this slit surrounds the w_v spirallikely. The remaining $f(R_v)$ with $n < v \leq N$ are also slits.

“Slit” means here always a system of a finite number of arcs in the sense of a graph without circles; cf. Figure 4 as an example.

Concerning the zeroes of the quadratic differential (2.6) with, generally speaking, forking the slits, cf., for example, [35, Chapter III] or [37].

The mapping $f(z)$ of Theorem 2.2 is not always unique (cf. after Theorem 2.3).

Special cases.

(i) For $m = n$ we have a classical theorem of Goluzin [22]. On the other side, we can obtain Theorem 2.2 from this result of Goluzin with a limit procedure [53]; cf. also [54].

(ii) In the case when the matrix elements $\gamma_{v\mu}$ are of the form

$$\gamma_{v\mu} = \gamma_v \gamma_\mu$$

we obtain the following theorem.

THEOREM 2.3. *Let G be a domain in the z -plane with the distinct interior points $z = \infty$ and $z_1, z_2, \dots, z_m$. Let the boundary components be $R_{m+1}, \dots, R_N$ ($0 \leq m < N$); these are assumed as closed analytic Jordan curves. Further let there be given the real constants $\gamma_1, \dots, \gamma_N$ which do not vanish altogether. Then there exists a hydrodynamically normalized schlicht conformal mapping $w = f(z)$ of G , such that on the image $f(R_\mu)$ with some constants C_μ it holds*

$$\sum_{v=1}^N \gamma_v \log |w - w_v| = C_\mu, \quad \mu = m+1, \dots, N, \quad (2.7)$$

with $w_v = f(z_v)$ for $v = 1, \dots, m$ and further points $w_{m+1}, \dots, w_N$ (which we cannot prescribe). In the case $\gamma_v \neq 0$ such a point w_v with $v > m$ is an exterior point of $f(G)$,

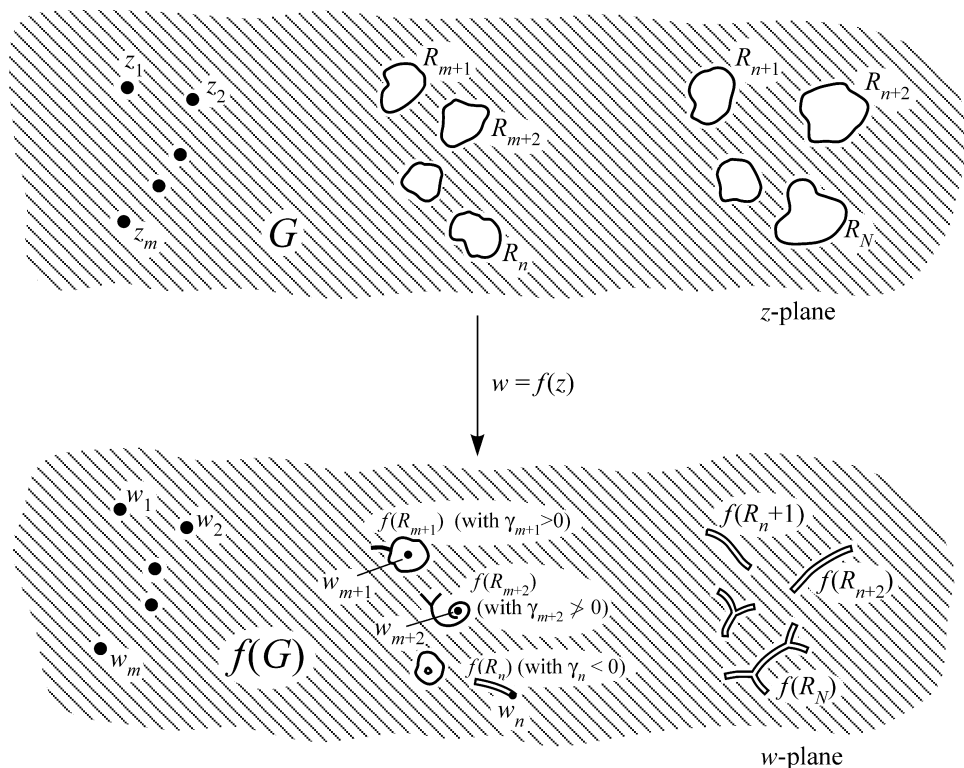


Fig. 4. For $m = 0$ the points $z_1, \dots, z_m$ fail; for $n = m$ the boundary components $R_{m+1}, \dots, R_n$ fail; for $N = n$ the boundary components $R_{n+1}, \dots, R_N$ are absent; in the case of $f(R_{m+1}), \dots, f(R_n)$ the situation for a special choice of the corresponding $\gamma_{m+1}, \dots, \gamma_n$ is illustrated.

surrounded by $f(R_v)$. In the case $\gamma_v = 0$ the image $f(R_v)$ is a slit (in the latter case the corresponding w_v is unnecessary).

The mapping $f(z)$ is not always unique (cf. Section 3).

This Theorem 2.3 follows from Theorem 2.2 if we identify the boundary components R_v , for which $\gamma_v = 0$, with $R_{n+1}, \dots, R_N$.

In the special case $\gamma_{m+1} = \dots = \gamma_N = 0$ we have again a classical theorem of Golusin; cf., for example, [23, Chapter IV, Section 3]. This theorem (resp. the corresponding extremal property) was very fruitful also because of its connection with the Grunsky coefficient conditions. Already in this special case of Golusin we have not always uniqueness for the mapping $f(z)$; cf. [52].

(iii) In the case $m = 0$ (that means $z_1, \dots, z_m$ are absent), $\gamma_1 = \dots = \gamma_{N-1} = 1$, $\gamma_N = 1 - N$ ($N \geq 2$) we obtain, for example, a classical canonical conformal mapping of Julia. This example and some other, e.g., of Grunsky, Walsh, are described in [53], [29, p. 106].

3. Nonuniqueness: “Verzweigungserscheinung” in the sense of Grötzsch

If we have an existence theorem for normalized slit-mappings, then generally we have also uniqueness, if the corresponding curve family has no singular points. This is not always true if there are singular points. This phenomenon was at first remarked in the paper [26] of Grötzsch (cf. also a similar result in [38, p. 73]). Grötzsch called this “Verzweigungserscheinung”, because we have then in some situations uniqueness, but after changing of parameters, e.g., after a deformation of the boundary components, suddenly a continuum of mappings appears.

We will explain this in the example of [26, cf. footnote on p. 905], the so called lemniscate slit mappings. For this purpose we will restrict ourselves here to the simplest case of the domain $G \equiv \{|z| > 1\}$ with two different and fixed finite inner points z_1 and z_2 . Then we have [26] the following theorem.

THEOREM 3.1. *There exists at least one schlicht conformal mapping $w(z)$ of G with $w(z_1) = +1$, $w(z_2) = -1$, $w(\infty) = \infty$ onto a domain which is bounded by a slit on the family of Cassinians (or lemniscates), defined by*

$$|w - 1| \cdot |w + 1| = \text{const.}$$

There are 3 possibilities (see Figure 5):

- (i) The slit does not contain the point $w = 0$.
- (ii) The slit is a smooth arc with endpoint $w = 0$, or the slit consists in 2 orthogonal smooth arcs with common endpoint $w = 0$.
- (iii) The slit consists in 3 or 4 smooth arcs with common endpoint $w = 0$, or the slit consists in a smooth arc with $w = 0$ as an interior point.

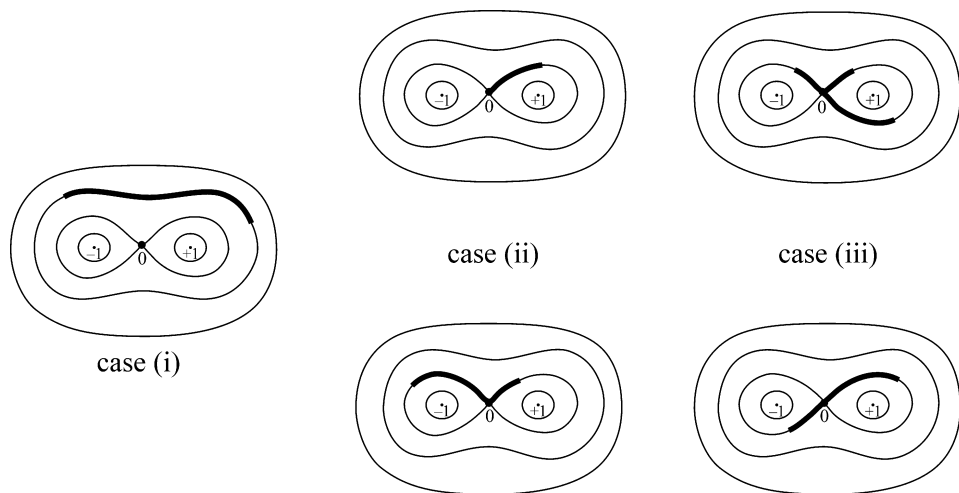


Fig. 5. $w = \text{plane}$.

The uniqueness question for the mappings of Theorem 3.1 is completely answered by the following theorem.

THEOREM 3.2. *In the cases (i) and (ii) the slit mapping of Theorem 3.1 is unique, in the case (iii) there exist infinitely many such slit mappings with respect to the Cassinians.*

The proof in [26] in connection with a corresponding extremal problem shows the great advantage of Grötzsch's method (or, equivalently, the method of extremal length) – it gives in a natural manner in the context of the discussion of the extremal property immediately also this Theorem 3.2. Contrary to this the uniqueness question remains open if we use a variational method to solve this extremal problem. On the other side, we have to confess that the existence proof in [26] for this slit mappings (in the multiply connected case) with the method of continuity is not so easy, even because uniqueness fails. For this reason a (very long) proof with any details was given in [89] in the case of another slit mapping.

Surprisingly it is possible to give a simple criterion to decide a priori (that means: in the situation in the z -plane) which of the cases (i)–(iii) will occur. And this criterion works very straightforward [52].

THEOREM 3.3. *If we define*

$$\alpha = 2\bar{z}_1\bar{z}_2, \tag{3.1}$$

$$4\beta = -\bar{z}_1\bar{z}_2 \cdot (z_1 + z_2) - 3(\bar{z}_1 + \bar{z}_2),$$

$$3\gamma = 2 + |z_1 + z_2|^2,$$

$$\begin{aligned} D = D(z_1, z_2) = & |\alpha|^6 - 12|\alpha|^4|\beta|^2 - 18|\alpha|^4\gamma^2 - 6|\alpha|^2|\beta|^4 \\ & - 180|\alpha|^2|\beta|^2\gamma^2 + 81|\alpha|^2\gamma^4 - 64|\beta|^6 + 36|\beta|^4\gamma^2 \\ & + 108\gamma(|\alpha|^2 + 2|\beta|^2 - \gamma^2)\Re(\alpha\bar{\beta}^2) - 54\Re(\alpha^2\bar{\beta}^4), \end{aligned} \tag{3.2}$$

then the cases $D < 0$, $D = 0$ and $D > 0$ coincide with the cases (i), (ii) and (iii), respectively. In particular, uniqueness of the slit mapping occurs exactly in the case $D \leq 0$.

Because the Goluzin slit mappings (cf. Section 2, (ii)) with $m = 2$, $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \dots = \gamma_N = 0$ correspond, after a simple similarity, exactly to the lemniscate slit mapping of Theorem 3.1, we have also in the case of these Goluzin mappings not always uniqueness (cf. [52, Section 12]).

4. Koebe's Kreisnormierungstheorem. Circle packings

The most natural generalization of the Riemann mapping theorem to the case of multiply-connected domains is the famous

KOEBE'S KREISNORMIERUNGSTHEOREM. *For every domain G of finite connectivity there exists a schlicht conformal mapping onto a domain whose boundary components are circles. The mapping is unique up to a following Möbius transformation.*

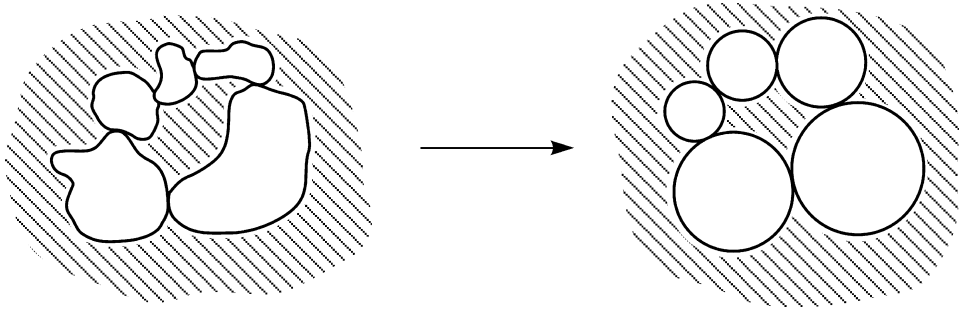


Fig. 6.

Koebe gave many proofs. A proof with Koebe's method of continuity can be found in [23], [29, p. 114]; cf. many further references in [17, p. 182]; there are also proofs with iteration processes), another idea of a proof in [9, p. 169].

Although Koebe's *Kreisnormierung* is so distinguished, an extremal property was found only in 1962 by Schiffer and Hawley. This paper was something of a sensation at that time. With this extremal property in [32, p. 488] also the existence was proved; in [12] the convergence of Koebe's iteration process for doubly-connected domains is treated. A representation of the *Kreisnormierung* mapping with a nonlinear differential equation was given in [52] where radial slit mappings are involved (unfortunately unknown accessory parameters appear).

As a byproduct of his *Kreisnormierungstheorem* in 1936 Koebe deduced as a limit case his mapping of a "Kontaktbereich" (contact domain) onto a "Kreiskontaktbereich"; cf. Figure 6 as a simple example.

Such a *Kontaktbereich* appears if the boundary components are changed such that a contact appears. A *Kontaktbereich* is of course not a domain in the usual sense, because it is still not generally connected. Cf. also the paper [111] of a "scientific grandson" of Koebe about some infinitely many connected domains, and in [60, p. 31, Zusatzbemerkung I] a symmetric case.

Koebe wrote in his paper on p. 162:

Damit ist ein merkwürdiges Sonderergebnis gewonnen, das wir auch folgendermaßen aussprechen können: Die Aufgabe, auf der Kugeloberfläche n Kreisflächen, deren Größe unbekannt bleibt, nebeneinander ohne gegenseitige Überdeckung so zu lagern, daß sie ein durch ein beliebiges Triangulationsschema vorgeschriebenes Kontaktschema erfüllen (Schließungsproblem), gestattet immer eine und, abgesehen von einer Kreisverwandtschaft, nur eine Lösung.

(In a footnote he announced a forthcoming paper which did not appear.) Koebe's paper was forgotten. The result appeared suddenly in many papers as a famous result of Andreev and Thurston. Now there exists a lot of papers with several nice new aspects about this topic. This appears now with the title "Circle Packing"; cf. the overview in [107].

5. Identities between the canonical conformal mappings

There exists a great collection of relations between some canonical conformal mappings, furthermore connections with some other fundamental solutions in the corresponding domain, and also with the Bergmann kernel function.

Let us consider as a first example the parallel slit mappings $g_\Theta(z)$ of Theorem 2.1. These are not independent (as a function of Θ for a fixed G). Namely Grötzsch [25] gave the following simple identity:

$$g_\Theta(z) = e^{i\Theta} [g_0(z) \cos \Theta - i g_{\pi/2}(z) \sin \Theta]; \quad (5.1)$$

cf. also [23, V, Section 2], [74, p. 242]. This means that we need only two of the functions $g_\Theta(z)$ to construct the others. (But it is of course not possible to construct all $g_\Theta(z)$ only from one of them with a simple procedure: If G is bounded by two segments on the real axis, then $g_0(z)$ is simply the identity, while $g_{\pi/2}(z)$ requires elliptic integrals.)

But there are some more “identities” between these parallel slit mappings, and furthermore also connections with some other mappings and domain functions. This great topic started in some sense with the fundamental paper [19] of Garabedian and Schiffer. We can mention here only examples; some further examples in [23], [74, p. 259], [99, p. 104].

These identities assume a technically simpler form with the functions M and N (resp. P and Q) introduced in Section 2. We have then

$$\frac{\partial}{\partial z} N(z, a) = \frac{\partial}{\partial a} N(a, z). \quad (5.2)$$

This means that the derivative $N'(z, a)$ is a symmetric function of z and a . We have additionally

$$\frac{\partial}{\partial z} M(z, a) = \overline{\frac{\partial}{\partial a} M(a, z)}. \quad (5.3)$$

Two simple relations between parallel slit mappings and the spiral slit mappings are given by

$$\begin{aligned} \frac{\partial}{\partial z} P(z, a, b) &= \overline{M(a, z)} - \overline{M(b, z)}, \\ \frac{\partial}{\partial z} Q(z, a, b) &= N(b, z) - N(a, z). \end{aligned} \quad (5.4)$$

In the proof for this identities the method of contour integration in the style of Grunsky is used.

Also Goluzin’s slit mappings of Theorem 2.3 have a representation with the functions P and G ; cf. [52, Section 13], [29]; another more complicated representation with the functions M and N is found in [52, Section 14].

Identities of another type were given in [100].

In [52] there is further explained a very general method to get with radial slit mappings a representation of an arbitrary polygonal mapping (not necessarily slit mapping), concerning the net of the isogonal trajectories of a given quadratic differential. This means that the sides of the polygon are arcs on isogonal trajectories of this quadratic differential. In [52] the procedure was explained for the case, the original domain G is an annulus with concentric circular slits. Unpleasant is the fact that accessory parameters and an integration is involved. In the case of a simple annulus G , that means for doubly-connected domains, the situation is very advantageous, because then the radial slit mappings (also the circular slit mappings) have a simple representation with elliptic functions; cf. [52] (there more references) with representations in the Weierstraß σ -function, [48] with Theta-functions. (In the simply-connected case, for the case of a disk G , the situation is of course again much more simpler because then the radial slit mappings and the circular slit mappings are rational functions.)

This procedure yields for example an explicit representation for the parabola slit mappings $p_{\Theta}(z, a, b)$ of Section 2 with elliptic functions in the case of an annulus G . For the above-mentioned accessory parameters we have a system of nonlinear equations. In the case of some symmetry the situation may become simpler; cf. [60, p. 64].

Finally we mention in this context that of course the ordinary *Schwarz–Christoffel formulas* are included. Then the above-mentioned quadratic differential is simply dw^2 , and the isogonal trajectories are lines. In particular, for an annulus G this yields representations with elliptic functions; cf. [1,48], [32, p. 478] and the classical references in [52]. (These classical references are still not always known; this shows the many new rediscoveries of these formulas.) The classical Schwarz–Christoffel formula for simply-connected domains is today of course included in many textbooks for Complex Analysis; cf. [70]. Surprisingly the proof is not always complete – in many textbooks it was not noticed that, e.g., the case of angle 0 at infinity needs a special consideration; a really exact proof in [48].

6. Connections with other fundamental solutions: Green’s function, Neumann’s function, harmonic measure. Orthonormal series, kernel function

Canonical conformal mappings have a distinguished boundary property and, generally, a singularity of a special, prescribed type. The same is true for some other domain functions, so-called fundamental solutions, such as Green’s function, Neumann’s function, etc. This is the reason for several connections between some canonical conformal mappings of multiply-connected domains (mainly parallel slit mappings, circular and radial slit mappings and related mappings) and these fundamental solutions, where additionally special harmonic measures have to be added as “corrections”. (In the simply-connected case the representation of Green’s function by the Riemann mapping function is classical.) There are today several books in which this topic is included. So we can restrict ourselves to list some of them: [3], [9, Appendix], [23,80–82,97].

There is another important domain function without singularity, the so-called Bergmann kernel function. We begin our sketch of the idea with introducing the scalar product (for a bounded G)

$$(f, g) = \iint_G f(z) \overline{g(z)} dx dy, \quad z = x + iy, \quad (6.1)$$

(also a contour integral representation is possible when the boundary values are smooth), for any two functions f, g in the class of all analytic functions with a finite norm

$$(f, f) = \iint_G |f'(z)|^2 dx dy < +\infty. \quad (6.2)$$

Then there exists a unique kernel function $K(z, \zeta)$ with the reproducing property, that means for all fixed $\zeta \in G$ and all f in the class:

$$(f(z), K(z, \zeta)) = f(\zeta). \quad (6.3)$$

This $K(z, \zeta)$ can be represented with the help of a complete orthonormal system $\{\varphi_v(z)\}$:

$$K(z, \zeta) = \sum \varphi_v(z) \overline{\varphi_v(\zeta)}. \quad (6.4)$$

We can produce such a system $\{\varphi_v(z)\}$ explicitly with the Schmidt procedure of orthonormalization.

The essential thing is now that also $\frac{1}{\pi}M'$ has this producing property:

$$\left(f(z), \frac{1}{\pi}M'(z, \zeta) \right) = f(\zeta). \quad (6.5)$$

This follows simply from the boundary property of M , that means of g_0 and $g_{\pi/2}$. Because of the uniqueness of the kernel function with the reproducing property (6.3), (6.5) yields

$$\frac{1}{\pi}M'(z, \zeta) = K(z, \zeta). \quad (6.6)$$

An analogous result follows for N' , therefore also for the parallel slit mappings $g_0, g_{\pi/2}$ itself. A related but more direct approach for the construction of the parallel slit mappings with an orthonormal system (without explicit use of the kernel function) was given in [17, p. 245]. For more details in the theory of the kernel functions cf. mainly [3, 80, 82].

7. Conformal mapping of domains G_∞ of infinite connectivity

This is a great special field with many new aspects. Koebe gave the first essential contribution [41, footnote 2 on p. 324]. He showed that for G_∞ the parallel slit mapping (cf.

Section 2) is not always uniquely determined. Later [42] he gave an instructive explicit example for this phenomenon. Surprisingly the construction of Koebe contained a gap, which was overlooked for a long time (also by Grötzsch as he confessed). Only in 1960 Reich [92] perceived this. Namely the constructed set is not closed, therefore unsuitable to be the boundary of a domain. Then in [51] it was shown that the Koebe example is correct after forming the closure of this set; cf. also [36] and in [51] remarks about an example of Tamura in [110].

Although the parallel slit mapping of a G_∞ is not always uniquely determined, after Koebe in the set of these parallel slit mappings is one “distinguished” with a minimal Dirichlet integral. A geometric characterization of these so-called minimal slit domains is difficult; cf. [67, p. 134] about the history. Grötzsch gave another characterization of the minimal slit domains; in [35, p. 81] in another form.

Grötzsch gave also many other contributions to the theory of conformal mappings of G_∞ [67]; many of his results in [99]. But in retrospect he said about 1962 to me: “Diese Theorie ist vielleicht doch nicht so organisch”.

In the case of the Koebe Kreisnormierungstheorem (cf. Section 4) for G_∞ there are many special old results, but the most general case is open. Only in [31,104] a proof was given in the case of countably many boundary components.

One can find another idea to prove the existence in the case of mappings onto an annulus with circular or radial slits in [93,94].

8. Kernel convergence. Dependence on parameters

It is an important problem to know how a canonical conformal mapping is changed if the given domain G is changed. That this dependence is in some sense continuous shows the Carathéodory theory of the so-called kernel convergence. Because this theory is very extensive we must restrict ourselves here to give the references, e.g., [23], [29, Chapter 3, Section 3]. We mention that, for example, concerning the parallel slit mapping of a circular domain, the parameters of the image domain (length and midpoints of the parallel slits) will change continuously if we change the parameters of the circle domain (radii and midpoints) continuously. Generally this aspect of continuous dependence on geometric parameters is of importance in the application of Koebe’s method of continuity.

To know more about this dependence on the parameters we have to use the theory of variations, e.g., with formulas like the classical Hadamard formula. The question of the real-analytic dependence on the parameters was at first attacked by Teichmüller [109]. This very difficult question was recently again studied in many special cases in [14,72].

9. Boundary behavior of the mappings

As usual we have considered the canonical conformal mappings only in the interior of the given domain G of finite connectivity. For the limits at the boundary points, the local boundary behavior, we have the same properties as in the case of a simply-connected G [90,91], without new aspects. For instance, if G is bounded by a finite number of Jordan

curves, the parallel slit mappings or Koebe's "Kreismormierung" have a continuous extension to the boundary. The reason for the same boundary behavior in the multiply-connected case is the following. We can choose in G , in the neighborhood of a boundary component, a ring domain. After an additional mapping with a suitable logarithm we have a simply-connected domain. So we get a reduction of the boundary behavior to the simply-connected case.

Another question is the global boundary behavior, for example, the question about the distances of the images of two boundary components. Here we have the results of Grötzsch about the minimum of the smallest distance and about the maximum of the greatest distance [35, Theorems 6.6 and 6.8]. Surprisingly the "dual" problems about the maximum of the smallest distance and about the minimum of the greatest distance are still open.

10. Integral equation methods

Very effective methods to give constructive existence proofs and also for numerical procedures for some simple canonical conformal mappings, such as parallel slit mappings, radial and circular slit mappings use integral equations. There are several possibilities. We can restrict ourselves here to give the reference [17], for more special situations [32, p. 461].

It should be also remarked that it is possible to give a constructive existence proof for the Koebe Geradenschlitztheorem (cf. Section 2) with integral equation methods: [50] (cf. also [71]).

11. Goluzin's functional equation

In the special case of a circular domain (the domain G is bounded by circles) it is possible to derive a functional equation for the parallel slit mapping: [20,21]. The essential tool is then of course the possibility for reflections at the circles.

12. Iteration procedures

Koebe was the first who introduced several iteration procedures to prove the existence of some canonical conformal mappings. Beside the existence itself this topic is connected with many interesting aspects. For example, this is a nice field for applications of distortion theorems.

There are several variants of iterative methods. Because [17] (cf. also [75,76] and references in [17]) is a good representation with many references we can restrict ourselves to the simplest case to sketch the idea. The underlying very simple idea is to obtain the corresponding canonical conformal mapping of a multiply-connected domain as a limit of infinitely many mappings of simply-connected domains, this because the canonical conformal mappings in the simply-connected case are much easier to obtain. In the case of the parallel slit mapping (cf. Section 2) this means, for example: In the first step we map the outside of that boundary component with the greatest variation of the imaginary part

(= greatest difference of the imaginary parts) onto the outside of a segment parallel to the real axis, while this map, as in all other steps, satisfies the hydrodynamical normalization. In the second step we seek again at the image domain that boundary component with the greatest variation of the imaginary part, and consider the corresponding conformal mapping of the outside of this component onto the outside of a segment parallel to the real axis, etc. The main idea in the simplest proof of the convergence of this procedure to the desired parallel slit mapping of our given domain is the fact that always the coefficient of $1/z$ in the development at $z = \infty$ will not decrease. In this way Grötzsch [24] (cf. also [17, p. 208] proved the convergence also for several other canonical mappings, if there exists also a suitable extremal property. The nice thing is that also explicit error estimates for the convergence of the iteration process are possible; cf. [17, p. 236] (it is possible to obtain sharper error estimates there with [39]).

In the case of Koebe's Kreisnormierungstheorem the possibility of reflections is the essential tool [17]; a new proof in the case of the Kreisnormierungstheorem for doubly-connected domains with using the extremal property of Schiffer and Hawley [101] was given in [12].

A complete other situation arises if we try to prove convergence of such iteration procedures in the case of hydrodynamically normalized (h.n.) slit mappings, but now with slits on different curve systems. This great problem was formulated in [27, p. 157] as a desideratum, which is still open. Indeed Grötzsch tried without success to settle this problem in the "simplest" case of the Koebe Geradenschlitztheorem (cf. Section 2) with two boundary components. (This was probably the last problem in function theory in which he was extremely interested. I often had discussions with him in his last years as a professor in Halle.) This means therefore the following problem (cf. also in [17, p. 238, "Gemischtschlitzabbildung"]). Let $G \ni \infty$ be a domain with two boundary components R_1 and R_2 . Let there be given two real numbers Θ_1 and Θ_2 , $\Theta_1 \not\equiv \Theta_2 \pmod{\pi}$. Then we consider the following iteration. First step: h.n. conformal mapping f_1 of the domain bounded by R_1 , such that $f_1(R_1)$ is a segment of inclination Θ_1 . Second step: h.n. mapping f_2 of the domain bounded by $f_1(R_2)$, such that $f_2(f_1(R_2))$ is a segment of inclination Θ_2 . Third step: h.n. mapping f_3 of the domain bounded by $f_2(f_1(R_1))$, such that $f_3(f_2(f_1(R_1)))$ is a segment of inclination Θ_1 , etc. Question: Does this iteration always converge to the h.n. mapping of the domain G onto a domain, bounded by two segments of inclination Θ_1 , resp. Θ_2 ?

Nevertheless in the paper [79], inspired by Grötzsch, a great success was attained: This iteration is indeed convergent to the desired mapping if R_1 and R_2 are lying in disks of radii r_1 and r_2 and with the distance d , where $d \geq 3.4 \cdot \max(r_1, r_2)$. Very interesting is the fact, that in the proof in [79] so many distortion theorems for conformal mappings intervene. Numerical experiments in [65] seem to suggest that convergence always takes place.

13. Factorization

In some sense congenial to the iteration theory of the last section is the theory of factorization of a conformal mapping of a multiply-connected domain. Here we have the representation of a conformal mapping of a domain with n boundary components as the

composition of n conformal mappings of simply-connected domains. The following nice theorem goes back to Erokhin [15] (cf. also for other proofs [17], papers of Hübner cited there, and [58,59] with further references).

THEOREM 13.1. *Let G be a domain in the complex plane z with the inner points $z = \infty$ and the boundary components $R_1, \dots, R_n$ ($n \geq 2$), in an arbitrary but then fixed order. Let be $w = f(z)$ a given fixed conformal mapping of G with the hydrodynamic normalization (2.1). Then $f(z)$ has a representation of the form*

$$f(z) = f_n(\dots f_2(f_1(z))), \quad (13.1)$$

where f_1 is a hydrodynamically normalized conformal mapping of the domain outside of R_1 , f_2 of the domain outside of $f_1(R_2)$, etc. For a given G and f the representation (13.1) is unique up to Möbius transformations between the “factors” f_n .

Surprisingly it seems hopeless to construct nontrivial examples, even in the case $n = 2$, for which it is possible to give the factors in form of explicit analytic expressions. In a new formulation, Theorem 13.1 reads in the simplest nontrivial case as follows. If a finite doubly-connected domain G with the outer boundary component R_1 and the inner boundary component R_2 is given, then there exists a closed analytic Jordan curve C such that after a conformal mapping $f(z)$ of the outside of R_2 onto the outside of C with $f(\infty) = \infty$ the curve C will become a level line of the Green’s function of the inside of $f(R_1)$, with respect to a suitable pole in this simply-connected domain. But the problem is to find this curve C .

If G and $f(z)$ have some symmetry then there are in Theorem 13.1 also factorizations with symmetry, cf. Section 14 for typical examples.

14. Canonical conformal mappings with symmetries: Mappings on the elliptic and on the hyperbolic plane

If the domain G has some symmetry we can ask for canonical mappings onto domains with the same symmetry. We will demonstrate this in the case of domains with symmetry with respect to the substitution

$$z^* = -1/\bar{z} \quad (14.1)$$

or

$$z^* = 1/\bar{z}. \quad (14.2)$$

In the first case (14.1) z and z^* are antipodal points on the Riemann sphere. We can speak (following Klein) in this case about the elliptic plane as the Riemann sphere with

identification of antipodal points. It is sometimes useful to use the corresponding natural elliptic metric, given by the spherical metric (line element)

$$\frac{|dz|}{1 + |z|^2}. \quad (14.3)$$

In the second case (14.2) z transforms onto z^* by reflection at the unit circle. It is then useful to consider the unit disk as the hyperbolic plane with the metric (line element)

$$\frac{|dz|}{1 - |z|^2}. \quad (14.4)$$

In the elliptic case we consider then “diametrically symmetric (d.s.) domains” G which transform onto themselves under the substitution (14.1). And we consider to these G d.s. mappings, that are mappings which transform antipodal points always onto antipodal points, such that the image domain is again d.s.

Analogously in the second hyperbolic case we consider “reflection-symmetric (r.s.) domains” G which transform onto themselves under the substitution (14.2). And then we take into account r.s. mappings, that are those which transform symmetric points with respect to the unit circle onto points with the same symmetries, such that the image domain has the same symmetry. In particular, in this case, points of the unit circle (which are inner points of the given domain) transform onto points of the unit circle. Because of the Schwarz reflection principle it is as such therefore enough to consider the mappings only inside the unit disk. (An analogous simple possibility does not exist in the elliptic case.)

In the elliptic case in [60, Chapter III], inspired by [27], the existence of some canonical slit mappings (uniquely with a normalization) was proved, where the slits are lying on a prescribed fixed d.s. curve family. This family must in [60] additionally have the property that it transforms onto itself under all rotations about the origin or under all stretches with center at the origin. Very special cases of such families are the circles concentric to the origin, and the rays starting at the origin. This means that we have in particular the existence of d.s. conformal mappings of any d.s. domain $G \ni 0$ onto a d.s. circular or radial slit domain, where the circular slits are concentric to 0 as image of 0, resp. where the radial slits are lying on rays starting at 0 as image of 0.

The proof in [60] is in the style of Grötzsch [27] and uses the method of continuity together with the corresponding uniqueness theorem. Unfortunately this proof does not work without this rotation property resp. stretching property. Therefore we formulate here as a desideratum: Prove or disprove the corresponding more general mapping theorem (e.g., with fixed point theorems).

In the hyperbolic case similar results are possible (not formulated in [60]).

There exists, for example, also a corresponding Kreisnormierungstheorem in the elliptic case [60] (Satz 3.3) and in the hyperbolic case, that means the existence of conformal mappings onto domains which are bounded by circles and are d.s. (resp. r.s.) (but the latter hyperbolic case is of course trivial).

Another possibility to obtain d.s. resp. r.s. canonical conformal mappings of a d.s. (resp. r.s.) domain is to formulate an extremal problem of Grötzsch–Teichmüller type. Then

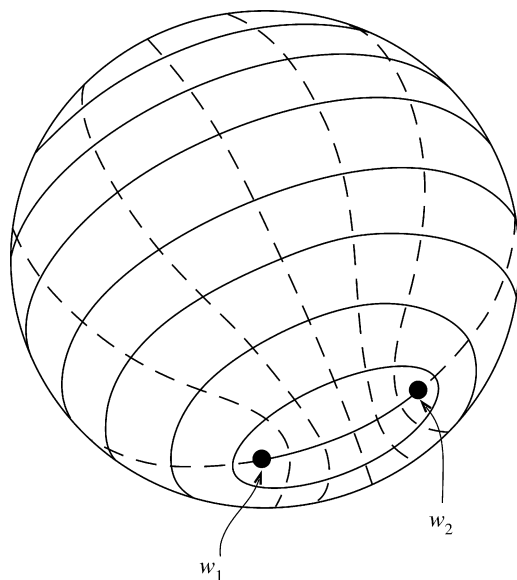


Fig. 7.

we get usually slit mappings with this symmetry. In [60] are proved many theorems in the elliptic and in the hyperbolic case with the method of continuity and using the uniqueness (in connection with the extremal property). We restrict ourselves to a simple typical example in the elliptic case.

THEOREM 14.1. *For every d.s. domain G with the (different) inner points z_1 and z_2 there exists a d.s. conformal mapping for which the images of the boundary components are slits on the family of the spherical conics with focal points at the images w_1 and w_2 of z_1 and z_2 (and at the d.s. points w_1^* and w_2^* ; cf. Figure 7). This mapping is unique up to an additional rotation of the Riemann sphere.*

These spherical conics are characterized by the property that the sum of the spherical distances to the focal points has a constant value. If we replace one focal point by the antipodal point, then we obtain the same family with the difference of the spherical distances. Therefore spherical ellipses and spherical hyperbolas are the same.

Another method to prove Theorem 14.1 and other such theorems in the elliptic case is an adapted variational method which preserves the symmetry [68]. But then we have the general dilemma using the variational method: The question of uniqueness remains open.

There are analogous results in the hyperbolic case [60]. Interestingly all 9 sorts of conics (with respect to the hyperbolic metric (14.4)) appear here in connection with extremal problems. This corresponds to the fact that in the hyperbolic plane as in the elliptic plane the conics correspond exactly to the simplest quadratic differentials on the hyperbolic plane, namely to those without zeroes [60, p. 28].

Also in the hyperbolic case existence proofs are possible with the variational method, now with that of [13] (cf. also remarks concerning [13] in [68]).

There are also factorization theorems in the style of Section 13, now in such a way, that also all “factors” have the same symmetry as the “product” [60, Chapter X]. More precisely: If there is given a conformal and d.s. mapping of a d.s. domain with $2n$ boundary components, then there is a factorization with n also d.s. mappings, where every of the n steps (= “factors”) is a d.s. mapping of the doubly-connected domains between two antipodal boundary components.

Analogous results hold in the hyperbolic case.

15. Canonical conformal mappings on a fixed Riemann surface

A new and much more difficult type of problem arises if we consider a fixed Riemann surface $\mathfrak{R}$ (or, more generally, a Riemann manifold) as “carrier”. (Up to now $\mathfrak{R}$ was simply the Riemann sphere. The situation in Section 14 can be seen as the case, $\mathfrak{R}$ is the elliptic or the hyperbolic plane.) Then we can ask for (schlicht) conformal mappings of a domain G on $\mathfrak{R}$ onto a domain which again lies on $\mathfrak{R}$. In particular, we have again the question for canonical conformal mappings. A great new difficulty appears now because then rigid domains G are possible.

We can find first remarks about this new topic in [26] at the end of Section 1, but especially in [27, p. 157], [28].

Let us consider a simple, typical example, namely the case of an annulus ($0 < r < |z| < R$ ($< +\infty$)) as fixed Riemann surface $\mathfrak{R}$. Then every domain G , consisting in $\mathfrak{R}$ with a finite number of concentric circular slits, is rigid. More precisely, the only possible conformal mappings of G with an image on $\mathfrak{R}$ are the rotations. Every system of canonical conformal mappings has to take this phenomenon in account.

In [85] a simple system of canonical conformal mappings was given for all such G which arise from the annulus $\mathfrak{R}$ after deleting of n disjoint continua $k_1, \dots, k_n$ lying in $\mathfrak{R}$: *For every such G exists a conformal mapping onto a domain which arises from $\mathfrak{R}$ after deleting of n slits on logarithmic spirals with center 0 and the same inclination.* This inclination depends on G and cannot be prescribed. In this system of spirals also the concentric circles and also the concentric radii are included. Surprisingly the corresponding uniqueness question (up to a following rotation) is an open problem, even in the simplest case $n = 1$.

In [8] we can find another canonical conformal mapping in an annulus $\mathfrak{R}$, in [87] a much more complicated system of such mappings.

Finally it should be remarked that extremal problems in connection with such canonical mappings can be solved again with Grötzsch’s strip method, resp. the method of the extremal metric; cf. also general remarks in [86], and the more general situation in [88]. This case of Riemann surfaces as carrier is also included in the GCTh of Jenkins [35].

16. Canonical conformal mappings with higher normalization

In the usual classical cases of canonical conformal mappings (parallel slit mapping, radial slit mapping, etc.) we have the simplest normalization. That means we ask (to obtain also

uniqueness) for mappings which satisfy, beside the desired geometric nature of the image domain, 3 complex or 6 real side conditions, corresponding to the fact that the Möbius transformations contain 3 complex parameters. The simplest case is the hydrodynamical normalization (2.1), that means the condition

$$w(z) = a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \quad (16.1)$$

with prescribing the 3 side conditions $w(\infty) = \infty$, $a_{-1} = 1$, $a_0 = 0$.

Grötzsch [26] called this situation a “niedere Normierung” (low normalization) and studied then the first case of a “höhere Normierung” (higher normalization), this in connection with an extremal problem. Namely, let G be a domain in the z -plane with the interior points ∞ , 0 and z_1 ($\neq 0$). He considered the class of all schlicht conformal mappings of G with the normalization

$$w(\infty) = \infty, \quad w(0) = 0, \quad w(z_1) = w_1, \quad \text{and} \quad (16.1) \text{ with } |a_{-1}| = 1, \quad (16.2)$$

where $w_1 > 0$ is a fixed value. This means 7 real conditions. Of course the value w_1 has to be restricted to a closed interval (defined with Grötzsch’s ellipse-slit and hyperbola-slit mapping). Then Grötzsch solved the extremal problem

$$|w'(0)| \rightarrow \max, \quad \text{resp.} \quad \min, \quad (16.3)$$

with his method (strip method for the extremality and method of continuity for the existence of the extremal mappings). This yields as a by-product canonical conformal slit mappings (characterized by a quadratic differential) with this higher normalization. This method impressively shows the advantage of the method of continuity, which needs in its usual form the uniqueness of the mappings (which appears here as a by-product of the extremality).

It should be remarked that it is today possible to replace Grötzsch’s method to construct the curve families for the slits by Riemannian manifolds (with a special welding procedure) simply by defining a suitable quadratic differential.

Contrary to Grötzsch’s method, by using the variational method we have to prepare at first variational formulas which satisfy the higher normalization. This is not easy. (And, as it was remarked, the uniqueness question remains open.) It is possible to avoid this difficulty by addition of a “penalty term” to the functional. This penalty term has the effect, that in the extremal situation also the normalization conditions beside the “first” usual 3 complex, resp. 6, real normalization conditions are satisfied. This was explained in [95] (for a more general situation with quasiconformal mappings); cf. also the references there. But here again the uniqueness question remains open, and additionally we have the disadvantage, that we are not sure to obtain all extremal mappings.

Other ideas to obtain canonical mappings with higher normalizations are used in [6,88].

17. Numerical realization of canonical conformal mappings

In the simply-connected case we have a great literature about the numerical constructions of the Riemann mapping and its inverse (the latter is from the numerical point of view indeed another problem). For this great field we can restrict ourselves here to give the exhaustive references [18,112]. There one can find also the known procedures for the multiply-connected case.

18. Generalizations for quasiconformal mappings

(a) In the first instance the question for canonical quasiconformal mappings seems senseless because we have then a too great set of mappings. But we have a completely different situation if we restrict ourselves to solutions $w(z) = u(x, y) + iv(x, y)$ of a fixed elliptic system of the form

$$v_y = au_x + bu_y, \quad -v_x = cu_x + du_y; \quad (18.1)$$

cf. [96] for assumptions for the given fixed 4 functions $a = a(x, y), \dots, d = d(x, y)$, defined in the domain G . Then we have an analogous situation for the solutions $w = w(z)$ as in the special case of conformal mappings with $a(x, y) \equiv d(x, y) \equiv 1$, $b(x, y) \equiv c(x, y) \equiv 0$. In [83] it was shown that it is possible to get again a great variety of canonical mappings (now always solutions of (18.1)) as in the conformal case. This direction goes back to ideas of Lavrent'ev; cf. [73,83,96], with further references in [83] to Schapiro, Dressel and Gergen, Bers and Nirenberg. The same is true for the case of nonlinear generalizations of (18.1); cf. [4].

As in the conformal case it is also possible to obtain canonical slit mappings for a system (18.1) by solving extremal problems with variational methods. Here the great difficulty to preserve the system (18.1) under a variation was overwhelmed in [96, Chapter VI].

(b) Of a specific interest in several connections, also in mathematical physics [55], [49, pp. 91, 151], is the special system

$$v_y = \frac{1}{p}u_x, \quad -v_x = \frac{1}{p}u_y \quad (18.2)$$

with a given function $p = p(x, y) = p(z)$. This system for $w(z) = u(x, y) + iv(x, y)$ can also be written in complex form:

$$w_{\bar{z}} = v(z)\bar{w}_z \quad \text{with } v(z) = \frac{p(z) - 1}{p(z) + 1}. \quad (18.3)$$

This system was studied at first mainly by Bers and Polozhii; cf. [96]. The solutions of (18.2), (18.3) were called p -analytical by Polozhii.

We will restrict ourselves here now to the case

$$p(z) = \begin{cases} 1 & \text{in } G, \\ Q \geq 1 & \text{in the complement of } G, \end{cases} \quad (18.4)$$

where $G \ni \infty$ is a domain which is bounded by a finite number of closed analytic Jordan curves. (We obtain the case $0 < Q \leq 1$ by exchanging the u - and the v -axis.) Then we have for every real Θ a schlicht and continuous mapping $g_{\Theta, Q}(z)$ of the whole plane, for which $e^{-i\Theta} g_{\Theta, Q}(z)$ satisfies (18.3), with hydrodynamical normalization

$$g_{\Theta, Q}(z) = z + a_{1, \Theta}(Q) \cdot z^{-1} + \dots \quad \text{at } z = \infty. \quad (18.5)$$

The latter normalization is possible because $g_{\Theta, Q}(z)$ is conformal in G .

By the way, there are only a few cases in which this mapping $g_{\Theta, Q}(z)$ is simply an affine mapping in the complement of G [57]; if G is simply-connected then ∂G must be an ellipse.

These functions $g_{\Theta, Q}(z)$ yield in some sense generalizations of the parallel slit mappings $g_{\Theta}(z)$ of Section 2. Namely, we have (cf. [49, p. 90])

$$\lim_{Q \rightarrow +\infty} g_{\Theta, Q}(z) = g_{\Theta}(z). \quad (18.6)$$

This follows from the property that $g_{\Theta, Q}(z)$ maximizes $\Re e^{-i\Theta} a$, in the class of all schlicht conformal mappings of G with the hydrodynamical normalization

$$z + \frac{a_1}{z} + \dots$$

and with a continuous Q -quasiconformal extension to the complement of G (cf. [49, p. 98]). This extremal property yields that $\Re e^{-i\Theta} a_{1, \Theta}(Q)$ is monotonously increasing as a function of Q , with the limit

$$\lim_{Q \rightarrow +\infty} \Re e^{-i\Theta} a_{1, \Theta}(Q) = \Re e^{-i\Theta} a_{1, \Theta}^*, \quad (18.7)$$

where $a_{1, \Theta}^*$ is the corresponding coefficient of $g_{\Theta}(z)$. This limit property (18.7) follows simply from the fact that $g_{\Theta}(z)$ can be approximated by a normalized conformal mapping of G with a quasiconformal extension (with some dilatation bound, which can be great). Finally, (18.6) follows from (18.7) because the maximal Θ -width must tend to 0 for $Q \rightarrow \infty$. This is a classical consideration ([24]; cf. also [39] with the possibility to obtain better concrete estimates).

Analogously we obtain mappings $g_{\Theta, Q}(z, a)$, now with the normalization as in (2.2) in the neighborhood of the finite point $a \in G$, and with the limit

$$\lim_{Q \rightarrow +\infty} g_{\Theta, Q}(z, a) = g_{\Theta}(z, a). \quad (18.8)$$

We can again define further functions

$$\begin{aligned} M_Q(z, a) &= \frac{1}{2} (g_{0, Q}(z, a) - g_{\pi/2, Q}(z, a)), \\ N_Q(z, a) &= \frac{1}{2} (g_{0, Q}(z, a) + g_{\pi/2, Q}(z, a)). \end{aligned} \quad (18.9)$$

These definitions have sense immediately as in the limit case $Q \rightarrow \infty$ (cf. (2.3)), if we restrict the point a to the domain G . But as it was shown in [62] there is also a definition possible in the case, a is chosen in the complement.

Additionally, we have mappings $j_{\Theta, Q}(z, a, b)$ which we have to define as $j_{\Theta}(z, a, b)$ in Section 2, but now with the property, that $\log j_{\Theta, Q}(z, a, b)$ is a solution of the complex differential equation (18.3) with (18.4). Again a limit relation as in (18.8) holds. We remark by the way in connection with this limit relation that the phenomenon “Verzweigungsercheinung” studied in Section 3 disappears for the mappings of Theorem 3.1 if we consider here in an analogous manner the system (18.3); cf. for this “regularization” [56].

(c) The functions $g_{\Theta, Q}$ and $j_{\Theta, Q}$ play an important role as complex potentials in fluid dynamics (e.g., ground water flow), electrostatics etc. in inhomogeneous materials in the plane, cf. [55, 56, 102], [49, pp. 91, 151]. In the case of electrostatics Q corresponds to the dielectric.

(d) Between the new mappings $g_{\Theta, Q}$, $j_{\Theta, Q}$ and some others and also some fundamental solutions there exist again a great system of identities, as in the in Section 5 mentioned limit case of the “pure” conformal mappings g_{Θ} , j_{Θ} , etc. This was shown in [62]. We restrict ourselves here to remark as an example that again (5.1) holds if we replace $g_{\Theta}(z)$ by $g_{\Theta, Q}(z)$.

(e) Now we will report about the possibility to obtain M_Q , N_Q in (18.9) and other fundamental solutions with a new orthonormal system, produced with a new scalar product [2, 33, 64]. Let again $G \ni \infty$ be a domain with a finite number of boundary components. We use the abbreviation

$$q = \frac{Q - 1}{Q + 1} \quad (18.10)$$

and denote by $\mathfrak{H}$ the family of all finite functions $F'(z)$, for which $F(z)$ is a single-valued and analytic function in G with $F(\infty) = 0$, and with

$$\iint_G |F'(z)|^2 dx dy < +\infty, \quad z = x + iy. \quad (18.11)$$

We produce a new function

$$F^*(z) = q F(z) + q \frac{1}{\pi} \iint_G \frac{\overline{F'(\zeta)}}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta. \quad (18.12)$$

Then the desired scalar product for two functions F'_1 and F'_2 in $\mathfrak{H}$ is

$$(F'_1, F'_2) = \iint_G (F'_1 \overline{F'_2} - \overline{F'_1} F'^*_2) dx dy. \quad (18.13)$$

(In (18.12) and (18.13) there are also representations with contour integrals possible.) From the general theory [80] there then follows the existence of a kernel function $K(z, \zeta)$ with the reproducing property

$$(F'(z), K(z, \zeta)) = F'(\zeta) \quad \text{for all } F' \in \mathfrak{H}. \quad (18.14)$$

This kernel function permits again, as in the classical case, a representation with an orthonormal system of $\mathfrak{H}$. One can produce such a system with the Schmidt orthonormalization. But now the calculations are of course much more complicated (although again straightforward) because of the more complicated scalar product. Also the then following explicit representation of our functions $M_Q(z, \zeta)$, $N_Q(z, \zeta)$ needs much more calculations. (The aim of the scalar product (18.13), which looks for the first time over-refined, is the representation of these functions M_Q , N_Q and the then following representation of the mapping $g_{\Theta, Q}$.) One can find more details in [66] in the special case of an especially convenient orthonormal system.

The procedure with the scalar product (18.13) is of course restricted to the system (18.3) with (18.4). There is another possibility to construct the mappings $g_{\Theta, Q}$ with an orthonormal system in the general case of coefficients $v(z)$ in (18.3) – cf. [62]. But the disadvantage is then that the functions of an orthonormal system themselves have to be solutions of the same elliptic system (18.3).

(f) There is also the possibility to construct our mappings $g_{\Theta, Q}$ with an integral equation [63] or with Fourier integrals [34].

19. A desideratum: Another way from conformal to quasiconformal mappings

Trivially, conformal mappings represent a special case of quasiconformal mappings. Therefore canonical conformal mappings can be obtained as a special case of canonical quasiconformal mappings. As was shown in Section 18, canonical conformal mappings arise also with a sort of limit process: We define in the “holes” of the domain G the dilatation bound Q and consider a suitable limit with $Q \rightarrow \infty$.

But now we will sketch as a heuristic principle, without proof, in some sense the reverse. Namely, following [61, middle of p. 286], it is also possible to get quasiconformal mappings by a limit process with conformal mappings.

If we have a (smooth) quasiconformal mapping, then locally a suitable infinitesimal square transforms affinely onto an infinitesimal rectangle. We can now approximate such an affine mapping of a square onto a rectangle by a conformal mapping of this square with many “holes” (e.g., disjoint disks in a “suitable” distribution) where the holes transform onto segments parallel to the direction of the rectangle (again the corners of the square have to transform onto the corners of the rectangle). Therefore we can obtain a given quasiconformal mapping of a domain in such a way that we delete from the domain in a suitable way disjoint small disks, where the number of these disks have to tend to infinity. The “density” of these disks in the neighborhood of a point depends on the dilatation there. This means for example that we finally obtain a quasiconformal mapping of the whole plane as a limit of conformal mappings arising from Koebe’s Geradenschlitztheorem. Compare further remarks in [61], also a physical interpretation.

20. Miscellaneous

(a) There are several other canonical conformal mappings, also in the doctoral dissertations with Koebe. Let us prescribe only the following canonical conformal mapping of

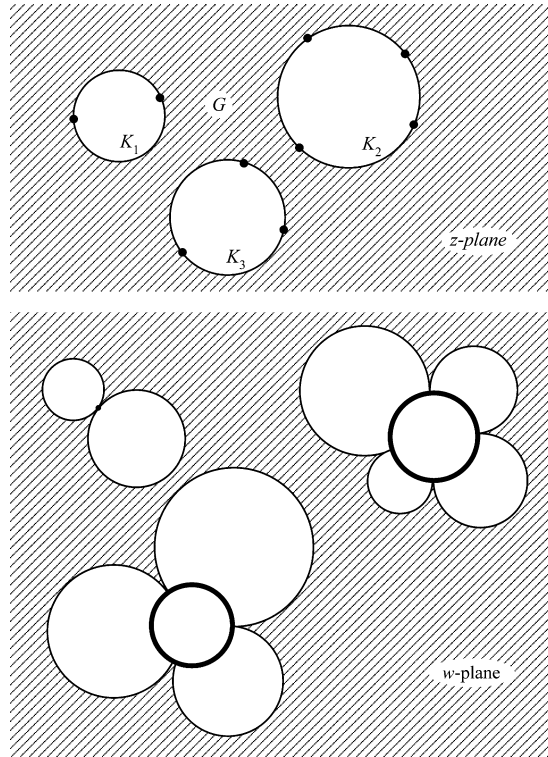


Fig. 8.

Sonnenschein [106]. Without loss of generality we can assume that the starting domain G in the z -plane is bounded by a finite number of circles K_ν . On each K_ν a given finite set of points is marked, which divide K_ν in arcs. Then there exists exactly one hydrodynamically normalized schlicht conformal mapping $w = w(z)$ of G , such that the mentioned arcs on the same K_ν always transform onto circular arcs which are orthogonal to the same circle (Figure 8). Sonnenschein considered also the more general case in which a disjoint finite number of arcs on the K_ν , which not necessarily cover K_ν completely, is given.

We mention two further possibilities for hydrodynamically normalized mappings of a domain $G \ni \infty$ of finite connectivity onto canonical or representative domains G^* . Without loss of generality we can assume that G is bounded by n closed analytic Jordan curves.

For example let G^* be domains bounded by n circular arcs. The curvature cannot be prescribed, but we prescribe at every boundary component of G two different points or preimages of the endpoints of the circular arcs.

Or let the boundary components of G^* be "stars". A star consists in k segments of the same length, starting at the same point. We prescribe at the corresponding boundary component of G now k different points which have to transform into the other endpoints of these segments. (We do not prescribe the middle point of the star, the length of the segments, and the angles.)

(b) We can see the question of canonical conformal mappings also from the following completely different point of view. Let G again be bounded by n closed analytic Jordan curves. Then, for example, the Koebe Kreisnormierung means the existence of a system of n sewing functions. Namely, we consider beside the mapping of G a system of n conformal mappings of the “holes” of G onto the “holes” of the desired circle domain. Then we have at the two sides of the boundary components of G two different mappings. The correspondence between the two preimages of the same boundary point of the circle domain defines a sewing. In this manner the Koebe Kreisnormierung is equivalent to a system of sewing functions. Also in the case of canonical slit mappings we can see an equivalent sewing problem. This yields a new approach to the theory of canonical conformal mappings with integral equation methods, because the latter are linked with the theory of conformal sewing, that is an application of the theory of boundary value problems with “shift” [77].

(c) Up to now we always considered a conformal mapping of only a single domain. But there exists also a great literature about the theory of systems of schlicht conformal mappings of a given system of domains $G_1, \dots, G_n$ onto nonoverlapping domains, mainly concerning extremal problems; cf. the monography [74] (unfortunately until now not translated). Because we can assume $G_1, \dots, G_n$ also nonoverlapping, we can “join” these domain by a system of bridges, so that we obtain only a single domain. In this way it is possible to obtain the conformal mappings of the system $G_1, \dots, G_n$ onto nonoverlapping domains as the limit of mappings of one domain. This idea is in particular useful to solve extremal problems. In a special case this was explained in [54]. What concerns the question of canonical conformal mappings of systems $G_1, \dots, G_n$ onto nonoverlapping domains: This question has mainly sense in the case of higher normalizations (cf. Section 16).

(d) In this chapter we have restricted ourselves to schlicht mappings. Concerning non-schlicht canonical conformal mappings cf. [23,29,110].

References

- [1] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI (1990); Russian orig., 2nd edn: Nauka, Moscow (1970).
- [2] H. Begehr and R.P. Gilbert, *Transformations, Transmutations, and Kernel Functions*, Vol. 2, Longman Scientific & Technical/ Wiley, New York (1993).
- [3] S. Bergman, *The Kernel Function and Conformal Mapping*, Amer. Math. Soc., Providence, RI (1950); 2nd edn (1970).
- [4] B. Bojarski and T. Iwaniec, *Quasiconformal mappings and nonlinear elliptic equations in two variables, I, II*, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astron. Phys. **22** (1974), 473–478, 479–484.
- [5] U. Böttger, B. Plümper and R. Rupp, *Complex potentials*, J. Math. Anal. Appl. **234** (1999), 55–66.
- [6] M. Brandt, *Ein Existenzsatz für schlicht-konforme Abbildungen endlich-vielfach zusammenhängender Gebiete bei höherer Normierung*, Bull. Soc. Sci. Lett. Łódź **29** (5) (1979), 1–13.
- [7] M. Brandt, *Ein Abbildungssatz für endlich-vielfach zusammenhängende Gebiete*, Bull. Soc. Sci. Lett. Łódź **30** 3 (1980), 1–12.
- [8] M. Brandt, *Ein Abbildungssatz für in einen konzentrischen Kreisring schlicht eingelagerte Gebiete*, Bull. Soc. Sci. Lett. Łódź **30** (4) (1980), 1–7.
- [9] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces* (with an Appendix by M. Schiffer), Interscience, New York–London (1950).
- [10] R. Courant, B. Maue and M. Shiffman, *A general theorem on conformal mapping of multiply connected domains*, Proc. Nat. Acad. Sci. USA **26** (1940), 503–507.

- [11] B. Dittmar, *Übertragung eines Extremalproblems von M. Schiffer und N.S. Hawley auf quasikonforme Abbildungen*, Math. Nachr. **94** (1980), 107–115.
- [12] B. Dittmar, *Bemerkungen zu einem Funktional von M. Schiffer und N.S. Hawley in der Theorie der konformen Abbildungen*, Math. Nachr. **97** (1980), 21–37.
- [13] P. Duren and M. Schiffer, *A variational method for functions schlicht in an annulus*, Arch. Ration. Mech. Anal. **9** (1962), 260–272.
- [14] C.J. Earle and S. Mitra, *Variation of moduli under holomorphic motions*, Proc. First Ahlfors–Bers Colloquium, State University New York, Stony Brook, NY, Contemp. Math. **256** (2000), 39–67.
- [15] V. Erokhin (Erohin), *On the theory of conformal and quasiconformal mapping of multiply connected regions*, Dokl. Akad. Nauk SSSR **127** (1959), 1155–1157 (in Russian).
- [16] C.H. Fitzgerald and F. Weening, *Existence and uniqueness of rectilinear slit maps*, Trans. Amer. Math. Soc. **352** (2000), 485–513.
- [17] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1964).
- [18] D. Gaier, *Konforme Abbildung mehrfach zusammenhängender Gebiete*, Jahresber. Deutsche Math.-Ver. **81** (1978), 25–44.
- [19] P. Garabedian and M. Schiffer, *Identities in the theory of conformal mapping*, Trans. Amer. Math. Soc. **65** (1949), 187–238.
- [20] G.M. Goluzin, *Solution of fundamental problems of mathematical physics in the plane in the case of the Laplace equation in multiply-connected domains, which are bounded by circles (method of functional equations)*, Mat. Sb. **41** (1934), 246–276 (in Russian).
- [21] G.M. Goluzin, *Conformal mapping of multiply-connected domains onto the plane with slits with the method of functional equations*, Conformal Mapping of Simply- and Multiply-Connected Domains, ONTI, Moscow–Leningrad (1937), 98–110 (in Russian).
- [22] G.M. Goluzin, *A method of variation in conformal mapping, II*, Mat. Sb. **21** (63) (1947), 83–117 (in Russian).
- [23] G.M. Golusin (Goluzin), *Geometrische Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1957) (transl. from the Russian) 2nd Russian edn: Nauka, Moscow (1966); Engl. transl.: Amer. Math. Soc. Providence, RI (1969).
- [24] H. Grötzsch, *Zur konformen Abbildung mehrfach zusammenhängender schlichter Bereiche (Iterationsverfahren)*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **83** (1931), 67–76.
- [25] H. Grötzsch, *Über das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **84** (1932), 15–36.
- [26] H. Grötzsch, *Über die Geometrie der schlichten konformen Abbildung*, 1. und 2. Mitteil., Sitzungsber. Preuß. Akad. Wiss. Berlin, Phys.-Math. Kl. (1933), 654–671, 893–908.
- [27] H. Grötzsch, *Zur Theorie der konformen Abbildung schlichter Bereiche*, 1. und 2. Mitteil., Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **87** (1935), 145–158, 159–167.
- [28] H. Grötzsch, *Zur Geometrie der konformen Abbildung*, Hallische Monographien **16** (1950), 5–11.
- [29] H. Grunsky, *Lectures on Theory of Functions in Multiply Connected Domains*, Vandenhoeck & Ruprecht, Göttingen (1978).
- [30] A.N. Harrington, *Conformal mappings onto domains with arbitrarily specified boundary shapes*, J. Anal. Math. **41** (1982), 39–53.
- [31] Z.-X. He and O. Schramm, *Fixed points, Koebe uniformization and circle packings*, Ann. Math. **137** (1993), 369–406.
- [32] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, Wiley, New York (1986).
- [33] E. Hoy, *Darstellungsformeln für gewisse quasikonforme Normalabbildungen*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **33** (1984), 87–95.
- [34] E. Hoy, *Eine Integralgleichung für quasikonforme Extremalabbildungen*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **36** (1987), 16–21.
- [35] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1958).
- [36] J.A. Jenkins, *On a paper of Reich concerning minimal slit domains*, Proc. Amer. Math. Soc. **13** (1962), 358–360.

- [37] J.A. Jenkins, *The method of the extremal metric*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier Science, Amsterdam (2002), 393–456.
- [38] G. Julia, *Leçons sur la représentation conforme des aires multiples connexes*, Gauthier-Villars, Paris (1934).
- [39] S. Kirsch, *Univalent functions with range restrictions*, Z. Anal. Anwendungen **19** (2000), 1057–1073.
- [40] P. Koebe, *Über die konforme Abbildung mehrfach zusammenhängender ebener Bereiche, insbesondere solcher Bereiche, deren Begrenzung von Kreisen gebildet wird*, Jahresber. Deutsche Math.-Ver. **15** (1906), 142–153.
- [41] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven*, 4. Mitteil., Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1909), 324–361.
- [42] P. Koebe, *Zur konformen Abbildung unendlich-vielfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1918), 60–71.
- [43] P. Koebe, *Abhandlungen zur Theorie der konformen Abbildung, IV. Abbildung mehrfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Acta Math. **41** (1918), 305–344.
- [44] P. Koebe, *Über die Strömungspotentiale und die zugehörigen konformen Abbildungen riemannscher Flächen*, Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1919), 1–46.
- [45] P. Koebe, *Riemannsche Mannigfaltigkeiten und nichteuklidische Raumformen*, 1.–8. Mitteil., Sitzungsber. Preuß. Akad. Wiss. Berlin Phys.-Math. Kl. (1927–1932).
- [46] P. Koebe, *Hydrodynamische Potentialströmungen in mehrfach zusammenhängenden ebenen Bereichen im Zusammenhang mit der konformen Abbildung solcher Bereiche (N-Decker-Strömung, N-Schaufel-Strömung, N-Gitter-Strömung)*, Ber. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **87** (1935), 287–318.
- [47] P. Koebe, *Kontaktprobleme der konformen Abbildung*, Ber. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **88** (1936), 141–164.
- [48] W. von Koppenfels and F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1959).
- [49] S.L. Kruschkal und R. Kühnau, *Quasikonforme Abbildungen—neue Methoden und Anwendungen*, Teubner, Leipzig (1983); in Russian: Nauka Sibirsk. Otd., Novosibirsk (1984).
- [50] V. Krylow (Kryloff), *Une application des équations intégrales à la démonstration de certains théorèmes de la théorie des représentations conformes*, Rec. Math. (Mat. Sb.) **4** (46) (1938), 9–30 (in Russian).
- [51] R. Kühnau, *Über ein KOEBESches Beispiel zur Theorie der minimalen Schlitzbereiche*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg, Math.-Nat. Reihe **14** (1965), 319–321.
- [52] R. Kühnau, *Über die analytische Darstellung von Abbildungsfunktionen, insbesondere von Extremalfunktionen der Theorie der konformen Abbildung*, J. Reine Angew. Math. **228** (1967), 93–132.
- [53] R. Kühnau, *Bemerkungen zu einigen neueren Normalabbildungen in der Theorie der konformen Abbildung*, Math. Z. **97** (1967), 21–28.
- [54] R. Kühnau, *Über die schlichte konforme Abbildung auf nichtüberlappende Gebiete*, Math. Nachr. **36** (1968), 61–71.
- [55] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien*, J. Reine Angew. Math. **231** (1968), 101–113.
- [56] R. Kühnau, *Einige Extremalprobleme bei differentialgeometrischen und quasikonformen Abbildungen, II*, Math. Z. **107** (1968), 307–318.
- [57] R. Kühnau, *Bemerkungen zu den GRUNSKYschen Gebieten*, Math. Nachr. **44** (1970), 285–293.
- [58] R. Kühnau, *Einige elementare Bemerkungen zur Theorie der konformen und quasikonformen Abbildungen*, Math. Nachr. **45** (1970), 307–316.
- [59] R. Kühnau, *Weitere elementare Bemerkungen zur Theorie der konformen und quasikonformen Abbildungen*, Math. Nachr. **51** (1971), 377–382.
- [60] R. Kühnau, *Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag Wiss., Berlin (1974).
- [61] R. Kühnau, *Zur Methode der Randintegration bei quasikonformen Abbildungen*, Ann. Polon. Math. **31** (1975–1976), 269–289.
- [62] R. Kühnau, *Identitäten bei quasikonformen Normalabbildungen und eine hiermit zusammenhängende Kernfunktion*, Math. Nachr. **73** (1976), 73–106.
- [63] R. Kühnau, *Eine Integralgleichung in der Theorie der quasikonformen Abbildungen*, Math. Nachr. **76** (1977), 139–152.

- [64] R. Kühnau, *Eine Kernfunktion zur Konstruktion gewisser quasikonformer Normalabbildungen*, Math. Nachr. **95** (1980), 229–235.
- [65] R. Kühnau, *Numerische Realisierung konformer Abbildungen durch “Interpolation”*, Z. Angew. Math. Mech. **63** (1983), 631–637 (pp. 632 and 634 are exchanged by mistake!).
- [66] R. Kühnau, *Entwicklung gewisser dielektrischer Grundlösungen in Orthonormalreihen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 165–170; Misprints: After (5) replace $\frac{z}{g_\vartheta(z)} \rightarrow \overline{g_\vartheta(z)}$, in (31) replace $\xi \rightarrow \zeta$, on p. 325, first line, replace $\mathfrak{c} \rightarrow \mathfrak{C}$, before “Satz 6” replace $\mathfrak{C} \rightarrow \lambda$.
- [67] R. Kühnau, *Herbert Grötzsch zum Gedächtnis*, Jahresber. Deutsche Math.-Ver. **99** (1997), 122–145.
- [68] R. Kühnau, *Variation of diametrically symmetric or elliptically schlicht conformal mappings*, J. Anal. Math. **89** (2003), 303–316.
- [69] R. Kühnau, *The conformal module of quadrilaterals and of rings*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier Science, Amsterdam (2005), 99–129 (this Volume).
- [70] R. Kühnau, *Bibliography of Geometric Function Theory (GFT)*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier Science, Amsterdam (2005), 809–828 (this Volume).
- [71] R. Kühnau and H. Renelt, *Ein Existenzbeweis für schlichte Lösungen linearer elliptischer Differentialgleichungssysteme durch eine Integralgleichung*, Math. Nachr. **79** (1977), 225–232.
- [72] M. Lanza de Cristoforis, *Analyticity of a nonlinear operator associated to the conformal representation in Schauder spaces. An integral equation approach*, Math. Nachr. **220** (2000), 59–77.
- [73] M.A. Lavrent’ev, *Variational Methods for Boundary Value Problems for Systems of Elliptic Equations*, Noordhoff, Groningen (1963); Russian orig.: Izdat. Akad. Nauk SSSR, Moscow (1962).
- [74] N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Nauka, Moscow (1972) (in Russian).
- [75] I. Lind, *An iterative method for conformal mappings of multiply-connected domains*, Ark. Mat. **4** (1963), 557–560.
- [76] I. Lind, *Some problems related to iterative methods in conformal mapping*, Ark. Mat. **7** (1967), 101–143.
- [77] G.S. Litvinchuk, *Boundary Value Problems and Singular Integral Equations with Shifts*, Nauka, Moscow (1977) (in Russian).
- [78] F. Maitani and D. Minda, *Rectilinear slit conformal mappings*, J. Math. Kyoto Univ. **36** (1996), 659–668.
- [79] R. Maskus, *Anwendung eines Iterationsverfahrens auf das KOEBESche Geradenschlitztheorem*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **14** (1965), 323–332.
- [80] H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1962).
- [81] I.M. Milin, *Univalent Functions and Orthonormal Systems*, Amer. Math. Soc., Providence, RI (1977); Russian orig.: Nauka, Moscow (1971).
- [82] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York–Toronto–London (1952); Reprint, New York (1975).
- [83] S.V. Parter, *On mappings of multiply connected domains by solutions of partial differential equations*, Commun. Pure Appl. Math. **13** (1960), 167–182.
- [84] U. Pirl, *Zum Normalformenproblem für endlich-vielfach zusammenhängende schlichte Gebiete*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **6** (1956–1957), 799–802.
- [85] U. Pirl, *Ein Abbildungssatz für Gebiete, die in eine gegebene Kreisringfläche schlicht eingelagert sind*, Math. Nachr. **41** (1969), 185–187.
- [86] U. Pirl, *Geometrische Funktionentheorie*, “25 Jahre Mathematik in der DDR”, VEB Deutscher Verlag Wiss., Berlin (1974), 329–332.
- [87] U. Pirl, *Normalformen für endlich-vielfach zusammenhängende in einen Kreisring eingelagerte Gebiete*, Math. Nachr. **76** (1977), 181–194.
- [88] U. Pirl und C. Michel, *Normalformen für schlichte endlich-vielfach zusammenhängende Gebiete bezüglich schlichter konformer Abbildung mit vier Fixpunkten*, Math. Nachr. **95** (1980), 119–133.
- [89] W. Pohl, *Aus Extremalproblemen über dem Wertebereich $f'(P)$ folgende Normalformen dreifach zusammenhängender Gebiete bezüglich einer Klasse schlichter Funktionen, die P und zwei Randkomponenten festhalten*, Dissertation, Math.-Nat. Fakultät Humboldt-Univ., Berlin (1974).
- [90] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin (1992).
- [91] Ch. Pommerenke, *Conformal maps at the boundary*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier Science, Amsterdam (2002), 37–74.

- [92] E. Reich, *A counterexample of Koebe's for slit mappings*, Proc. Amer. Math. Soc. **11** (1960), 970–975.
- [93] E. Reich and S.E. Warschawski, *On canonical conformal maps of regions of arbitrary connectivity*, Pacific J. Math. **10** (1960), 965–985.
- [94] E. Reich and S.E. Warschawski, *Canonical conformal maps onto a circular slit annulus*, Scripta Math. **25** (1960), 137–146.
- [95] H. Renelt, *Extremalprobleme bei quasikonformen Abbildungen unter höheren Normierungen*, Math. Nachr. **66** (1975), 125–143.
- [96] H. Renelt, *Quasikonforme Abbildungen und elliptische Systeme erster Ordnung in der Ebene*, Teubner, Leipzig (1982); English transl.: Wiley, Chichester (1988).
- [97] B. Rodin, L. Sario and M. Nakai, *Principal Functions*, van Nostrand, Toronto–London–Melbourne–Princeton (1968).
- [98] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Springer-Verlag, Berlin–Heidelberg–New York (1970).
- [99] L. Sario and K. Oikawa, *Capacity Functions*, Springer-Verlag, Berlin–Heidelberg–New York (1969).
- [100] M. Schiffer, *Faber polynomials in the theory of univalent functions*, Bull. Amer. Math. Soc. **54** (1948), 503–517.
- [101] M. Schiffer and N.S. Hawley, *Connections and conformal mapping*, Acta Math. **107** (1962), 175–274.
- [102] M. Schiffer and G. Schober, *Representation of fundamental solutions for generalized Cauchy–Riemann equations by quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **2** (1976), 501–531.
- [103] M. Schiffer and D.C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton, NJ (1954).
- [104] O. Schramm, *Transboundary extremal length*, J. Anal. Math. **66** (1995), 307–329.
- [105] O. Schramm, *Conformal uniformization and packings*, Israel J. Math. **93** (1996), 399–428.
- [106] A. Sonnenschein, *Über einige konforme Abbildungen mehrfach zusammenhängender schlichter Bereiche*, Dissertation, Math.-Nat. Abteilung Philosophischen Fakultät der Universität Leipzig (1935).
- [107] K. Stephenson, *Circle packing and discrete analytic function theory*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier Science, Amsterdam (2002), 333–370.
- [108] O. Teichmüller, *Über Extremalprobleme der konformen Geometrie*, Deutsche Math. **6** (1941), 50–77 (also in Gesammelte Abhandlungen).
- [109] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern*, Deutsche Math. **7** (1944), 309–336.
- [110] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo (1959); 2nd edn: Chelsea, New York (1975).
- [111] R. Wagner, *Ein Kontaktproblem der konformen Abbildung*, J. Reine Angew. Math. **196** (1956), 99–132.
- [112] R. Wegmann, *Methods for numerical conformal mapping*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier Science, Amsterdam (2005), 479–506 (this Volume).

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CHAPTER 5

Univalent Holomorphic Functions with Quasiconformal Extensions (Variational Approach)

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Abstract

Univalent holomorphic functions with quasiconformal extensions play a fundamental role in Teichmüller space theory and complex metric geometry of these spaces as well as in geometrical complex analysis. This survey presents the variational theory of univalent functions with quasiconformal extensions and their applications.

0. Introduction

0.1. Interaction between univalent functions and Teichmüller space theory

Univalent holomorphic functions with quasiconformal extensions play a fundamental role in the theory of Teichmüller spaces and in complex metric geometry of these spaces as well as in geometrical complex analysis.

Every conformal structure on a Riemann surface X is determined by a Beltrami differential μ on X . If X is hyperbolic, its universal covering surface can be modeled by the unit disk Δ . One can extend the lifting of μ to Δ by zero on $\Delta^* = \widehat{\mathbb{C}} \setminus \overline{\Delta}$, getting an (injective) conformal map $\Delta^* \rightarrow \widehat{\mathbb{C}}$ with quasiconformal extension onto Δ . This provides the holomorphic Bers embedding of the Teichmüller space $\mathbb{T}(X)$ of X as a bounded domain formed by the Schwarzian derivatives of the corresponding univalent functions $\Delta^* \rightarrow \widehat{\mathbb{C}}$ in the complex Banach space of quadratic differentials on X . In particular, the universal Teichmüller space $\mathbb{T}$ corresponding to $X = \Delta$ is intrinsically connected with the class of all univalent holomorphic functions on Δ or on Δ^* .

The complex analytic theory of Teichmüller spaces has many applications in various fields of mathematics.

0.2. General remarks on analytic functionals

Univalent functions arising in this way can be normalized in a standard way. It is natural to deal with the maps

$$F(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} : \Delta^* \rightarrow \widehat{\mathbb{C}} \setminus \{0\} \quad (0.1)$$

and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{1}{F(1/z)} : \Delta \rightarrow \mathbb{C}. \quad (0.2)$$

Such univalent functions form the well-known classes Σ and S . Quasiconformal homeomorphisms of the plane whose restrictions to Δ^* or Δ are of the form (0.1) and (0.2) constitute the dense subclasses of Σ and S , respectively, in the topology of locally uniform convergence.

An investigation of the functions with k -quasiconformal extensions with a given $k \in (0, 1)$ has an independent interest. The corresponding subclasses of Σ and S are denoted by $\Sigma(k)$ and $S(k)$, provided additionally $F(0) = 0$ and $f(\infty) = \infty$. An extra normalization condition is needed to get the uniqueness of the solutions to the corresponding Beltrami equation $\bar{\partial} w = \mu \partial w$ on $\mathbb{C}$.

Univalent functions (conformal maps) possess various remarkable features due to their global injectivity. These properties can be expressed qualitatively by estimating suitable

functionals. Such functionals usually have some geometric or physical sense. The Taylor coefficients provide a canonical example of such functionals.

An investigation of the extremal problems on various classes of univalent functions have a long history and still occupy a prominent place in Geometric Function Theory. Different powerful methods provided by the theory of holomorphic functions allow us either to find the desired quantitative estimates for a given functional or in many cases reduce this to evaluating a finite number of constants (similar to the Christoffel–Schwarz integral). Some of these methods will be mentioned briefly below.

Various aspects of the theory of univalent functions are presented, for example, in the books [Ah5, Al, Du, Go, Goo, Je, Po1, Po2].

0.3. *Remarks on variational methods*

The variational methods play an important role in many fields of mathematics. These methods provide both qualitative description of extremals and the desired quantitative estimates and give in many cases the complete solutions of the extremal problems.

These methods also continue to be the most powerful in the theory of univalent functions. For such functional classes, the methods of variations were developed in the classical works of Hadamard, Schiffer, Lavrentiev, Goluzin and many other mathematicians.

0.4. *New phenomena*

Univalent functions with quasiconformal extensions are interesting also in their own right. One can consider the more general case of quasiconformal maps of plane regions which are conformal (in the sense that $\bar{\partial}w = 0$) on arbitrary subsets of these domains. This enables us to connect variational problems for quasiconformal maps with variational problems for univalent functions. A study of this connection is of great interest. For example, the solutions of various extremal problems for the normalized conformal maps of a plane region D (or of a region on an arbitrary Riemann surface) with Jordan boundary could be obtained as the limit of solutions of somewhat analogous problems in the appropriate classes of k -quasiconformal homeomorphisms of the whole plane (or surface) that are conformal in this region, letting k approach 1.

Another reason is that quasiconformal maps play an important role in the study of rather general elliptic partial differential equations. The estimates obtained for quasiconformal extensions can be generalized to homeomorphic extensions of more general type, in particular, to quasiconformal homeomorphisms whose dilatations are bounded by a nonconstant function $k_0(z) \geq 0$ with $\|k_0\|_\infty < 1$.

A new phenomenon which has origins in the general theory of quasiconformal maps concerns certain natural problems involving quasiconformal homeomorphisms with an extremely different normalization. Namely, one can consider the families of univalent functions with quasiconformal extensions fixing the values of the maps and of their derivatives of prescribed orders at a finite number of distinguished points. Such families are not empty for the suitable choices of those fixed values. Solving the extremal problems in such classes

requires the existence of admissible variations. The latter involve quasiconformality in an essential way.

Such a situation is impossible in general for holomorphic functions. For example, due to Cartan's uniqueness theorem (and by Schwarz's lemma), any holomorphic map f of a bounded domain $D \subset \mathbb{C}$ into itself with a fixed point z_0 in D at which $f'(z_0) = 1$, i.e., such that

$$f(z) = z_0 + f'(z_0)(z - z_0) + O((z - z_0)^2) \quad \text{near } z_0,$$

is reduced to the identity map (see, e.g., [Ru]).

0.5. Grunsky coefficients

The Grunsky operator $\mathcal{G} = (\alpha_{mn}(f))_{m,n=1}^\infty$, defined on both classes Σ and S , has become an important and successful tool in various topics of Geometric Function Theory based on the necessity and sufficiency of the Grunsky inequalities for univalence of a holomorphic function. This operator naturally relates also to the Fredholm eigenvalues and to geometric features of the boundary curves.

The Grunsky coefficients defining holomorphic functions on the universal Teichmüller space connect geometric function theory with the theory of Teichmüller spaces. For example, these coefficients intrinsically relate to complex metric geometry of universal Teichmüller space $\mathbb{T}$.

The interaction of both these theories provides a new fruitful approach also to classical problems of Geometric Function Theory. For example, the solution of the Kühnau–Niske problem on the bounds for Taylor coefficients $a_n(f)$ of univalent functions f with k -quasiconformal extensions is based on the important fact that, in contrast to $a_n(f)$, the Grunsky coefficients $\alpha_{mn}(f)$ of f are intrinsically connected with the complex geodesics in the universal Teichmüller space, acting naturally on the extremal holomorphic disks $\Delta_\mu = \{\phi_{\mathbb{T}}(t\mu/\|\mu\|_\infty) : t \in \Delta\}$, while each $a_n(f^{t\mu})$ ranges on a holomorphic, in general not geodesic, disk in $\mathbb{T}$ (see Section 5).

0.6. Related quadratic differentials

Another general principle in the theory of conformal and quasiconformal maps discovered by Teichmüller (see [Te2,Je]) is that solutions of the extremal problems relate to holomorphic quadratic differentials.

This survey presents different aspects in geometric theory of univalent functions having quasiconformal extensions and results produced by interaction of this theory with the methods of the Teichmüller space theory. We present also the different approach which relies on the alternative variational methods. Some of these methods are direct extensions of the classical methods in the theory of univalent functions.

The last section is devoted to geometric features of Teichmüller spaces which rely on the intrinsic properties of univalent functions. The paper can be regarded as a continuation of [Kru23].

1. The existence theorems for special quasiconformal deformations: Old and new

The purpose of this section is to prove the existence of special kinds of quasiconformal variations of the complex plane, which are conformal outside of a given set and satisfy certain prescribed properties. First of all, these results illustrate that dropping holomorphicity of the maps even on a thin set changes completely the features and eliminates strong rigidity, which intrinsically characterizes the holomorphic maps. On the other hand, these key theorems ensure the nonemptiness of various classes of holomorphic maps with quasiconformal extension.

1.1. Two local theorems

THEOREM 1.1 [Kru5, Chapter 4]. *Let D be a simply connected domain on the Riemann sphere $\widehat{\mathbb{C}}$. Assume that there are a set E of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, \dots, z_n$ distinguished in D . Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be nonnegative integers assigned to $z_1, z_2, \dots, z_n$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.*

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$ and for any given collection of numbers w_{sj} , $s = 0, 1, \dots, \alpha_j$, $j = 1, 2, \dots, n$, which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \leq \varepsilon, \quad |w_{1j} - 1| \leq \varepsilon, \quad |w_{sj}| \leq \varepsilon, \quad s = 0, 1, \dots, \alpha_j, \quad j = 1, \dots, n,$$

there exists a quasiconformal self-map h of D which is conformal on $D \setminus E$ and satisfies

$$h^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, \dots, \alpha_j, \quad j = 1, \dots, n.$$

Moreover, the Beltrami coefficient $\mu_h(z) = \partial_{\bar{z}}h/\partial_z h$ of h on E satisfies $\|\mu_h\|_\infty \leq M\varepsilon$. The constants ε_0 and M depend only upon the sets D , E and the vectors $(z_1, \dots, z_n)$ and $(\alpha_1, \dots, \alpha_n)$.

If the boundary ∂D is Jordan or is $C^{l+\alpha}$ -smooth, where $0 < \alpha < 1$ and $l \geq 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \leq l$, respectively.

This is a special case of a general theorem for the Riemann surfaces of a finite analytical type proved in [Kru5, Chapter 5]. In particular, it extends Theorem 1.1 to finitely connected Jordan subdomains of $\widehat{\mathbb{C}}$.

To formulate the general theorem, let us consider the *marked* Riemann surfaces of finite conformal type (g, h) , i.e., closed Riemann surfaces of genus g with n punctures determined by their canonical representations in $SL(2, \mathbb{C})$ up to isomorphisms which correspond to homotopies of the surfaces fixing the punctures. Here $g \geq 0$, $n \geq 0$ and $m = 3g - 3 + n > 0$. As conformal moduli of these surfaces, we take any local complex coordinates $\tau = (\tau_1, \dots, \tau_m)$ in the neighborhoods of the corresponding points of the Teichmüller space $T(g, n)$, which define its complex holomorphic structure. All such coordinates are locally holomorphically equivalent.

Let X and X' denote two similarly oriented homeomorphic marked Riemann surfaces of the type (g, n) , on which the uniformizing complex parameters w and ζ are determined,

which vary on the universal coverings $\tilde{X}$ and $\tilde{X}'$ of these surfaces, respectively. Let E denote a set of positive two-dimensional Lebesgue measures on the surface X .

Suppose that we are given the divisor $\mathbf{a} = \sum_{j=1}^h \alpha_j p_j$, where $p_j \in X$ and $\alpha_j \geq 0$ are integers. Let $w_1, \dots, w_k$ denote fixed values of the parameter w corresponding to the points $p_1, \dots, p_k$.

THEOREM 1.2. *Suppose that the surfaces X and X' have moduli $\tau = (\tau_1, \dots, \tau_m)$ and $\tau' = (\tau'_1, \dots, \tau'_m)$, respectively, and that $|\tau - \tau'| < \varepsilon$. Let $\zeta_{s,j}$, for $j = 1, \dots, k$ and $s = 0, 1, \dots, \alpha_j$, denote numbers such that $\zeta_{0j} \in X$, $|\zeta_{0j} - w_j| < \varepsilon$, $|\zeta_{1j} - 1| < \varepsilon$, and $|\zeta_{sj}| < \varepsilon$ for $f = 1, \dots, k$ and $s = 2, \dots, \alpha_j$. Then for sufficiently small ε in $(0, \varepsilon_0)$, there exists a Beltrami differential $\mu(w) d\bar{w}/dw$ on X such that $\mu(w) = 0$ on E and a quasiconformal homeomorphism $\zeta = f(w)$ with the Beltrami coefficient $\mu_f(w) = \mu(w)$ is a map of X onto X' with the following properties: $f^{(s)}(w_j) = \zeta_{sj}$ (for $j = 1, \dots, k$ and $s = 0, 1, \dots, \alpha_j$) and $\|\mu\|_\infty \leq M\varepsilon$, where the constants ε_0 and M depend only on X, E and $\mathbf{a}$.*

1.2. Sketch of the proof of Theorem 1.1

This proof provides an upper bound for ε_0 .

By applying additional conformal maps, the proof reduces to examination of two cases: (a) D is the plane $\mathbb{C}$ and $f(\infty) = \infty$; (b) D is the half-plane $\text{Im } z > 0$ and $f(\infty) = \infty$. We may also assume without loss of generality that $D \setminus E \neq \emptyset$, that the set E is bounded, that the points z_j (for $j = 1, \dots, n$) do not belong to E , and that $\min \rho(z_j, E') = \rho_0 > 0$.

Assuming $D = \widehat{\mathbb{C}}$, $f(\infty) = \infty$, we define for $\rho \in L_p(E)$, $p \geq 2$, the operators

$$T_E \rho = -\frac{1}{\pi} \iint_E \frac{\rho(\xi) d\xi d\eta}{\zeta - w}, \quad \Pi \rho = \partial_w T_E = \frac{1}{\pi} \iint_E \frac{\rho(\xi) d\xi d\eta}{(\zeta - w)^2} \quad (1.1)$$

(the second integral exists as a principal Cauchy value). We seek the required automorphism $f = f^\mu$ of the form $f(z) = z + T\rho_E(z)$, with the Beltrami coefficient $\mu = \mu_f$ supported in E . Then

$$\rho = \mu + \mu \Pi \mu + \mu \Pi (\mu \Pi \mu) + \dots \in L_p(E)$$

for some $p > 2$. By virtue of the properties of the operators T_E and Π , we have, for $\|\mu\|_\infty < \varepsilon_1 < 1$,

$$h(w) = w + T\mu(w) + \omega(w) \quad \text{with } \|\omega\|_{C(\Delta_{R'})} \leq M_1(\kappa, R) \|\mu\|_\infty^2, R < \infty.$$

Fix $\varepsilon_1 < 1$ and $R > \max(\sup_{z \in E_0} |z|, \max_j |z_j|)$ and put

$$\left. \frac{d^s}{dz^s} (f(z) - z) \right|_{z=z_j} = w'_{sj}.$$

Then

$$w'_{sj} = -\frac{s!}{\pi^{-1}} \iint_{E_0} \mu(\zeta)(\zeta - z_j)^{-s-1} d\xi d\eta + \omega^{(s)}(z_j),$$

where $s = 0, 1, \dots, \alpha_j$, $j = 1, 2, \dots, n$. This system of equations defines a nonlinear operator

$$W\mu = H\mu + \Omega\mu, \quad (1.2)$$

where $W\mu = (w'_{m,j})$, $H\mu = (h^{(m)}(z_j))$, $\Omega\mu = (\omega^{(m)}(z_j))$ are the d -component (complex) vectors, $d = n + \sum_{j=1}^n \alpha_j$, acting on the set $\{\mu: \|\mu\|_{L_\infty(D)} < \varepsilon_1, \mu(z) = 0 \text{ for } z \in D \setminus E\}$, and H is the Fréchet derivative of W .

On the linear complex span $A(E)$ of the functions $\varphi_{s,j}(\zeta) = (\bar{\zeta} - \bar{z}_j)^{-s-1}$, $\zeta \in E$, vanishing for $\zeta \in D \setminus E$ ($s = 0, 1, \dots, \alpha_j$, $j = 1, 2, \dots, n$), we introduce the norm $\|\varphi\|_{A(E)} = \|\varphi\|_{L_\infty(D)}$ and shall consider only $\mu(\zeta) \in A(E)$, with $\|\mu\| < \varepsilon_1$. Substituting $\mu(\zeta) = \sum_{j=1}^n \sum_{s=0}^{\alpha_j} c_{sj} \varphi_{sj}(\zeta)$ with unknown constants c_{sj} into the equation $H\mu = a$ for prescribed $a \in C^d$, one obtains for the determination of the $c_{s,j}$ a linear algebraic system of equations, the determinant of which equals $\gamma \det(\iint_E \phi_{s,j} \bar{\phi}_{k,r} d\xi d\eta)$ with $\gamma \neq 0$, and differs from zero because of the linear independence of the functions $\phi_{m,j}(\zeta)$. Hence this system has a unique solution and, therefore, the operator $H: A(E) \rightarrow C^d$ is uniquely invertible in C^d . Then from (1.2), we have for the desired quantity $\mu(z)$ the operator equation $\mu = -H^{-1}\Omega\mu + v$ ($v = H^{-1}a$), where a is the vector with the given components $w'_{m,j}$.

Assume now that $C_1 = \|H^{-1}\|_{C^d}$, $C_2 = w\sqrt{d}$ and choose $\varepsilon_2 < \min\{(\varepsilon_1/2C_2), 1/4 \times C_1 C_2^2\}$ and $\varepsilon_0 = \varepsilon_2/\sqrt{d}$. Then provided that $\max_{m,j} |w'_{m,j}| < \varepsilon_0$, i.e., $|q| < \varepsilon_2$, the continuous operator $Q\mu = -H^{-1}\Omega\mu + v$ maps the closed convex set $B_{M_2\varepsilon_0}$ into itself. Therefore, due to the well-known Bol-Brouwer theorem (see, e.g., [LS, p. 507]), the map $\lambda = Q\mu$ has a fixed point μ_0 in $B_{M\varepsilon}$; i.e., $\mu_0 = -H^{-1}\Omega\mu_0 + v$. The automorphism $f(z)$ of the plane $\widehat{\mathbb{C}}$ with this Beltrami coefficient μ_0 satisfies the assertion of Theorem 1.1.

When D is the upper half-plane $\{\text{Im } z > 0\}$, we continue the desired homeomorphism $w = f(z)$ by the symmetry into the lower half-plane $D^* = \{\text{Im } z < 0\}$, obtaining $f(\bar{z}) = \overline{f(z)}$, whence $\mu(\bar{z}) = \overline{\mu(z)}$, and along with initial conditions in the points z_j the condition $f^{(m)}(\bar{z}_j) = \bar{w}_{m,j}$ for the same m and j must be satisfied. Accordingly, we take $h(z) = z + T_{E \cup E^*}\mu$, where $E^* = \{\zeta \in D^*: \bar{\zeta} \in E\}$, and consider now the vector space $A(E)$ spanned by the functions

$$\begin{aligned} \varphi_{m,j}(\zeta) &= (\bar{\zeta} - \bar{z}_j)^{-m-1} \quad \text{and} \quad \phi_{m,j}(\zeta) = (\bar{\zeta} - z_j)^{-m-1}, \quad \zeta \in E \cup E^*, \\ \varphi_{m,j}(\zeta) &= \phi_{m,j}(\zeta) = 0 \quad \text{outside } E \cup E^*, \quad m = 0, 1, \dots, \alpha_j, \quad j = 1, \dots, n, \end{aligned}$$

with the norm $\|\phi\|_{A(E)} = \|\phi\|_{L_\infty(E \cup E^*)}$. Applying the previous arguments to

$$\mu(\zeta) = \sum_{j=1}^n \sum_{s=0}^{\alpha_j} [c_{s,j} \varphi_{s,j}(\zeta) + \overline{c_{s,j}} \phi_{s,j}(\zeta)]$$

one obtains the assertion of Theorem 1.2 for the given case.

Finally, the case of the boundary points reduces, by applying conformal maps of D onto the upper half-plane, to constructing an automorphism of the half-plane with added prescribed values w_{sj} at several points $z_j \in \mathbb{R}$ so that $\operatorname{Im} w_{sj} = 0$.

1.3. Quasiconformal deformations decreasing L_p -norm

Most of the adopted methods for solving variational problems in the Banach spaces of holomorphic functions in the disk (or in other subdomains of $\widehat{\mathbb{C}}$) use in an essential way the integral representations of these functions by means of corresponding measures. These applications usually involve great difficulties, especially in the case when a problem admits several local extrema.

We provide an alternative approach which will be illustrated in the case of some spaces $L_p(G)$, $G \subset \widehat{\mathbb{C}}$. It relies on constructing quasiconformal deformations h satisfying $\|h \circ f\|_p \leq \|f\|_p$ for corresponding $f \in L_p(G)$ as well as some other prescribed conditions. We set

$$A_p(G) = \{f \in L_p: f \text{ holomorphic on } G\}, \quad \|f\|_{A_p} = \|f\|_p, \quad 1 < p < \infty,$$

where G is a ring domain bounded by a curve $L \subset \Delta$ and by the unit circle $S^1 = \partial\Delta$. The degenerated cases $E = \Delta \setminus \{0\}$ and $E = S^1$ correspond to the Bergman space B^p and the Hardy space H^p , respectively. Let $\mathbf{d}^0 = (0, 1, \dots, 0) =: (d_k^0) \in \mathbb{R}^{n+1}$.

THEOREM 1.3 [Kru19]. *Given a function $f_0(z) = \sum_{k=j}^{\infty} c_k^0 z^k \in H^{2m} \cap H^\infty$ (with $c_j^0 \neq 0$, $0 \leq j < n$, $m \in \mathbb{N}$), which is not a polynomial of degree $s \leq n$, then there exists a positive number ε_0 so that, for every point $\mathbf{d}' = (d'_{j+1}, \dots, d'_n) \in \mathbb{C}^{n-j}$ and every $a \in \mathbb{R}$ satisfying $|\mathbf{d}'| \leq \varepsilon$, $|a| \leq \varepsilon$ with $\varepsilon < \varepsilon_0$, there is a quasiconformal automorphism h of $\widehat{\mathbb{C}}$, which is conformal at least in the disk $D_0 = \{w: |w - c_0^0| < \sup_\Delta |f_0| + |c_0^0| + 1\}$, and satisfies*

- (i) $h^{(k)}(c_0^0) = k!d_k = k!(d_k^0 + d'_k)$, $k = j+1, \dots, n$ (i.e., $d_1 = 1 + d'_1$ and $d_k = d'_k$ for $k \geq 2$); in other words,

$$\begin{aligned} h(w) &= d_0 + d_j(w - c_0^0)^j + \dots + d_n(w - c_0^0)^n \\ &\quad + d_{n+1}(w - c_0^0)^{n+1} + \dots, \quad w \in D_0, \end{aligned}$$

with given $d_{j+1}, \dots, d_n$, and

- (ii) $\|h \circ f_0\|_{2m}^{2m} = \|f_0\|_{2m}^{2m} + a$.

The map h can be chosen to have the Beltrami coefficient $\mu_h = \partial_{\bar{w}}h/\partial_w h$ with $\|\mu_h\|_\infty \leq M\varepsilon$. The quantities ε_0 and M depend only on f_0 , m and n .

In particular, Theorem 1.3 ensures the existence of quasiconformal homeomorphisms h with $\|h \circ f_0\|_{2m} \leq \|f_0\|_{2m}$; moreover, one can do this by varying independently also a well-defined finite number of the Taylor coefficients of the maps h and f_0 .

This key theorem can be applied to solve some coefficient conjectures for nonvanishing holomorphic functions.

SKETCH OF THE PROOF OF THEOREM 1.3. Fix $R \geq \sup_{\Delta} |f_0| + |c_0| + 1$ and take the annulus $E = \{w: R < |w - c_0^0| < R + 1\}$. Similarly to Theorem 1.1, we again seek the required automorphism $h = h^\mu$ of the form

$$h(w) = w - \frac{1}{\pi} \iint_E \frac{\rho(\zeta) d\xi d\eta}{\zeta - w} = w + T\mu(w) + \omega(w) \quad (1.3)$$

with the Beltrami coefficient $\mu = \mu_h$ supported in E and $\|\mu\|_\infty < \kappa < 1$. Then in (1.3), $\|\omega\|_{C(\Delta_{R'})} \leq M_1(\kappa, R') \|\mu\|_\infty^2$ for any $R' < \infty$.

We will now essentially use the property of quasiconformal maps that if $\mu(z; t)$ is a C^1 -smooth $L_\infty(\mathbb{C})$ function of a real (respectively complex) parameter t , then $\partial_w h^{\mu(\cdot, t)}$ and $\partial_{\bar{w}} h^{\mu(\cdot, t)}$ are smoothly $\mathbb{R}$ -differentiable (respectively, $\mathbb{C}$ -differentiable) L_p functions of t , and, consequently, the function $t \mapsto h^{\mu(\cdot, t)}(z)$ is C^1 -smooth as an element of $C(\Delta_{R'})$ for any $R' < \infty$.

Letting

$$\langle v, \varphi \rangle = \frac{1}{\pi} \iint_E v(\zeta) \varphi(\zeta) d\xi d\eta, \quad v \in L_\infty(E), \quad \varphi \in L_1(E),$$

the representation (1.3) results in

$$h(w) = w + \sum_1^\infty \langle \mu, \varphi_k \rangle (w - c_0^0)^k + \omega(w), \quad \varphi_k(w) = \frac{1}{(w - c_0^0)^{k+1}},$$

which provides the first group of equalities to be satisfied by the desired Beltrami coefficient μ :

$$k!d_k = \langle \mu, \varphi_k \rangle + \omega^{(k)}(c_0^0) = \langle \mu, \varphi_k \rangle + O(\|\mu\|_\infty^2), \quad k = j + 1, \dots, n. \quad (1.4)$$

On the other hand, combining the previous representation of h with (ii), we get

$$\|h \circ f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \operatorname{Re} \langle \mu, \phi \rangle + O_m(\|\mu\|_\infty^2), \quad (1.5)$$

where

$$\phi(\zeta) = -m \int_G \frac{|f_0(w)|^{2m-2} \overline{f_0(w)}}{f_0(w) - \zeta} dG_w. \quad (1.6)$$

The function ϕ is holomorphic in the disk $D_R^* = \{w \in \widehat{\mathbb{C}}: |w - c_0^0| > R\}$. It belongs to the subspace A_{2m}^0 formed in $A_{2m}(G)$ by holomorphic functions φ in D_R^* , and $\phi(z) \not\equiv 0$.

A rather complicated analysis involving the variational technique and Parseval's equality for the functions $f_0(z)^n$ implies that under the assumptions of Theorem 1.3, the function ϕ

does not reduce to a linear combination of fractions $\varphi_0, \dots, \varphi_l$ with $l \leq n$. Therefore the remainder

$$\begin{aligned}\psi(z) &= \phi(\zeta) - \sum_0^n b_k(\zeta - c_0^0)^{-k-1} \\ &= \left(\sum_0^{j-1} + \sum_s^\infty \right) b_k(\zeta - c_0^0)^{-k-1}, \quad s \geq n+1,\end{aligned}$$

does not vanish identically in Δ_R^* .

We now choose the desired Beltrami coefficient μ of the form

$$\mu = \sum_0^n \xi_k \bar{\varphi}_k + \xi_s \bar{\varphi}_s + \tau \bar{\psi}, \quad \mu|_{\mathbb{C} \setminus E} = 0, \quad (1.7)$$

with unknown constants $\xi_j, \xi_{j+1}, \dots, \xi_n, \tau$ to be determined from equalities (1.4) and (1.5). Substituting the expression (1.7) into (1.4) and (1.5) and taking into account the mutual orthogonality of φ_k on E , we obtain for determining ξ_k and τ nonlinear equations

$$k!d_k = \xi_k r_k^2 + O(\|\mu\|^2), \quad k = j+1, \dots, n, \quad (1.8)$$

where $r_k^2 := \langle \bar{\varphi}_k, \varphi_k \rangle$, and

$$\|h \circ f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \operatorname{Re} \left\langle \xi_j \bar{\varphi}_j + \sum_0^n \xi_k \bar{\varphi}_k + \tau \bar{\psi}, \phi \right\rangle + O(\|\mu\|^2). \quad (1.9)$$

The only remaining equation is a relation for $\operatorname{Re} \xi_s, \operatorname{Im} \xi_s, \operatorname{Re} \tau, \operatorname{Im} \tau$. To distinguish a unique solution, we add to (1.8) and (1.9) three real equations. Namely, we will seek ξ_s satisfying $\langle \xi_s \bar{\varphi}_s + \sum_0^n \xi_k \bar{\varphi}_k, \sum_0^s b_k \varphi_k \rangle = 0$, and let τ be real. Then (1.9) is reduced to

$$\|h \circ f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m} = \tau \sum_k r_k^2 + O(\|\mu\|^2). \quad (1.10)$$

Separating the real and imaginary parts in all above equations, one obtains $2(n-j)+1$ real equalities, which define a nonlinear C^1 -smooth (in fact, $\mathbb{R}$ -analytic) map

$$\mathbf{y} = W(\mathbf{x}) = W'(\mathbf{0})\mathbf{x} + O(|\mathbf{x}|^2)$$

of the points $\mathbf{x} = (\operatorname{Re} \xi_j, \operatorname{Im} \xi_j, \dots, \operatorname{Re} \xi_n, \operatorname{Im} \xi_n, \tau)$ in a small neighborhood U_0 of the origin in $\mathbb{R}^{2(n-j)+1}$, taking the values

$$\mathbf{y} = (\operatorname{Re} d_j, \operatorname{Im} d_j, \dots, \operatorname{Re} d_n, \operatorname{Im} d_n, \operatorname{Re} d_s, \operatorname{Im} d_s, \|h \circ f_0\|_{2m}^{2m} - \|f_0\|_{2m}^{2m})$$

also near the origin of $\mathbb{R}^{2(n-j)+1}$.

Its linearization $\mathbf{y} = W'(\mathbf{0})\mathbf{x}$ defines a linear map $\mathbb{R}^{2(n-j)+1} \rightarrow \mathbb{R}^{2(n-j)+1}$ whose Jacobian equals up to a constant factor to $r_j^2 \cdots r_n^2 \sum_k r_k^2 \neq 0$. Therefore, $\mathbf{x} \mapsto W'(\mathbf{0})\mathbf{x}$ is a linear isomorphism of $\mathbb{R}^{2(n-j)+1}$ onto itself, and one can apply to W the inverse mapping theorem, which implies the assertion of Theorem 1.3. $\square$

The boundedness of f_0 can be replaced by the much general assumptions to get that covector (1.6) is not degenerate.

1.4. Finite boundary interpolation by univalent functions

We mention also the following nice theorem which is useful for quasiconformal extensions of univalent holomorphic functions and for approximation of Teichmüller spaces.

THEOREM 1.4. *Given two collections of distinct points $z_1, \dots, z_n \in S^1 = \partial \Delta$ and $w_1, \dots, w_n \in S^1$ ordered cyclically, there exists a function f , $f(0) = 0$, univalent and analytic in the closed disk $\bar{\Delta}$ and such that $f(z_j) = w_j$ for all $j = 1, \dots, n$, and if $z \in S^1 \setminus \{z_1, \dots, z_n\}$, then $|f(z)| < 1$.*

The proof of this theorem see in [CHMG] and [MGT]. It was applied to different questions in [KG, Vel2].

A similar assertion was announced in [Sa].

2. Grunsky coefficient inequalities, Carathéodory metric, Fredholm eigenvalues and asymptotically conformal curves

The Grunsky operator (matrix) $\mathcal{G} = (\alpha_{mn})_{m,n=1}^\infty$ has become an important and successful tool in various topics of Geometric Function Theory based on the necessity and sufficiency of the Grunsky inequalities for univalence of a holomorphic function. This operator naturally relates to the Fredholm eigenvalues and to geometric features of the boundary curves.

2.1. Main theorem

We shall use the following notations. Let D be a hyperbolic simply connected Jordan domain in $\widehat{\mathbb{C}}$. We shall regard the functions $\mu \in L_\infty(D)$ and $\varphi \in L_1(D)$, respectively, as the Beltrami $(-1, 1)$ -forms and integrable quadratic differentials supported on D and define for a holomorphic map g of the unit disk Δ into D the induced measurable forms on Δ :

$$g^* \mu = (\mu \circ g) \frac{\bar{g}'}{g'}, \quad g^* \varphi = (\mu \circ g) (g')^2.$$

We denote by $A_1(D)$ the subspace of $L_1(D)$ formed by holomorphic functions in D , and put

$$\langle \mu, \varphi \rangle_D = \iint_D \mu(z) \varphi(z) dx dy, \quad \mu \in L_\infty(D), \varphi \in L_1(D); z = x + iy,$$

and

$$A_1^2(D) = \{\varphi \in A_1(D): \varphi = \omega^2, \omega \text{ holomorphic}\};$$

the last set consists of the integrable of holomorphic functions in D with zeros of even orders in D . Let $\mathbb{B}(D)$ denote the Banach space of hyperbolically bounded holomorphic functions in D with the norm

$$\|\varphi\|_{\mathbb{B}(D)} = \sup_D \lambda_D^{-2}(z) |\varphi(z)|,$$

where $\lambda_D(z)|dz|$ is the hyperbolic metric on D of curvature -4 .

The well-known Grunsky univalence criterion [Gru1] says that a $\widehat{\mathbb{C}}$ -holomorphic function

$$f(z) = z + b_0 + b_1 z^{-1} + \dots \quad (2.1)$$

in a neighborhood of the point at infinity extends to an injective holomorphic function in the disk

$$D^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}: |z| > 1\}$$

if and only if its *Grunsky coefficients* $\alpha_{mn}(f)$, defined from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta^*)^2,$$

satisfy for any $\mathbf{x} = (x_1, x_2, \dots) \in l^2$ the inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq \|\mathbf{x}\|^2. \quad (2.2)$$

Here the single-valued branch of the logarithmic function is chosen, which vanishes as $z = \zeta \rightarrow \infty$, and $\|\mathbf{x}\| = (\sum_1^{\infty} |x_n|^2)^{1/2}$. Note that f does not vanish in Δ^* .

It is known also that for the functions (2.1) having k -quasiconformal extensions from Δ^* onto $\widehat{\mathbb{C}}$, the inequality (2.2) is sharpened as follows:

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right| \leq k \|\mathbf{x}\|^2; \quad (2.3)$$

on the other hand, any holomorphic function f in Δ^* satisfying (2.3) is univalent and quasiconformally extendible to $\widehat{\mathbb{C}}$ with a dilatation $k' \geq k$ (see, e.g., [Ku7, Po1]; [KK1, pp. 82–84]).

We shall denote the class of univalent functions in Δ^* of the form (2.1) which do not share there the value $w = 0$ by Σ , and let $\Sigma(k)$ consist of $f \in \Sigma$ having k -quasiconformal extensions to $\widehat{\mathbb{C}}$.

First we give a complete description of the class of functions for which the value

$$\kappa(f) = \sup \left\{ \left| \sum_{m,n=1}^{\infty} \sqrt{mn} c_{mn} x_m x_n \right| : \mathbf{x} \in l^2, \|\mathbf{x}\| = 1 \right\} \quad (2.4)$$

(called the *Grunsky constant* of f) coincides with the least (extremal) dilatation $k(f)$ among the possible extensions of f , i.e., the inequality (2.3) is both necessary and sufficient to have a k -quasiconformal extension. This solves the question which was raised by different authors starting from [Ku7].

THEOREM 2.1. *The equality*

$$\kappa(f) = \inf \{ \|\mu\|_{\infty} : w^{\mu}|_{\Delta^*} = f \} \quad (2.5)$$

holds if and only if the function f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with the Beltrami coefficient μ_0 satisfying the condition

$$\sup |\langle \mu_0, \varphi \rangle_{\Delta}| = \|\mu_0\|_{\infty}, \quad (2.6)$$

where supremum is taken over the set C^0 of holomorphic functions $\varphi \in A_1^2(\Delta)$ with norm $\|\varphi\|_{A_1(\Delta)} = 1$.

This result is established in [Kru6, Kru11]. Note that the elements of C^0 are of the form

$$\varphi(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2} \quad (2.7)$$

with $\mathbf{x} = (x_1, x_2, \dots) \in l^2$, $\|\mathbf{x}\| = 1$.

In particular, for any function $f \in \Sigma$ having k -quasiconformal extension onto $\overline{\Delta}$ with the Beltrami coefficient μ of the form

$$\mu(z) = k |\varphi(z)| / \varphi(z), \quad \varphi \in A_1^2(\Delta) \setminus \{0\}, \quad (2.8)$$

we have equality

$$\kappa(f) = k. \quad (2.9)$$

Reich established that condition (2.6) plays a crucial role also in another problem (see [Re3]).

2.2. Geometric features

From a geometric point of view, the above theorems are intrinsically connected with invariant metrics on the universal Teichmüller space $\mathbb{T}$. Recall that this space can be modeled as a bounded domain in the complex Banach space $\mathbb{B} = \mathbb{B}(\Delta^*)$ of hyperbolically bounded functions with the norm

$$\|\psi\|_{\mathbb{B}} = \sup_{\Delta^*} (|z|^2 - 1)^2 |\psi(z)|,$$

and the points of this domain are the Schwarzian derivatives

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2, \quad z \in \Delta^*,$$

of the maps $f \in \bigcup_k \Sigma(k)$. The space $\mathbb{T}$ is obtained from the Banach ball

$$\text{Belt}(\Delta)_1 = \{ \mu \in L_\infty(\mathbb{C}): \mu|_{\Delta^*} = 0, \|\mu\| < 1 \}$$

by the equivalence relation identifying the Beltrami coefficients $\mu(z) = \partial_{\bar{z}} w / \partial_z w$ of quasiconformal homeomorphisms $w = w^\mu$ of $\widehat{\mathbb{C}}$ with the same Schwarzian $S_{w^\mu}|_{\Delta^*}$. The map

$$\phi_{\mathbb{T}}: \mu \mapsto S_{w^\mu}|_{\Delta^*}, \quad \text{Belt}(\Delta)_1 \rightarrow \mathbb{T} \subset \mathbb{B}$$

is surjective and holomorphic.

Every Beltrami coefficient $\mu \in \text{Belt}(\Delta)_1$ defines a conformal structure on the disk Δ and on the sphere $\widehat{\mathbb{C}}$, i.e., a vector field of infinitesimal ellipses, or equivalently, a class of conformally equivalent Riemannian metrics $ds^2 = \lambda(z)|dz + \mu d\bar{z}|^2$, $\lambda(z) > 0$. Thus the ball $\text{Belt}(\Delta)_1$ can be regarded as the space of all conformal structures on $\widehat{\mathbb{C}}$ without their additional identification. For more details see Section 7.

Note that the coefficients $\alpha_{mn}(f)$ and, hence, the sums in the left-hand side in (2.3) depend holomorphically on the Schwarzian derivatives $\psi = S_f \in \mathbb{B}$, because each α_{mn} is a polynomial of the first Taylor coefficients $b_1, \dots, b_p$, $p \leq \min(m, n)$, which are holomorphic functions of ψ . Using the holomorphic maps $\mathbb{T} \rightarrow \mathbb{C}$, determined through the sums in (2.3), one obtains that geometrically Theorem 2.1 means that the Carathéodory metric on the immersion of holomorphic disk $\{t\mu_0: t \in \Delta\} \subset L_\infty(\Delta)$ into the universal Teichmüller space $\mathbb{T}$, i.e., on the Teichmüller extremal disk

$$\Delta_{\mu_0} = \{ \phi_{\mathbb{T}}(t\mu_0): t \in \Delta \} \subset \mathbb{T},$$

coincides with the intrinsic Teichmüller metric of this space (see Section 4). Moreover, it turns out that the maps mentioned above provide the maximizing sequences for the Carathéodory distances $c_{\mathbb{T}}(\mathbf{0}, \phi_{\mathbb{T}}(t\mu_0))$.

The universal Teichmüller space and its invariant metrics are intimately connected with the subjects of Geometric Function Theory. We shall illustrate this below.

The following important consequence of Theorem 2.1 concerns the Carathéodory metric of the space $\mathbb{T}$.

THEOREM 2.2. *Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain with quasiconformal boundary, and let the function $\varphi \in A_1(D) \setminus \{0\}$ have in D zeros of even order only. Then in the holomorphic disk*

$$\Delta_\varphi = \left\{ \phi \left(t \frac{|\varphi \circ g| |g'|^2}{(\varphi \circ g) g'^2} \right) : t \in \Delta \right\} \subset \mathbb{T}, \quad (2.10)$$

where g is a conformal map of the unit disk Δ onto D , the Carathéodory metric and the Teichmüller–Kobayashi metric of the space $\mathbb{T}$ coincide.

SKETCH OF THE PROOF. By applying the mentioned conformal map, the proof is reduced to the case of the disk Δ . In this case $\Delta_\varphi = \{\phi(t\bar{\varphi}/|\varphi|)\}$, and a maximizing sequence $\{h_n\} \subset \text{Hol}(\mathbb{T}, \Delta)$ for $c_{\mathbb{T}}(\phi(t\mu_0), 0)$ is formed by the functions

$$h(\psi) = \sum_{m+n=2}^{\infty} \sqrt{mn} \alpha_{mn}(\psi) x_m x_n,$$

where $\psi = S_{f^t\mu_0} \in \mathbb{T}$, $\mathbf{x} = (x_n) \in \mathbf{I}^2$ and $\|\mathbf{x}\| = 1$. Applying to their lifts $\hat{h} = h \circ \phi : \text{Belt}(\Delta) \rightarrow \Delta$ to the ball $\text{Belt}(\Delta)$ some variational arguments concerning the integral representation of the functions $f^v \in \Sigma(k)$, one derives that

$$\alpha_{mn}(f^v) = -\frac{1}{\pi} \iint_{\Delta} v(z) z^{m+n-2} dx dy + O(\|v\|^2),$$

and, consequently, the differential of $\hat{h}$ at zero is given by

$$d\hat{h}(0)v = \frac{1}{\pi} \iint_{\Delta} v(z) \sum_{m+n=2}^{\infty} \sqrt{mn} x_m x_n z^{m+n-2} dx dy.$$

Combining this with (2.7) and Schwarz's lemma, one obtains the equality $|h \circ \phi(t\bar{\varphi}/|\varphi|)| = |t|$, which is equivalent to the assertion of Theorem 2.2.

The same arguments give that in fact the Carathéodory and the Teichmüller–Kobayashi metrics coincide in more general disks $\{\phi(t\mu_0) : t \in \Delta\}$ with $\|\mu_0\| = 1$, which correspond to $(\mu_0 \circ g)\bar{g}'/g'$ satisfying (2.6). $\square$

2.3. Equivalence of conditions (2.5) and (2.9) for asymptotically conformal curves

There arises naturally the question, for which functions f the condition (2.9) is nevertheless a necessary one. It was first studied in [Ku19], where it was proved that if a function $f \in \Sigma(k)$ is holomorphic in the closure $\bar{\Delta}^*$, then the equality $\kappa(f) = k$ can really hold only when (2.8) is valid; its proof is based on the fine properties of the least nontrivial positive Fredholm eigenvalue λ_1 of the curve $f(S^1)$ (see Section 2.6) and of the Faber polynomials.

Theorem 2.1 enables us to essentially decrease the required degree of smoothness of the boundary curves. The next theorem gives an affirmative answer for a wide class of boundary curves, which are called asymptotically conformal curves. It seems that the result is close to a complete one.

Let us first recall some definitions. Orientation preserving homeomorphisms $w = h(z)$ of the unit circle $S^1 = \{|z| = 1\}$ onto itself satisfying the equality

$$\lim_{\tau \rightarrow 0} \frac{h(\theta + \tau) - h(\theta)}{h(\theta) - h(\theta - \tau)} = 1, \quad \theta = \arg z, h(\theta) = \arg w, \quad (2.11)$$

uniformly in θ , are called *asymptotically symmetric* (on S^1); their quasiconformal extensions onto $\bar{\Delta}$ are called *asymptotically conformal* on S^1 (cf. [Ca]).

A Jordan curve L is called *asymptotically conformal* if for any pair of points $a, b \in L$, we have

$$\max_{z \in L(a,b)} \frac{|a - z| + |w - b|}{|a - b|} \rightarrow 1 \quad \text{as } |a - b| \rightarrow 0,$$

where the point z lies on L between a and b .

Such curves are quasicircles without corners and can be rather pathological (see, e.g., [Po2, p. 249]). All C^1 -smooth curves are asymptotically conformal.

There are certain analytic characterizations of these curves; for example, if f maps conformally the unit disk Δ onto the interior of a Jordan curve L , then the following conditions are equivalent:

- (i) L is asymptotically conformal;
- (ii) f has a quasiconformal extension to $\widehat{\mathbb{C}}$ whose Beltrami coefficient $\mu(z)$ satisfies

$$\lim_{r \rightarrow 1+} \operatorname{ess\,sup}_{|z| \leq r} |\mu(z)| = 0;$$

- (iii) Schwarzian derivative S_f satisfies

$$\lim_{|z| \rightarrow 1-} (1 - |z|^2)^2 S_f(z) = 0.$$

For a proof of these and certain other equivalent characterizations we refer to Pommerenke's book [Po2] and references cited there.

Other examples of univalent functions with asymptotically conformal restrictions to S^1 are provided by conformal maps f of Δ^* onto the Jordan domains whose Schwarzian derivatives S_f are from $A_1(\Delta^*)$.

As is shown in [Kru12], the set of such maps contains the functions $f \in \Sigma(k)$ for k sufficiently close to 1, such that the interval $\{tS_f\}$, $0 < t < 1$, contains the points $t_0 S_f$ which are the Schwarzians of locally univalent holomorphic functions in Δ^* , and these functions are not univalent in the whole domain Δ^* .

THEOREM 2.3. *If the boundary quasicircle $L = f(\partial\Delta)$ is asymptotically conformal, then (2.8) is both necessary and sufficient for $f \in \Sigma^0(k)$ to have (2.5).*

The proof of Theorem 2.3 is given in [Kru17]; more special cases of smooth curves were considered earlier in [Ku21] and [Kru11].

As a corollary of Theorem 2.3, one obtains the following theorem.

THEOREM 2.4. *For every function $f \in \Sigma(k)$, which maps the disk Δ^* onto a domain with asymptotically conformal curve boundary and does not admit quasiconformal extensions with the Beltrami coefficient of the form (2.8), we have the strong inequality $\kappa(f) < k$.*

Moreover, using Strebel's frame mapping criterion (see [St2,EL2]), one can establish that each $f \in \Sigma(k)$ mapping the unit circle $S^1 = \partial\Delta$ onto an asymptotically conformal curve admits a unique extremal quasiconformal extension to $\bar{\Delta}$ of Teichmüller's type, i.e., whose Beltrami coefficient $\mu_0 = k|\varphi|/\varphi$ with $\varphi \in A_1(\Delta) \setminus \{0\}$ (see [Kru17]). Hence, Theorem 2.4 holds only for the maps $f \in \Sigma(k)$ whose extremal Beltrami coefficients are of the form

$$\mu_0 = k|\varphi|/\varphi, \quad \varphi \in A_1(\Delta) \setminus A_1^2(\Delta),$$

i.e., their defining quadratic differentials φ have at least one zero in Δ of odd order.

As the simplest case, we have the following quite surprising fact:

COROLLARY 2.5. *Let for $t \in \Delta$,*

$$f_p(z) = z \left(1 - \frac{t}{z^{p/2}} \right)^{2/p}, \quad |z| \geq 1, p = 2, 3, \dots,$$

so $f_p \in \Sigma(|t|)$. If $p \geq 2$ is even, then $\kappa(f_p) = |t|$ for each t , while for every odd $p \geq 3$ the strict inequality

$$\kappa(f_p) < \inf \{ \|\mu\| : w^\mu|_{\Delta^*} = f \} < |t|$$

holds.

This was established by a different method in [Ku18,Ku21].

2.4. Two examples

The main purpose in the proof of Theorem 2.3 is to guarantee the absence of degenerating sequences $\{\varphi_n\} \subset C = \{\varphi \in A_1(\Delta) : \|\varphi_n\| = 1\}$. This is ensured by the Strebel condition for frame maps, which is both sufficient and necessary for extremality of a quasiconformal map. The situation concerning the presence of degenerating sequences is here the same as in the general theory of extremal quasiconformal maps, but now it is necessary to take those sequences which are in C^0 , so that (2.5) holds.

Our first example shows that *in general the condition (2.8), contrary to (2.5), is not necessary*. Its construction is closely related to Reich's example in [Re1], which provides a

quasiconformal automorphism w of the disk Δ with $\mu_w = k\bar{\varphi}_0/|\varphi_0|$, where $\varphi = \psi^2 \in C^0$ and for $\bar{\varphi}_0/|\varphi_0|$ there is a degenerating sequence $\{\varphi_n\}$, also belonging to C^0 . Having extended μ_w by zero onto Δ^* , we come to $\hat{f} \in \Sigma(k)$ for which (2.5) and (2.8) hold, though the Strebel condition is not valid.

On the other hand, we consider the affine stretching

$$F_K(z) = Kx + iy = \frac{K+1}{2}z + \frac{K-1}{2}\bar{z}, \quad K > 1; z = x + iy,$$

of the half-strip $\Pi_+ = \{z: 0 < x < \infty, 0 < y < 1\}$; evidently, $\mu_{F_K}(z) \equiv (K-1)/(K+1) = k$. Taking the sequence

$$\omega_n(z) = \frac{1}{n}e^{-z/n}, \quad z \in \Pi_+, n = 1, 2, \dots, \quad (2.12)$$

we obtain that $\omega_n \rightarrow 0$ uniformly in Π_+ , and at the same time

$$\iint_{\Pi_+} |\omega(z)| dx dy = 1, \quad \left| \iint_{\Pi_+} \omega(z) dx dy \right| = 1 - O\left(\frac{1}{n}\right). \quad (2.13)$$

Having Δ mapped conformally onto Π_+ by using a function $z = g(\zeta)$, we construct $f^\mu \in \Sigma(k)$ with μ equal to zero in Δ^* and equal to $k\bar{\varphi}_0/|\varphi_0|$ in Δ , where $\varphi_0 = g'^2$. Then the corresponding sequence

$$\varphi_n = (\omega_n \circ g_0)g'^2, \quad n = 1, 2, \dots, \quad (2.14)$$

belongs to C^0 and is degenerating for μ , but by virtue of (2.13) the equality (2.6) holds, and hence $\kappa(f^\mu) = k$; however $\varphi_0 \notin A_1(\Delta)$.

The next example is a slight modification of the previous one. It shows that there exist $\mu_0 \in B(\Delta)$ satisfying the conditions (2.5) and (2.6), with $\mu_0(z) \neq \text{const}$ in Δ ; such μ_0 also are the extremals of the Grunsky functional (2.4). Consider the map $f_q(z) = q(x) + iy$ of the same half-strip Π_+ onto itself, which is the stretching of this half-strip along the x -axis with the variable stretching coefficient

$$q(x) = \frac{2x+1}{x+1}$$

increasing monotonically on $[0, \infty]$ from 1 to 2. The Beltrami coefficient of this stretching is equal to

$$\mu_q(z) = -\frac{q(x)-1}{q(x)+1} = -\frac{x}{3x+1};$$

thus $\arg \mu_q(z) = \frac{1}{2}\pi$ and $\|\mu_q\|_\infty = \frac{1}{3}$.

Now for the sequence (2.12), we have

$$\begin{aligned}
 I_n &:= \left| \iint_{\Pi_+} \mu_q(z) \omega_n(z) dx dy \right| = \frac{1}{n} \left| \int_0^1 e^{-iy/n} dy \right| \int_0^\infty \frac{x e^{-x/n}}{3x+1} dx \\
 &= |1 - e^{i/n}| \left(\frac{n}{3} - \frac{1}{9} \int_0^\infty \frac{e^{-x/n}}{x + \frac{1}{3}} dx \right) \\
 &= \frac{1}{3} \left(1 + O\left(\frac{1}{n}\right) \right) \left(1 + O\left(\frac{\log n}{n}\right) \right), \tag{2.15}
 \end{aligned}$$

and again using a conformal map g of Δ onto P_+ , we construct the map $f^{\mu_0} \in \Sigma(\frac{1}{3})$ with μ_0 equal to zero in Δ^* and equal to $(\mu_q \circ g) \bar{g}'/g'$. The sequence (2.14) is maximizing in $L_1(\Delta)$ for $\|\mu_0\|$, because it follows from (2.15) that

$$\lim_{n \rightarrow \infty} \left| \iint_{\Delta} \mu_0(z) \omega_n(z) dx dy \right| = \frac{1}{3} = \|\mu_0\|_\infty.$$

Here $\sup_{\Delta} |\mu_0(z)| = \frac{1}{3}$ is attained at a unique boundary point, wherefore μ_0 can vary in Δ anyhow, without loss of (2.5).

Different examples of the extremals of the Grunsky functional based on the geometrical arguments were constructed by Kühnau (see, e.g., [Ku19]).

The second example is interesting also because it gives explicitly the geodesic holomorphic disk $\{\phi(3t\mu_0) : t \in \Delta\}$ in the universal space $\mathbb{T}$, which is not a Teichmüller disk and in which the Carathéodory metric and the Teichmüller–Kobayashi metrics coincide by Theorem 2.2.

2.5. The Teichmüller–Kühnau extension of univalent functions

2.5.1. We wish now to describe the most general situation in which the conditions (2.5) and (2.9) remain equivalent. Our goal is to characterize all the maps $f \in \Sigma(k)$ which satisfy

$$\kappa(f) = k(f) = k \tag{2.16}$$

and have extremal quasiconformal extensions to $\overline{\Delta}$ of Teichmüller's type, i.e., with the Beltrami coefficients of the form

$$\mu_{\hat{f}}(z) = k \frac{|\varphi(z)|}{\varphi(z)}, \quad \varphi \in A_1(\Delta) \setminus \{0\}.$$

We call the extremal quasiconformal extensions of the maps $f \in \Sigma$ with the Beltrami coefficients of the form $\mu(z) = k|\psi|^2/\psi^2$ the *Teichmüller–Kühnau* extensions.

Assume that the maps $f \in \bigcup_k \Sigma(k)$ satisfy the following condition concerning boundary dilatation, which is a special form of Strebel's frame mapping condition mentioned above:

(α) each point $z \in S^1$ has a neighborhood $U_0 \subset \mathbb{C}$ such that f admits a quasiconformal extension $\tilde{f}_0$ across $U_0 \cap S^1$ to a $U_0 \cap D$ with dilatation $k_{U_0}(\tilde{f}_0) = \|\mu_{\tilde{f}_0}\|_\infty < \kappa(f)$.

We call any such $\tilde{f}_0$ a local frame map for f and set

$$q_{U_0}(f) = \inf\{k_{U_0}(\tilde{f}_0): \tilde{f}_0 \text{ frame}\}, \quad q(f) = \sup\{q_{U_0}(f): z_0 \in S^1\}.$$

Recall that Strebel's condition is sufficient and necessary for existence and uniqueness of the Teichmüller extremal map in the set of quasiconformal extensions of f onto $\bar{\Delta}$ (see [St2,EL2]).

THEOREM 2.6. Assume that $f \in \Sigma(k)$ with $k(f) = k$ satisfies (α) at each point of S^1 . Then the following conditions are equivalent:

- (a) $\kappa = k$,
- (b) f admits a Teichmüller–Kühnau extension to $\bar{\Delta}$ with Beltrami coefficient $\mu(z) = \kappa|\psi|^2/\psi^2$.

The implication (b) $\Rightarrow$ (a) follows from Theorem 2.1 and does not require the assumption (α) (moreover, having (b), one gets (α) by Theorem 2.1 and necessity of Strebel's frame mapping condition as well).

The proof of the inverse (a) $\Rightarrow$ (b) follows the lines of Theorems 2.3 and 2.4 and is given in [Kru18].

Using similar arguments, one can prove the following theorem.

THEOREM 2.7. Assume that $f \in \Sigma$ satisfies $k(f) = \kappa(f)$, then the following conditions are equivalent:

- (b') f admits a Teichmüller–Kühnau extension to $\widehat{\mathbb{C}}$,
- (c) f admits the property (α) at all points of S^1 .

As a consequence of these theorems, one gets a complete description of possible extremal extensions of univalent functions to $\widehat{\mathbb{C}}$:

COROLLARY 2.8. We have:

- (i) If $f \in \Sigma$ satisfies $k(f) = \kappa(f)$ and $q(f) < \kappa(f)$, then f admits a Teichmüller–Kühnau extension.
- (ii) If f satisfies $k(f) = \kappa(f)$ and does not satisfy the condition (b) of Theorem 2.1, then there exists a point $z_0 \in S^1$ with $q_{U_0}(f) = \kappa(f)$.
The extremal extension of f is either unique or not, in both cases not of Teichmüller type.
- (iii) If $\kappa(f) < q(f) < k(f)$, then f has a unique Teichmüller extension $\tilde{f}$ with $\mu_{\tilde{f}} = k(f)|\varphi|/\varphi$, $\varphi \in A_1(\Delta) \setminus C^0$.

(iv) If $q(f) = k(f)$, then f has only non-Teichmüller extensions (again either unique or not).

PROOF. In view of Theorems 2.1 and 2.6 and of the frame mapping criterion, only (ii) requires a separate proof. To this end, observe that by (2.9),

$$\sup_{\varphi \in C^0} |\langle \mu_f, \varphi(z) \rangle|_{\Delta} = k(f),$$

and on the other hand, f cannot admit the property (α) at all points of S^1 . This yields the existence of the indicated point z_0 . $\square$

Let us mention also the following important consequence of Theorems 2.6 and 2.7:

COROLLARY 2.9. *The Teichmüller–Kühnau maps have nontrivial representatives in the asymptotic universal Teichmüller space.*

For definition and properties of asymptotic Teichmüller spaces we refer, e.g., to [GaL]. We illustrate the above theorems by two examples.

EXAMPLE 1. Let G be a circular lune bounded by two circular arcs L_1 and L_2 whose joint endpoints are a, b . Let the inner angles at these points be equal to $\alpha\pi$ ($0 < \alpha < 2, \alpha \neq 1$). Put $L = L_1 \cup L_2$ and choose it so that the conformal map f of Δ^* onto the complement of G is normalized via (2.1). The extremal extension of f is reduced to horizontal affine stretching in the logarithmic plane (after a fractional linear transformation), and $f \in \Sigma(k)$ with $k = |1 - \alpha|$.

It follows from Theorem 2.1 that $\kappa(f) = k$ (cf. [Kru18]); therefore, by Corollary 2.8(ii), the preimages $f^{-1}(a)$ and $f^{-1}(b)$ on S^1 have the neighborhoods U_0 with $q_{U_0} = |1 - \alpha|$. This can be established also directly, cf. [Ku22, Kru18].

EXAMPLE 2. Let G be the square centered at the origin with vertices a_1, a_2, a_3, a_4 so that the conformal Christoffel–Schwarz map f of Δ^* onto the complement of G is of the form (2.1). Due to [Ku22], $\kappa(f) = k(f) = 1/2$. In this case, again (ii) holds and the extremal extension is of non-Teichmüller’s type.

2.5.2. Theorem 2.1 provides a basic tool for applications of the Grunsky inequalities technique to the Teichmüller space theory and to univalent functions with quasiconformal extensions. It reveals a crucial role of holomorphic quadratic differentials with zeros of even orders. Some of these applications will be given in the following sections. An appropriate extension of Theorem 2.1 to differentials with zeros of odd orders is given in [Kru21].

2.6. The Fredholm eigenvalues

The Grunsky matrix $(c_{mn}(f))$ is closely related to the Fredholm eigenvalues λ_j of the curve $L = f(S^1)$, which was discovered by Schiffer [Schi6]. In the case of a smooth curve L , these are the eigenvalues of the double-layer potential, i.e., of the equation

$$h(z) = \frac{\lambda}{\pi} \int_L h(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta, \quad z \in L.$$

In certain questions the least nontrivial positive eigenvalue $\lambda_1 = \lambda_L$ plays a crucial role. This eigenvalue can be defined for any oriented Jordan curve $L \subset \widehat{\mathbb{C}}$ by the equality

$$\frac{1}{\lambda_L} = \sup \frac{|D_G(h) - D_{G^*}(h)|}{D_G(h) + D_{G^*}(h)},$$

where G and G^* are, respectively, the interior and exterior of L , and the supremum is taken over all functions h continuous on $\widehat{\mathbb{C}}$ and harmonic on $\mathbb{C} \setminus L$, and $D(h) = \iint (h_x^2 + h_y^2) dx dy$ is the Dirichlet integral.

A remarkable result of Schiffer [Schi6] and Kühnau [Ku19] says that

$$\frac{1}{\lambda_L} = \kappa(f). \quad (2.17)$$

On the other hand, due to [Ah1], the *reflection coefficient* q_L of L satisfies $\frac{1}{\lambda_L} \leq q_L$; hence,

$$\kappa(f) \leq q_L.$$

It is important for various questions to have sharp or even approximate values of λ_L and of quasireflection coefficients of curves and arcs. This problem was originated by Kühnau. His deep results are crucial in this direction. We shall provide a somewhat different approach in Section 6.

3. Distortion theory for univalent functions with quasiconformal extension

3.1. General distortion problems for univalent functions with quasiconformal extension

Let us begin with the following general problem, and at the same time, introduce the necessary notations. For details we refer, e.g., to the book [KK1].

Many variational problems of geometric function theory concerning conformal and quasiconformal maps can be included in the general scheme of obtaining the range domains of analytical functionals having a certain geometric or physical meaning. We give details of this situation, as follows.

Let E be a measurable subset of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, whose complement $E^* = \widehat{\mathbb{C}} \setminus E$ has positive measure, and

$$B(E^*) = \{\mu \in L_\infty(E^*): \|\mu\| < 1, \mu|_E = 0\}. \quad (3.1)$$

Denote by $Q(E)$ the class of normalized quasiconformal automorphisms f^μ of the sphere $\widehat{\mathbb{C}}$ with the Beltrami coefficients $\mu = \mu_f \in B(E^*)$, and introduce on $Q(E)$ the topology of uniform convergence on compacts in $\mathbb{C}$. Here, any normalization is allowed which provides uniqueness of a quasiconformal map with given μ . We also put

$$Q_k(E) = \{f \in Q(E): \|\mu_f\| \leq k\}, \quad 0 \leq k < 1.$$

If $\text{mes } E = 0$, then $Q(E)$ coincides with the class of all normalized quasiconformal automorphisms of $\widehat{\mathbb{C}}$. The second case which also is of special interest concerns the situation when E is a region. Let there be defined a nontrivial holomorphic functional

$$F(f): Q(E) \rightarrow \mathbb{C},$$

which means that it is complex Gateaux differentiable (and therefore also strongly, i.e., Frechet differentiable, cf., e.g., [HiF, Chapter 3]). As was already mentioned, the Gateaux derivative $F'_f(G)$ of any such functional admits integral representation with a finite Borel measure (with compact support in $\mathbb{C}$) and extends to the set of all the functions that are integrable with respect to this measure.

Due to the openness of the map F , the set of its values $V(F) = F(Q(E))$ is a subdomain of $\mathbb{C}$; we require that this domain have at least three boundary points, and lift F on $B(E^*)$ up to the holomorphic functional $\phi(\mu) = F(f^\mu): B(E^*) \rightarrow \mathbb{C}$.

Then one can define the hyperbolic metric $\rho_v(\cdot, \cdot)$ on $V(F)$ by projecting the Poincaré hyperbolic metric $\rho(\cdot, \cdot)$ of the unit disk, using the holomorphic universal covering map $\Delta \rightarrow V(F)$. The metric ρ is again normalized so that its Gaussian curvature $K(\rho) = -4$ and thus its differential element $ds = |dz|/(1 - |z|^2)$.

One can define for the Banach ball (3.1) its hyperbolic Kobayashi metric $k_B(\cdot, \cdot)$ and its Carathéodory metric $c_T(\cdot, \cdot)$. Using the chain rule for the Beltrami coefficients, it is possible to show that these metrics agree with the Teichmüller metric for any two points of $B(E^*)$. In particular, for any $\mu \in B(E^*)$ we have

$$c_B(\mu, 0) = k_B(\mu, 0) = \tau_B(\mu, 0) = \rho(\|\mu\|, 0) = \frac{1}{2} \log K(f^\mu), \quad (3.2)$$

where $K(f^\mu) = (1 + \|\mu\|)/(1 - \|\mu\|)$ is the deviation of the map f from the conformal one, i.e., the maximal dilatation of f^μ .

3.2. Lehto's majoration principle and its improvements. General range value theorems

The features of (bounded) holomorphic functionals were first revealed by Lehto in [Leh2], where he considered the univalent functions on the disk with quasiconformal extension (see Corollary 3.3), though he did not establish when the bounds are sharp and which is the form of the extremal maps.

We provide below, following [KK1, Part 1], two general theorems with a complete description of the extremal points.

Assume, for simplicity of writing, that $F(I) = 0$, where I is again the identity map $Iz \equiv z$; otherwise pass to the functional $F_1(f) = F(f) - F(I)$.

Note also that any normalization of the maps $f \in Q(E)$ is admissible, ensuring the uniqueness of the normalized quasiconformal map f^μ with given Beltrami coefficient $\mu \in B(E^*)$. For example, one can use the following normalizations:

- (1) $f(z_0) = z_0, f(z_1) = z_1, f(\infty) = \infty$ ($z_0, z_1 \in \mathbb{C}$);
- (2) $f(z_0) = z_0, f'(z_0) = 1, f(\infty) = \infty$ if z_0 is an inner point of E ;
- (3) hydrodynamical normalization $f(z) = z + a_1 z^{-1} + \dots$, if ∞ is an inner point of E .

The varied function $f^* \in Q(E^*)$ with the Beltrami coefficient close to μ in $L_\infty(E^*)$ is represented by

$$f^*(z) = f(z) + \pi^{-1} \iint_{f(E^*)} v(\xi) g(f(z), \xi) d\xi d\eta + O(\|v\|^2), \quad (3.3)$$

with the corresponding variational kernel

$$g(w, \zeta) = \frac{1}{w - \zeta} + g_1(w, \zeta);$$

here g_1 is a holomorphic function of both variables w, ζ .

The following general theorem is obtained as a corollary of the properties of invariant metrics and topological coverings. It provides various results containing the universal distortion estimates for many classes of conformal and quasiconformal maps. In many cases these bounds are sharp; moreover, the form of boundary maps become clear at once.

THEOREM 3.1 [KK1]. *The set $F(Q_k(E))$ of values of the functional F on the class $Q_k(E)$ is located entirely in the closed hyperbolic disk*

$$U_k = \left\{ w \in V(F): \rho_V(w, 0) \leq \frac{1}{2} \log K \right\},$$

where $K = (1 + k)/(1 - k)$. Moreover, the equality

$$\rho_V(F(f^\mu), 0) = \frac{1}{2} \log K$$

for $\mu \neq 0$ can only hold for the maps μ_0 with $t = ke^{i\alpha}$, and

$$\mu_0(z) = \frac{\overline{F'_I(g(I, z))}}{|F'_I(g(I, z))|}, \quad (3.4)$$

where $\alpha \in \mathbb{R}$ and $F'_I(g(I, z))$ is the value of the Gateaux derivative of the functional F on the variation kernel of the class $Q(E)$ for $f = I$; then $F(Q_k(E)) = U_k$.

Indeed, from the holomorphy of $\widehat{F}(\mu) = F(f^\mu)$ and contractibility of the Kobayashi metric, taking into account (3.2), one immediately obtains

$$\rho_v(\widehat{F}(\mu), 0) \leq k_{B(E^*)}(\mu, 0) = \frac{1}{2} \log K(f^\mu). \quad (3.5)$$

If equality holds in (3.5) for some $\mu_* \neq 0$, then we take the function $h(t) = \widehat{F}(t\mu_*/\|\mu_*\|)$ which should coincide with the universal holomorphic (nonramified) covering $\Delta \rightarrow V(F)$, and using h , lift the map $\widehat{F}(\mu)$ to the (single-valued) holomorphic map

$$\widehat{F}_*(\mu) = h^{-1}\widehat{F}(\mu) : B(E^*) \rightarrow \Delta$$

covering $\widehat{F}$. Then (3.5) implies that the Carathéodory distance $c_{B(E^*)}(t\mu_*/\|\mu_*\|, 0)$, $|t| < 1$, is attained by the map $\widehat{F}_*$. Applying the standard variation of quasiconformal maps to $h_0(t) = \widehat{F}(t\mu_0) : \Delta \rightarrow \Delta$, with μ_0 determined by (3.4) yields the concluding assertion of the theorem.

From the various corollaries of Theorem 3.1, we only present the following three statements. Let

$$L_{E^*}(F) = \frac{1}{\pi} \iint_{E^*} |F'_I(g(I, z))| dx dy. \quad (3.6)$$

This quantity characterizes many important properties of the functional F . Note that

$$L_{E^*}(F) = \|\widehat{F}'(0)\|,$$

where $\widehat{F}'(0)$ is the derivative of $\widehat{F}(\mu) = F(f^\mu)$ at the zero point $\mu = 0$.

COROLLARY 3.2. *Equality in (3.5) holds at least for one (and then for all) $K > 1$ if and only if*

$$L_{E^*}(F) = |h'(0)|, \quad (3.7)$$

where h is a holomorphic universal covering map $\Delta \rightarrow V(F)$; in other words, if $L_{E^*}(F)$ equals the conformal radius of the domain $V(F)$.

The next corollary strengthens Lehto's majorant principle, since it suits arbitrary classes of the maps and shows the form of extremal functions.

COROLLARY 3.3. *Let $V(F)$ be bounded, and*

$$\max_{Q_k(E)} |f(f)| = \|F\|_k, \quad \sup_{Q(E)} |F(f)| = \|F\|_1.$$

Then we have

$$\|F\|_k \leq k \|F\|_1; \quad (3.8)$$

moreover, equality occurs only if $V(F)$ is a disk, and the extremal functions have the Beltrami coefficients of the form (3.4).

The next consequence yields an immediate construction of extremal functions.

COROLLARY 3.4. *Let E be a simply connected domain with rectifiable boundary. Let*

$$\varphi(z) = F'_t(g(I, z)) \neq 0 \quad \text{in } E^*,$$

and let $V(F)$ be a disk. Then, for each $t = ke^{i\alpha} \in \Delta$, we have for the points $z \in E^*$ the equality

$$f_t(z) \equiv f^{t\mu_0}(z) = \eta^{-1} \circ \left[\alpha \left(\int_{z_0}^z \sqrt{\varphi} dz + t \int_{z_0}^z \sqrt{\varphi} d\bar{z} \right) \right]. \quad (3.9)$$

Here $z_0 \in E$ is a fixed point

$$\eta(w) = \int_{w_0}^w [F'_{f_t}(g(f_t, w))]^{1/2} dw, \quad w_0 = f_t(z_0), \quad (3.10)$$

where the fixed branches of roots in E^* are chosen in (3.9) and (3.10), and α is a constant depending on t , which is uniquely defined from the condition

$$\max_{Q_k(E)} \operatorname{Re}\{e^{i\alpha} F(f)\} = \operatorname{Re}\{e^{i\alpha} F(f_t)\}.$$

The values of f_t in E are obtained from (3.9) by means of Cauchy's integral formula.

3.3. Generalization: The maps with dilatations bounded by a nonconstant function

One can consider a more general situation, when the Beltrami coefficients μ_f are bounded on E^* by a measurable nonconstant function $\tau(z)$, $0 < \tau(z) \leq 1$.

Take the weighted space $L_\infty(E^*, \tau)$ of the measurable functions on E^* with the norm $\|\mu\|_{\infty, \tau} = \|\mu(z)/\tau(z)\|_\infty$ and extend them by zero to E . Let

$$B(E^*; \tau) = \{\mu \in L_\infty(E^*; \tau): \|\mu\|_{\infty, \tau} < 1\},$$

and let $Q(E)$ be a class of normalized quasiconformal automorphisms of $\widehat{\mathbb{C}}$ whose Beltrami coefficients $\mu \in B(E^*, \tau)$, i.e., $|\mu(z)| \leq \tau(z)$ almost everywhere on E^* . Let

$$Q_k(E; \tau) = \{f \in Q(E; \tau): \|\mu\|_{\infty, \tau} \leq k\}.$$

Note that $L_\infty(E^*, \tau)$ is a closed subspace in $L_\infty(E^*)$; thus $B(E^*, \tau) \subset B(E^*)$ and $Q(E, \tau) \subset Q(E)$, $Q_k(E, \tau) \subset Q(E)$ for all $k \in [0, 1)$.

For a holomorphic nonconstant functional $F(f): Q(E, \tau) \rightarrow \mathbb{C}$, $F(I) = 0$, with hyperbolic range domains $V(F) = F(Q(E; \tau))$, we have the following generalization of Theorem 3.1.

THEOREM 3.5 [KK1]. *The set $F(Q_k(E; \tau))$ of values of $F(f)$ on the class $Q_k(E, \tau)$ is located in the closed hyperbolic disk*

$$U_k(\tau) = \left\{ w \in V(F; \tau): \rho_V(0, w) \leq \frac{1}{2} \log K \right\}, \quad K = \frac{1+k}{1-k}.$$

Equality

$$\rho_V(F(f^\mu), 0) = \frac{1}{2} \log \frac{1 + \|\mu\|_{\infty, \tau}}{1 - \|\mu\|_{\infty, \tau}}$$

for $\mu \neq 0$ can occur only if $\|\tau\|_{\infty} = 1$ and only for the maps $f^{t\mu_0}$ with $t = ke^{i\alpha}$, $\alpha \in \mathbb{R}$, and

$$\mu_0(z) = \tau(z) \frac{\overline{F'_1(g(I, z))}}{|F'_1(g(I, z))|}; \quad (3.11)$$

then $F(Q_k(E; \tau)) = U_k(\tau)$ for all k .

Again, if F is bounded, then

$$\max_{Q_k(E; \tau)} |F(f)| \leq k \sup_{Q(E; \tau)} |F(f)|,$$

with equality, even for one $k > 0$, only if $V(F; \tau)$ is a disk, $\|\tau\|_{\infty} = 1$, and the extremal maps have the Beltrami coefficients of the form (3.11).

3.4. Examples

Let us restrict ourselves to the following illustrations of Theorem 3.1.

(1) Assume that $Q_k(E) = S(k)$; this means that $E = \Delta$, $f(z) = z + a_2 z^2 + \dots$, $f(\infty) = \infty$. Consider the functional $F(f) = a_2(f)$. In this case we have

$$g(f, \zeta) = \frac{f^2}{\zeta^2(f - \zeta)}$$

and

$$L_{\Delta^*}(a_2) = \pi^{-1} \iint_{|\zeta| > 1} |\zeta|^{-3} d\xi d\eta = 2 = \max_S |a_2|.$$

By Theorem 3.1 we have, for any $k \in (0, 1)$,

$$\max_{S(k)} |a_2| \leq 2k. \quad (3.12)$$

Using Corollary 3.4, one establishes that equality holds only for the functions

$$f_t(z) = \begin{cases} z(1 - tz)^{-2}, & |z| \leq 1, \\ |z|^2(\bar{z}^{1/2} - tz^{1/2})^{-2}, & |z| \geq 1, |t| = k. \end{cases} \quad (3.13)$$

This was first found (by another method) in [Ku7].

(2) Let

$$F(f) = a_3 - a_2^2 = \frac{1}{6}S_f(0) \quad \text{on } S(k).$$

In this case, $L_\Delta(F) = \max_S |F(f)| = 1$; thus the values of F on $S(k)$ range over the closed disk $\bar{\Delta}_k = \{|w| \leq k\}$ for any $k < 1$. The boundary functions are of the form $f_{t,2}(z) = \sqrt{f_t(z)}$, where f_t is again from (3.13).

(3) Consider on $S(k)$ the functional

$$F(f) = zf'(z)/f(z), \quad z \in \Delta,$$

related to starlikeness of the functions $f|_\Delta$. Passing to $\log F$ (where the branch of $\log$ is chosen equal to 0 as $z \rightarrow 0$), we get

$$L_{\Delta^*}(\log F) = \frac{|z|}{\pi} \iint_{|\zeta| > 1} \frac{d\xi d\eta}{|\zeta(\zeta - z)^2|} = \log \frac{1 + |z|}{1 - |z|}.$$

Combining this with the known estimate

$$\frac{1 - |z|}{1 + |z|} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|}, \quad f \in S,$$

one obtains the following sharp bounds for $f \in S(k)$:

$$\left(\frac{1 - |z|}{1 + |z|} \right)^k \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \left(\frac{1 + |z|}{1 - |z|} \right)^k, \quad (3.14)$$

where equality on the right-hand side and on the left-hand side is attained by the maps with Beltrami coefficients

$$\mu_\alpha(z) = ke^{i\alpha} \frac{\bar{z}}{|z|} \frac{|\zeta(\zeta - z)^2|}{(\bar{\zeta} - \bar{z})^2}, \quad \mu_z^*(z) = -\mu_\alpha(z),$$

respectively (cf. [Gu1]). The maps by themselves are reproduced using the formula (3.9).

(4) Consider on the class Q_K of K -quasiconformal automorphisms of $\widehat{\mathbb{C}}$ with the fixed points $0, 1$ and ∞ the functional $F(f) = f(z)$, where z is a fixed point from $\mathbb{C}^* = \mathbb{C} \setminus \{0, 1\}$. In this case

$$g(f, \zeta) = \frac{f(f-1)}{\zeta(\zeta-1)(f-\zeta)}$$

and

$$L_C(F) = \frac{|z||z-1|}{\pi} \iint_{\mathbb{C}} \frac{d\xi d\eta}{|\zeta(\zeta-1)(\zeta-z)|}.$$

It is shown in [Ag] (see also [Kr2, KK1, Va3]) that

$$d\rho_{\mathbb{C}^*}(z) = \left(\frac{|z||z-1|}{\pi} \iint_{\mathbb{C}} \frac{d\xi d\eta}{|\zeta(\zeta-1)(\zeta-z)|} \right)^{-1} |dz|.$$

Thus, $F(Q_K)$ is the closed hyperbolic disk

$$\left\{ w \in \mathbb{C}: \rho_{\mathbb{C}^*}(w, z) \leq \frac{1}{2} \log K \right\},$$

while the boundary functions are defined by (3.4) and (3.9). In this case the assertion of Theorem 3.1 is nothing more than the necessity part in Teichmüller's theorem on quasi-invariance of cross-ratios, cf. [Kr2, KK1].

4. General distortion theorems for univalent functions with quasiconformal extension

4.1. General variational problem

Let D be a simply connected domain in $\widehat{\mathbb{C}}$ with quasiconformal boundary and $D^* = \widehat{\mathbb{C}} \setminus \overline{D}$ its exterior. Denote by $S(D)$ the class of normalized univalent analytic functions in D , and let

$$S_k(D) = \{f \in D: f \text{ } k'\text{-quasiconformally extends to } D^*, k' \leq k\}.$$

The functions from $S(D)$ and $S_k(D)$ are normalized in the usual way, ensuring uniqueness, for instance, by means of one of the following conditions: $f(z) = z - z_0 + O(|z - z_0|^2)$ near a certain point $z_0 \in D \setminus \{\infty\}$; $f(z) + O(|z|^{-1})$ near $z = \infty$ if $\infty \in D$; f leaves two fixed points z_0 and z_* ($z_0 \in D, z_* \in \overline{D}$) and if $z_0 \neq \infty$, then the quasiconformal extensions of f to D^* are also submitted to the additional condition $f(z_{**}) = z_{**}$ at a point $z_{**} \in \partial D$ (then, in particular, $S(D)$ contains the identical map I). For quasiconformal automorphisms w of the sphere C with $w|_D \in S(D)$ we preserve the notation w^μ ; now

$$\mu \in B(D^*) = \{v \in L_\infty(\mathbb{C}): v|_D = 0, \|v\| < 1\}.$$

Define a (nonconstant) holomorphic functional $F(f)$ on the class $S(D)$. It is necessary to find the maximum of $|F(f) - F(I)|$ on $S_k(D)$ or, somewhat generally, the set of values $F(S_k(D))$.

Consider the Gateaux derivative $F'_f(h)$ of the functional F and assume that its value

$$\varphi_0(z) = F'_I(g(I, z)) \quad (4.1)$$

of the kernel g of the variation

$$f^\mu(\zeta) = \zeta - \frac{1}{\pi} \iint_{D^*} \mu(z) g(\zeta, z) dx dy + O(\|\mu\|^2)$$

of identity in $S(D)$ is a *rational* function with the poles in $\bar{D}^* \cup \{z_0\}$ and such that

$$\iint_{D^*} |\varphi_0| dx dy < \infty.$$

It is a common situation in the theory of univalent functions that one more often deals with functionals of the form

$$F(f) = \tilde{F}(f(z_1), \dots, f^{(\alpha_1)}(z_1); \dots; f(z_n), \dots, f^{(\alpha_n)}(z_n)). \quad (4.2)$$

Here $z_1, \dots, z_n$ are distinguished points from $D \setminus \{z_0\}$; $\alpha_1, \dots, \alpha_n$ are nonnegative integers and $\tilde{F}$ is a holomorphic function of its arguments in a certain domain $B \subset \mathbb{C}^d$ ($d = \sum_1^n \alpha_j + n$) containing the origin.

THEOREM 4.1. *If in the holomorphic disk*

$$\Delta_{\varphi_0} = \left\{ \Phi \left(t \frac{\overline{(\varphi_0 \circ G)G'^2}}{|(\varphi_0 \circ G)G'^2|} \right) : t \in \Delta \right\} \subset \mathbb{T},$$

where G is a conformal map $\Delta \rightarrow D$, the Carathéodory metric of the space $\mathbb{T}$ coincides with its Teichmüller–Kobayashi metric, then there exists a number $k_0(F) > 0$ such that for all $k \leq k_0(F)$, the inequality

$$\max_{\|\mu\| \leq k} |F(f^\mu) - F(I)| \leq \max_{|t| = k} |F(f^{t\bar{\varphi}_0/|\varphi_0|}) - F(I)| \quad (4.3)$$

holds.

This theorem together with Theorem 2.2 provides the following result which is convenient for applications.

THEOREM 4.2. *Let the function φ_0 , defined by (4.1), have zeros of only even order in D^* . Then for $k \leq k_0(F)$ the inequality (4.3) holds.*

The proof of Theorem 4.2 is obtained by extending to the universal Teichmüller space $\mathbb{T}$ of arguments, as outlined in [Kru23, Section 2.9]. But now the situation becomes more complicated due to the fact that the space $\mathbb{T}$ is infinite-dimensional, and while determining the analogue of the central Lemma 2.14 in [Kru23] one has to satisfy the infinite number of orthogonality conditions.

4.2. Generalizations of Theorem 4.1

Consider more general functionals $F(f)$ which depend also on the values of extensions of f in the finite number of distinguished points $\zeta_1, \zeta_2, \dots, \zeta_m \in D^*$, i.e., for instance, instead of (4.2) take

$$F(f) = \widetilde{F}(f(z_1), \dots, f^{(\alpha_1)}(z_1); \dots; f(z_n), \dots, f^{(\alpha_n)}(z_n); f(\zeta_1), \dots, f(\zeta_m)), \quad (4.4)$$

where $\widetilde{F}$ is again a holomorphic function in a suitable domain in $\mathbb{C}^{d'}$.

Now the function $\varphi_0 = F'_I(g(I, z))$ can also have simple poles in the distinguished points ζ_j and instead of the universal space $\mathbb{T}$ one needs to take the Teichmüller space $\mathbb{T}(\Delta^* \setminus \{\zeta'_1, \dots, \zeta'_m\})$ for the disk Δ^* with m punctures $\zeta'_j = G^{-1}(\zeta_j)$, where G^{-1} is again a conformal map $D^* \rightarrow \Delta^*$.

THEOREM 4.3. *In the holomorphic disk*

$$\Delta_{\varphi_0} = \left\{ \Phi \left(t \frac{\overline{(\varphi_0 \circ G)G'^2}}{|(\varphi_0 \circ G)G'^2|} \right) : |t| < 1 \right\} \subset \mathbb{T}(\Delta^* \setminus \{\zeta'_1, \dots, \zeta'_m\}),$$

let the Carathéodory metric coincide with the Teichmüller–Kobayashi metric. Then there exists a number $k_0(F) > 0$ such that for all $k \leq k_0(F)$ the inequality similar to (4.3) holds.

Unfortunately, for the present we cannot formulate a result which is analogous to Theorem 4.1, since there is no analogy to Theorem 2.2. It could yield *exact* distortion theorems spreading at once over conformal and quasiconformal portions of the domain of definitions of the maps. In fact, there are no results of such type at present, provided we do not take into account a few rather special cases. Nevertheless, Theorem 4.3 is not conditional, since it is also applied to the functionals (4.2), (4.4) in the significant case when one takes a normalization of extensions of the functions $f \in S_k(D)$, including, for instance, a condition $f(\zeta_1) = \zeta_1$ for a certain point $\zeta_1 \in D^*$ (aside from conditions in D) or when $\varphi_0 = F'_I(g(I, \cdot))$ has no poles in the points $\zeta \in D^*$ (see examples in Section 5).

4.3. Lower bound for $k_0(F)$

If the functional F on $S(D)$ is bounded, then one can find an explicit bound k'_0 from below for the value $k_0(F)$ figuring in the previous theorems so that for $k \leq k'_0$ the statements

of these theorems hold. This estimate is obviously nonexact, but it enables us to use the theorems effectively.

THEOREM 4.4. *Let $\sup_{S(D)} |F(f)| = M(F)$. Then*

$$k_0(F) \geq a \frac{\|F'_I\|}{\|F'_I\| + M(F) + 1} \equiv k'_0(a), \quad (4.5)$$

where a is any number from $(0, 1/2)$ and

$$\|F'_I\| = \frac{1}{\pi} \iint_{D^*} |F'_I(g(I, z))| dx dy. \quad (4.6)$$

The proof of this theorem is similar to Theorem 5.2 and consists of checking the fact that for $k \leq k'_0(a)$, for the extremal map f_0 an analogue of the necessary Lemma 2.5 in [Kru23] is valid. The other arguments in the proofs of Theorems 4.3 and 4.4 do not influence the estimate (4.5).

For $D = \Delta$, $z_0 = 0$ and normalization $f(z) = z + O(|z|^2)$, $z \rightarrow 0$, the class $S(D)$ becomes the known class S of univalent functions in Δ , already considered in the previous sections. Normalizing the functions $f \in S_k(D)$ by $f(\infty) = \infty$, one obtains the class $S(k)$. Theorems 4.3 and 4.4 provide an extension of the result of Section 5 to functionals of the form

$$F(f) = a_n + H(a_{m_1}, \dots, a_{m_s}; f(\zeta_1), \dots, f(\zeta_p)), \quad (4.7)$$

where $\zeta_1, \dots, \zeta_p \in \Delta^* \setminus \{\infty\}$, $m_1, \dots, m_s \geq 2$, and H is a holomorphic function in a certain domain of $\mathbb{C}^{s+p}$ containing the origin; also $H(\mathbf{0}) = \text{grad } H(\mathbf{0}) = 0$.

4.4. Two more illustrative examples

G.M. Goluzin proposed already in the 1940s to consider linear combinations of the coefficients $\sum_{n=2}^N \gamma_n a_n$. However, until now no quantitative estimates in the general case (i.e., when the functional does not simply reduce to a_n) have been obtained.

Take $F(f) = \sum_{n=3}^N \gamma_n a_n + H$, where perturbation H is of the same type as in (4.7) and N is odd. In this case,

$$\varphi_0(z) = \frac{1}{z^{N+1}} \sum_{n=3}^N \gamma_n z^{N-n}.$$

If φ_0 has zeros of only even order in Δ^* , i.e., $\varphi_0 = z^{-N-1} \psi^2$, where ψ is a polynomial, then in $S(1)$ for $k \leq k_0(F)$ (which is evaluated by Theorem 4.4),

$$\max_{\|\mu\| \leq k} |F(f^\mu)| \leq \max_{|t|=k} |F(f^{t\bar{\varphi}_0/|\varphi_0|})|.$$

This result is also carried over to the more general functional $\sum_{n=3}^{\infty} \gamma_n a_n + H$ with the corresponding γ_n , to converge the series.

5. The coefficient problem for univalent functions with quasiconformal extensions. Small dilatations

In the previous section we succeeded in applying the method proposed in [Kru23, Section 2.9] for solving general variational problems for analytic functions with quasiconformal extension. One of the most intriguing questions in this theory is the exact estimate of Taylor coefficients. It is traditionally of special interest in geometric theory of univalent functions because these coefficients provide the intrinsic features of univalence.

5.1. Main theorems

While the coefficient problem has been completely solved in the class of all normalized univalent functions on the disk [DB], the question remains open for functions with quasiconformal extension.

The complete result here is established only for the functions with k -quasiconformal extension, where k is sufficiently small; see [Kru9, Kru15].

Let S again be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ univalent in the unit disk $\Delta = \{|z| < 1\}$, and let the class $S(k)$ consist of $f \in S$ admitting k -quasiconformal extensions onto the whole Riemannian sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with additional normalization $\hat{f}(\infty) = \infty$. Denote

$$f_1(z) = \frac{z}{(1 - ktz)^2}, \quad |z| < 1, |t| = 1,$$

$$f_{n-1}(z) = \{f_1(z^{n-1})\}^{1/(n-1)} = z + \frac{2kt}{n-1} z^n + \dots, \quad n = 3, 4, \dots$$

Consider on S a functional F of the form

$$F(f) = a_n + H(a_{m_1}, a_{m_2}, \dots, a_{m_s}),$$

where $a_j = a_j(f)$; $n, m_j \geq 2$ and H is a holomorphic function of s variables in an appropriate domain of $\mathbb{C}^s$. We assume that this domain contains the origin $\mathbf{0}$ and that $H, \partial H$ vanish at $\mathbf{0}$.

The mentioned result of [Kru9] is the following theorem.

THEOREM 5.1. *For any functional of the above form, there exists $k(F) > 0$ such that, for $k \leq k(F)$,*

$$\max_{S(k)} |F(f)| = |F(f_{n-1})| \quad (5.1)$$

for some $|t| = 1$.

As a corollary, one immediately gets for $f \in S(k)$ the sharp estimate

$$|a_n| \leq \frac{2k}{n-1} \quad (5.2)$$

for $k \neq k_n$, with equality only for the function f_{n-1} . This solves the renowned problem of Kühnau and Niske on the best asymptotic estimates for the coefficients of univalent functions with quasiconformal extension (see [KuN]). The estimate (5.2) is interesting only for $n \geq 3$, because for $n = 2$ there is the well-known bound $|a_2| \leq 2k$ for all $k \in [0, 1]$ with equality for the function f_1 .

We now improve Theorem 5.1, supplementing it with an explicit estimate for the quantity $k(F)$. The main result here is:

THEOREM 5.2 [Kru14]. *Let $\sup_S |F(f)| = M_n$. Then the equality (5.1) holds for all*

$$k \leq \frac{1}{2 + (n-1)(M_n + 1)} =: k_0(F). \quad (5.3)$$

The bound (5.3) is not sharp and can be improved.

COROLLARY 5.3. *The estimate (5.2) is valid for all*

$$k \leq \frac{1}{n^2 + 1}. \quad (5.4)$$

PROOF. Take $F(f) = a_n$. Since $M_n = n$, by de Branges' theorem [DB], one immediately deduces from (5.3) that in this case

$$k_0(F) = \frac{1}{n^2 + 1}. \quad \square$$

For simplicity, we consider here the functionals F with holomorphic H depending on a finite number of coefficients a_m (provided the series expansion of H converges in some complex Banach space). The result shows that the main contribution here is given by the linear term a_m . The estimate (5.3) determines for which k this is true.

5.2. Proof of Theorem 5.2

The method is the same as in the principle given in [Kru23, Section 2.9] and in the preceding Section 3, with necessary modifications. We shall show that for k satisfying (5.3) one can apply the arguments similar to ones employed in the proof of Theorem 2.10 in [Kru23] and of Theorem 4.3.

On $\Delta^* = \{z \in \widehat{\mathbb{C}}: |z| > 1\}$ we have the Beltrami coefficients $\mu_f = \partial_{\bar{z}} f / \partial_z f$ of the extensions f^μ of functions $f \in S(k)$; these coefficients range over the ball

$$B(\Delta^*) = \{\mu \in L_\infty(\mathbb{C}): \mu|_\Delta = 0, \|\mu\|_\infty < 1\}.$$

Let $B(\Delta^*)_k = \{\mu \in B(\Delta^*): \|\mu\| \leq k\}$.

Note that the Beltrami coefficient for f_{n-1} can be taken to be $kt\mu_n$, where $|t| = 1$ and

$$\mu_n(z) = \frac{|z|^{n+1}}{\bar{z}^{n+1}}. \quad (5.5)$$

We shall also use the following notations. For a functional $L: S \rightarrow \mathbb{C}$ define

$$\widehat{L}(\mu) = L(f^\mu), \quad \mu \in B(\Delta^*).$$

If L is complex Gateaux differentiable, $\widehat{L}$ is a holomorphic functional on $B(\Delta^*)$. All our functionals have this property.

For $\mu \in L_\infty(\Delta^*)$, $\varphi \in L_1(\Delta^*)$, we define

$$\langle \mu, \varphi \rangle = -\frac{1}{\pi} \iint_{\Delta^*} \mu \varphi \, dx \, dy.$$

For small k , the functions $f^\mu \in S(k)$ can be represented by

$$f^\mu(\zeta) = \zeta - \frac{\zeta^2}{\pi} \iint_{\Delta^*} \frac{\mu(z) \, dx \, dy}{z^2(z - \zeta)} + O(\|\mu\|^2), \quad (5.6)$$

where the estimate of the remainder term is uniform on compact subsets of $\mathbb{C}$. Thus $\widehat{F}(\mu) = \langle \mu, z^{-n-1} \rangle + O_n(\|\mu\|^2)$ and hence,

$$\|\widehat{F}'(\mathbf{0})\| = \sup \left\{ \left| \left\langle \mu, \frac{1}{z^{n+1}} \right\rangle \right| : \|\mu\| \leq 1 \right\} = \frac{1}{\pi} \iint_{\Delta^*} \frac{dx \, dy}{|z|^{n+1}} = \frac{2}{n-1}.$$

Now, applying the Schwarz lemma to the function

$$h_\mu(t) = \widehat{F}(t\mu) - \widehat{F}'(\mathbf{0})t\mu : \Delta \rightarrow \mathbb{C},$$

where $\mu \in B(\Delta^*)$ is fixed, we get

$$|\widehat{F}(\mu) - \widehat{F}'(\mathbf{0})\mu| \leq (M_n + \|\widehat{F}'(\mathbf{0})\|)\|\mu\|^2 = \left(M_n + \frac{2}{n-1}\right)\|\mu\|^2. \quad (5.7)$$

Consider the auxiliary functional

$$\widehat{F}_p(\mu) = \widehat{F}(\mu) + (p-1)\xi \left\langle \mu, \frac{1}{z^{p+1}} \right\rangle, \quad (5.8)$$

where $p \neq n$ is fixed and $|\xi| < \frac{1}{2}$. Then $\sup_{B(\Delta^*)} |\widehat{F}_p(\mu)| < M_n + 1$, and similarly to (5.7),

$$\left| \widehat{F}_p(\mu) - \widehat{F}'(\mathbf{0})\mu - (p-1)\xi \left\langle \mu, \frac{1}{z^{p+1}} \right\rangle \right| \leq \left(M_n + 1 + \frac{2}{n-1}\right)\|\mu\|^2. \quad (5.9)$$

We shall require that

$$\left(M_n + 1 + \frac{2}{n-1}\right) \|\mu\|^2 < \frac{1}{n-1} \|\mu\| \quad (5.10)$$

or, equivalently,

$$\|\mu\| \leq \frac{1}{2 + (n-1)(M_n + 1)} = k_0(F). \quad (5.11)$$

Consider now any function f_0 in $S(k)$ maximizing $|F|$ over $S(k)$ (the existence of such functions follows from compactness). Let μ_0 be an extremal dilatation of f_0 , i.e.,

$$\|\mu_0\|_\infty = \inf\{\|\mu\|_\infty \leq k: f^\mu|_\Delta = f_0|_\Delta\}.$$

Note that $\|\mu_0\|_\infty = k$ by the maximum modulus principle. Suppose that $\mu_0 \neq kt\mu_n$, where $|t| = 1$, and μ_n is defined by (5.5). We show that this leads to contradiction for k satisfying (5.3). First of all, we may establish the following important property of extremal maps:

LEMMA 5.4. *If k satisfies (5.3), then, for all $2 \leq p \neq n$,*

$$\left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle = 0.$$

PROOF. Note that, from (5.6)

$$\left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle = \lim_{\tau \rightarrow 0} \frac{a_p(f^{\tau\mu_0})}{\tau}.$$

Consider the classes $S(\tau k_0)$ where $k_0 = k_0(F)$ is defined in (5.3) and $0 < \tau < 1$. It follows from (5.6) and from the known properties of the norm

$$h_p(\xi) = \iint_{\Delta^*} |z^{-n-1} + (p-1)\xi z^{-p-1}| dx dy$$

that is, as $\tau \rightarrow 0$, $\xi \rightarrow 0$,

$$\begin{aligned} \max_{B(\Delta^*)_{\tau k_0}} |\widehat{F}_p(\mu)| &= \frac{\tau k_0}{\pi} \iint_{\Delta^*} \left| \frac{1}{z^{n+1}} + \frac{(p-1)\xi}{z^{p+1}} \right| dx dy + O_n(\tau^2) \\ &= \max_{B(\Delta^*)_{\tau k_0}} |\widehat{F}(\mu)| + \tau o_p(\xi) + O_p(\tau^2 \xi) + O_n(\tau^2), \end{aligned} \quad (5.12)$$

where the bound for the remainder term $O_n(\tau^2)$ depends by (5.9) only on M_n and K_0 . On the other hand, from (5.8)

$$|\widehat{F}_p(\tau\mu_0)| = |\widehat{F}(\tau\mu_0)| + \tau(p-1)|\xi| \left| \left\langle \mu_0, \frac{1}{z^{p+1}} \right\rangle \right| + O(\tau^2 \xi^2) \quad (5.13)$$

with suitable choices of $\xi \rightarrow 0$. Comparing (5.12) and (5.13) we deduce that $\langle \mu_0, z^{-p-1} \rangle = 0$, which completes the proof of Lemma 5.4. $\square$

This lemma is one of the central points in the proofs of Theorems 5.1 and 5.2. The crucial point in the proof of Lemma 5.4 is that we now have to check that simultaneously an *infinite* (countable) number of orthogonality conditions remain valid for all k satisfying (5.3); cf. Section 4.

Consider the Grunsky coefficients of the function $\sqrt{f(z^2)}$ which are defined from the series expansion

$$\log \frac{(f(z^2))^{1/2} - (f(\zeta^2))^{1/2}}{z - \zeta} = - \sum_{m,n=1} \omega_{mn} z^m \zeta^n,$$

taking the branch of logarithm which vanishes at 1. The diagonal coefficients $\omega_{n-1,n-1}(f)$ are related to the Taylor coefficients of f by

$$\omega_{n-1,n-1} = \frac{a_n}{2} + P(a_2, \dots, a_{n-1}), \quad (5.14)$$

where P is a polynomial without constant or linear terms (see [Hu1]). Moreover, for $f \in S(k)$, there is the well-known bound $|\omega_{n-1,n-1}| \leq k/(n-1)$ with equality only for the functions f_{n-1} .

Therefore, the map $\Lambda_{n-1} = \{(n-1)\omega_{n-1,n-1}(\mu)\}\mu_n$ is holomorphic and fixes the disk $\{t\mu_n: |t| < 1\}$. The differential of Λ_{n-1} at $\mu = 0$ can be easily computed from (5.6), (5.14). It is an operator $P_n: L_\infty(\Delta^*) \rightarrow L_\infty(\Delta^*)$ given by

$$P_n(\mu) = \beta_n \langle \varphi_n, \mu \rangle \mu_n, \quad \varphi_n = 1/z^{n+1}.$$

Let us define $P_n(\mu) = \alpha(k)\mu_n$. Since, by assumption, f_0 is not equivalent to f_{n-1} , we have

$$\left\{ \Lambda_{n-1} \left(\frac{t}{k} \mu_0 \right) : |t| < 1 \right\} \subsetneq \{ |t| < 1 \}.$$

Thus, by the Schwarz lemma,

$$|\alpha(k)| < k. \quad (5.15)$$

Now consider the function

$$v_0 = \mu_0 - \alpha(k)\mu_n.$$

Arguing similarly as in the proofs of Theorems 2.10 and 2.11 in [Kru23], one can establish that v_0 eliminates integrable holomorphic functions on Δ^* . In other words,

$$v_0 \in A_1(\Delta)^\perp = \{ \mu \in L_\infty(\Delta^*) : \langle \mu, \varphi \rangle = 0 \text{ for all } \varphi \in A_1(\Delta^*) \},$$

where $A_1(\Delta^*)$ is the subspace in $L_1(\Delta^*)$ of functions φ which are holomorphic on Δ^* and satisfy the condition $\varphi(z) = O(|z|^{-3})$ as $|z| \rightarrow \infty$.

Now we use the well-known properties of extremal quasiconformal maps, which imply that for any $v \in A_1(\Delta)^\perp$ we must have

$$\|\mu_0\|_\infty = \inf\{|\langle \mu_0 + v, \varphi \rangle| : \varphi \in A_1(\Delta^*), \|\varphi\| = 1\} \leq \|\mu_0 + v\|_\infty.$$

Consequently,

$$\|\mu_0\|_\infty = k \leq \|\mu_0 - v_0\|_\infty = \|\alpha(k)\mu_n\|_\infty = |\alpha(k)|,$$

which contradicts (5.15). Hence f_0 is equivalent to f_{n-1} and we can take $\mu_0 = kt\mu_n$ for some $|t| = 1$. This completes the proof of Theorem 5.2.

5.3. Complementary remarks and open questions

The estimates (5.1)–(5.3) also hold in the class $S_1(k)$ of functions $f \in S$ with k -quasiconformal extensions $\hat{f}$ normalized by $\hat{f}(1) = 1$.

The proof is similar, only (5.6) should be replaced with the corresponding representation formula for $f \in S_1(k)$ [Kru5, Chapter 5]:

$$f^\mu(\zeta) = \zeta - \frac{\zeta^2(\zeta - 1)}{\pi} \iint_{\Delta^*} \frac{\mu(z) dx dy}{z^2(z - 1)(z - \zeta)} + O(\|\mu\|_\infty^2) \quad \text{as } \|\mu\| \rightarrow 0.$$

Similar results are valid for the class $\Sigma(k)$ of functions

$$g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}, \quad z \in \Delta^*,$$

with k -quasiconformal extensions to $\widehat{\mathbb{C}}$ which fix the origin.

The next two problems still remain open:

(1) Does there exist an estimate of coefficients a_n ($n \geq 3$) for $f \in S(k)$ which holds for $k \leq k_0$ with a single $k_0 > 0$?

(2) Can one find the exact estimates of coefficients a_n for univalent functions on the disk with quasiconformal extension in the general case when the dilatation $k < 1$ is arbitrary?

For $f \in S(k)$, one gets from (5.7) the estimate

$$|a_n| \leq \frac{2k}{n-1} + \left(n + \frac{2}{n-1}\right)k^2$$

for any k , $0 \leq k < 1$ (cf. [KK1, Part 1, Chapter 2]).

One might be able to obtain a bound $|a_n| \leq (2 + c(k_0))k/(n-1)$ which would improve Gökür's [Go] estimate

$$|a_n| \leq A(k)kn^{-1/2-\alpha(k)},$$

where $A(k)$ and $-\alpha(k)$ increase on $[0, 1)$, $\alpha(0) = 1/2$.

Grinshpan [Gri1] established the exact growth order, with respect to n , of the coefficients a_n of $f \in S$ with k -quasiconformal extension, without any additional normalization: $|a_n| \leq cn^k$.

6. Other variational methods

6.1. A general method of quasiconformal variations

6.1.1. The basic variational method for quasiconformal homeomorphisms of Riemann surfaces and their subdomains described in [Kru23] can be applied to special quasiconformal maps which are conformal on some portions of the domains of these maps and are quasiconformal on the complementary subsets. The case of maps of planar domains has a special interest since they are intrinsically connected with classical Geometric Function Theory. The quasiconformality allows us to work with the classes of maps possessing an arbitrary finite number of the normalization conditions contrary to strong rigidity of conformal maps.

6.1.2. Suppose that D and D' are two finitely-connected domains on the extended complex plane $\widehat{\mathbb{C}}$ of the same analytic type. Let E be a subset of D of a positive two-dimensional Lebesgue measure m_2 such that $m_2(D \setminus E) > 0$. Let $b_1, b_2, \dots, b_n$ denote distinct finite points to which one assigns nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_n$, respectively. We suppose that $\alpha_j = 0$ for $b_j \in \overline{D \setminus E}$ and that $n + \sum_{j=1}^n \alpha_j \geq 3$. Set $b = (b_1, \dots, b_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, and let $Q_k(E, \beta, \alpha, W)$ denote the class of quasiconformal homeomorphisms f of D onto D' with dilatations $k(f) \leq k < 1$ that are conformal (i.e., $\partial_{\bar{z}} f = 0$) on the distinguished set E and that satisfy the conditions

$$f^{(s)}(b_j) = w_{sj}, \quad j = 1, \dots, n, \quad \{w_{sj}\} \equiv W, \quad (6.1)$$

where s ranges over a subset of the integers $0, 1, \dots, \alpha_j - 1$ (may be empty) and takes the value α_j , while w_{sj} are given numbers such that $w_{0j} \in D'$.

Assume that this class is not empty (cf. Theorem 1.1). Let us consider the problem of finding the maximum on $Q_k(E, \beta, \alpha, W)$ of a real differentiable functional of the form

$$J(f) = J(w_{01}^*, w_{11}^*, \dots, w_{\beta_1, 1}^*; w_{02}^*, \dots, w_{\beta_2, 2}^*; \dots; w_{0m}^*, \dots, w_{\beta_m, m}^*)$$

with $\text{grad } J \neq 0$. Here $w_{sj}^* = f^{(s)}(z_j)$ (for $j = 1, \dots, m$ and $S = 0, 1, \dots, \beta_j$), the points $z_1, \dots, z_m$ are fixed points in D distinct from $b_1, \dots, b_n$, and $\beta_1, \dots, \beta_m$ are given nonnegative integers such that $\beta_j = 0$ if $z_j \in \overline{D \setminus E}$.

The existence of extremals of J is ensured by compactness of such classes of maps. The general properties of solutions are given by the following theorem.

THEOREM 6.1 [Kru4, Kru5]. *Any quasiconformal homeomorphism $w = f_0(z)$ maximizing J on $Q_k(E, \beta, \alpha, W)$ has the following properties: there exists a constant c and a meromorphic function φ_0 on D' , possibly having poles of orders not exceeding $\alpha_j + 1$ at the*

points w_{0j} ($j = 1, \dots, n$) such that the Beltrami coefficient $\mu_{f_0^{-1}}$ of the inverse map f_0^{-1} equals to zero on $f_0(E)$ and has at points $w \in D' \setminus f_0(E)$ the form

$$|\mu_{f_0^{-1}}(w)| = k, \quad \arg \mu_{f_0^{-1}}(w) = -\arg[c\varphi_*(w) + \varphi_0(w)], \quad (6.2)$$

where

$$\varphi_*(w) = \sum_{j=1}^m \sum_{s=0}^{\beta_j} \frac{\partial J}{\partial w_{sj}^*} \frac{d^s}{dz^s} \left(\frac{1}{f_0(z) - w} \right) \Big|_{z=z_j} \quad \text{with } w_{sj}^* = f_0^{(s)}(z_j). \quad (6.3)$$

If the domain D' is bounded by analytic curves, then the quadratic differential

$$[c\varphi_*(w) + \varphi_0(w)] dw^2$$

is analytic and real valued on its boundary.

The proof of this theorem involves the variational technique developed by Krushkal. Note that if $D = D' = \mathbb{C}$ or both these domains are bounded by a finite number of circles, then φ_0 and φ_* are rational functions of a special kind. In the case when the domains are Jordan, one can additionally prescribe the values w_{0l} at a finite number of boundary points b_l .

6.1.3. Theorem 6.1 yields a qualitative description of extremal maps and establishes the form of the Beltrami coefficients of inverse maps. For small k , one can use the variational formulas for approximate construction of the mapping functions and replace in (6.2) and (6.3) (with accuracy of order k^2) the unknown quantities $w_{sj}^* = f_0^{(s)}(z_j)$ by the known values corresponding to the identity map.

This provides various sharp asymptotical bounds for many quasiconformal maps with small dilatations. For example, we immediately have the sharp estimate

$$|a_n| \leq \frac{2k}{n-1} + M_1 k^2, \quad k \rightarrow 0,$$

for coefficients of the functions $f \in S(k)$, or that in this class,

$$\log \left| \frac{f(z)}{z} \right| \leq \frac{4k}{\pi} \int_0^{|z|} \mathbf{K}(x) dx + M_2 k^2 \quad \text{for } |z| \leq 1,$$

with the constants M_1 and M_2 not depending on f . Here $\mathbf{K}(x)$ is the complete elliptic integral of the first kind.

This method of variations provides the distortion bounds (though in many cases only asymptotic ones) for both conformal and quasiconformal parts of the preimage domain D and under normalization involving an arbitrary finite number of conditions.

A crucial point here is to obtain the estimates valid for any $k < 1$ which we had earlier under the standard normalizations (i.e., with minimal conditions ensuring existence and

uniqueness). Other variational methods presented below provide such estimates, again for the maps with a standard normalization.

6.2. Schiffer's method

6.2.1. This fruitful method is actually an appropriate modification of the classical Schiffer's method for univalent functions to quasiconformal homeomorphisms, which provides one of the basic tools in Geometric Function Theory. On the other hand, it also relates closely to the standard variational methods in quasiconformal theory described in [Kru23]. We present it as the following general theorem omitting some nonessential details.

Let $\mathcal{F}(\Omega)$ be a family of normalized quasiconformal maps $\Omega \rightarrow \widehat{\mathbb{C}}$, where $\Omega \subseteq \widehat{\mathbb{C}}$ is a subdomain, for simplicity, a simply connected one. We shall denote by $\mathcal{F}_Q(\Omega)$ the subfamily of the maps from $f \in \mathcal{F}(\Omega)$ whose Lavrentiev's characteristics (dilatations)

$$p(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} \geq 1$$

are restricted by a given bounded function: $p(z) \leq Q(z) \leq Q_0 < \infty$; cf. [La2].

Suppose that the maps $f \in \mathcal{F}$ admit a variation

$$f^*(z) = f(z) + th(z; f) + O(t^2) \in \mathcal{F}, \quad t \rightarrow 0, t \in \mathbb{C}, \quad (6.4)$$

so that the ratio $O(t^2)/t^2$ is uniformly bounded on the compact subsets of Ω .

Let $J(f)$ be a nonconstant real-valued Frechét differentiable functional $\mathcal{F} \rightarrow \mathbb{R}$ so that, accordingly to (6.4),

$$J(f^*) = J(f) + \operatorname{Re}\{tJ'[h(\cdot; f)]\} + O(t^2). \quad (6.5)$$

The derivative $J'(h)$ is again a linear functional, and, due to the Riesz representation theorem, it can be represented by the corresponding complex Borel measure m supported on compact subsets $e \Subset \Omega$, i.e.,

$$J'(h) = \int_e h(z) dm(z). \quad (6.6)$$

This representation allows us to extend this linear functional onto the space of all locally integrable functions on Ω .

Our goal is to describe the extremal functions f_0 maximizing $J(f)$ on $\mathcal{F}_Q(\Omega)$. Their existence follows from compactness of these subfamilies in the topology of uniform convergence on the closed subsets of Ω .

Let us assume that the function

$$g(w) = J'[h(\cdot; f_0)] \quad \text{with } w = f_0(z), \quad (6.7)$$

does not vanish identically on Ω , and define

$$\psi(w) = \int \sqrt{J'[h(\cdot; \tau)]} d\tau, \quad \Psi_{f_0}(z) = \psi \circ f_0(z). \quad (6.8)$$

The function Ψ is multivalued on Ω , but locally holomorphic, excluding a discrete set of singular points.

For many important functionals J in the theory of conformal and quasiconformal maps, all singularities arising in such way are the poles. In this case, the properties of the corresponding extremals f_0 of the functional J are given by the following Schiffer's theorem ([Schi4], see also [Ren1, Scho3, McL]).

THEOREM 6.2. *Suppose that the dilatation bound $Q(z)$ is piecewise constant on Ω and assumes the values Q_j on the open subsets $\Omega_j \subset \Omega$, $j = 1, \dots, N$, so that $\bigcup_j^N \Omega_j = \Omega$. Then for every extremal map f_0 , we have:*

- (a) *the corresponding functions $\Psi_{f_0}(z) - k_j \overline{\Psi_{f_0}(z)}$ with $k_j = (Q_j - 1)/(Q_j + 1)$ are locally holomorphic on the sets Ω_j , $j = 1, \dots, n$;*
- (b) *the image $f_0(\Omega)$ is bounded by the horizontal subarcs of quadratic differential $\psi(w) dw^2$, i.e., by analytic arcs satisfying $\psi(w) dw^2 > 0$.*

The proof of this theorem can be obtained also by variational method for quasiconformal maps described in [Kru5], using Biluta's boundary variation (cf. [Bi1, BiK, Kru23]). It relies on the factorization $f_0 = g_0 \circ F_0$, where F_0 is a quasiconformal automorphism of Ω with the same Beltrami coefficient μ_{f_0} and g_0 is an extremal of the corresponding functional on conformal maps $g : F_0(\Omega) \rightarrow \mathbb{C}$. Note that in the general case, the composition of an extremal quasiconformal with extremal conformal map does not give an extremal f_0 .

6.2.2. If in Theorem 6.2 at least one of the values $k_j = 0$, one obtains a family of conformal maps with quasiconformal extensions. Such a case is of a special interest.

A crucial point here is that holomorphic (meromorphic) quadratic differentials defining an extremal map in those families are analytically connected. This fact was first observed by Schiffer. It provides a ground for obtaining the complete solutions of several extremal problems for arbitrary $k \in (0, 1)$, combining conformal and quasiconformal variations.

6.3. Some applications: The Schiffer–Schober and McLeavey distortion theorems

Let us now restrict ourselves to the cases when Ω is either the sphere $\widehat{\mathbb{C}}$, or the disk $\Delta_R = \{|z| < R\}$, $0 < R < \infty$, or the disk $D_r^* = \{z \in \widehat{\mathbb{C}}: |z| > r\}$, $r \geq 0$.

In particular, one obtains the customarily used classes $\mathcal{F}_Q(\Omega) = S(k)$ or $\mathcal{F}_Q(\Omega) = \Sigma(k)$ with arbitrary $k = (Q - 1)/(Q + 1)$, $Q > 1$, and the more general classes $S(R, k)$ of quasiconformal homeomorphisms $f : \Delta_R \rightarrow \mathbb{C}$ with $f|_{\Delta} \in S$ and $\Sigma(r, k)$ of $f : \Delta_r^* \rightarrow \widehat{\mathbb{C}}$ with $f|_{\Delta^*} \in \Sigma$; here $1 < R < \infty$ and $0 < r < 1$.

Applying Theorem 6.2, Schiffer and Schober have obtained various new distortion estimates for the maps from $S(k)$ and $\Sigma(k)$ valid for arbitrary $k \in (0, 1)$ (see [ScSc1–ScSc3]). For example, the following deep result is true.

THEOREM 6.3. Let $f \in S(k)$ and $f(z) = z + \sum_2^\infty a_n z^n$ for $z \in \Delta$. Then

$$|a_2| \leq 2 - 4\kappa^2, \quad (6.9)$$

where

$$\kappa = \frac{1}{\pi} \arccos k \in (0, 1/2]. \quad (6.10)$$

Equality occurs only for the function $f(z)$, which equals

$$\frac{4z}{(1-z)^2} \left[\left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^\kappa + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}} \right)^\kappa \right]^{-2} \quad \text{if } |z| < 1,$$

and

$$4(1-k^2) \left\{ \frac{1-z}{\sqrt{z}} \left[\left(\frac{\sqrt{z}+1}{\sqrt{z}-1} \right)^\kappa - \left(\frac{\sqrt{z}-1}{\sqrt{z}+1} \right)^\kappa \right] + k \frac{1-z}{\sqrt{z}} \left[\left(\frac{\sqrt{z}+1}{\sqrt{z}-1} \right)^\kappa - \left(\frac{\sqrt{z}-1}{\sqrt{z}+1} \right)^\kappa \right] \right\}^{-2}$$

if $|z| \geq 1$ and for its rotations $e^{-i\theta} f(e^{i\theta} z)$, $\theta \in (0, 2\pi)$.

This result was obtained also in [Ku8] and [KuTh]. Applying Theorem 6.3 to

$$f_\zeta(z) = \frac{f((z+\zeta)/(1+\bar{\zeta}z)) - f(z)}{(1-|\zeta|^2)f'(\zeta)} \in S(k),$$

with $\zeta \in \Delta$, yields the bound of the functional $J(f) = zf''(z)/f'(z)$ on $S(k)$ related to convexity (see, e.g., [Scho3]):

COROLLARY 6.4. For any function $f \in S(k)$, we have

$$\left| (1-|z|^2) \frac{zf''(z)}{f'(z)} - 2|z|^2 \right| < 4 - \frac{8 \arccos k}{\pi^2}, \quad z \in \Delta. \quad (6.11)$$

This approach was extended by McLeavey [McL] to univalent functions on the disk with quasiconformal extensions to $\hat{\mathbb{C}}$ whose dilatations are bounded by a radial (circularly symmetric) function $Q(|z|)$ with $\|Q\| < \infty$. She obtained for these classes the analogs of the classical Grunsky and Goluzin inequalities and of their consequences which play a crucial role in Geometric Function Theory. The estimates of McLeavey are sharp. Using another method, Kühnau has obtained similar results in more general classes of $Q(z)$ -quasiconformal maps with a variable bound of dilatations. These results are presented below in Section 6.5 (see also [Ku12]).

6.4. Variations of Kühnau

6.4.1. Kühnau's fundamental research provided far-reaching developments of some basic methods in classical Geometric Function Theory to univalent holomorphic functions with quasiconformal extensions. We present briefly some of his main results. For details, the reader is referred, for example, to the book [KK1, p. 2], and to the bibliography provided there.

Schiffer's variational method was widely developed by Kühnau and Gutlyansky. We describes their approach separately.

Kühnau's general method concerns the following situation. Given a homeomorphism $w(z)$ of the sphere $\widehat{\mathbb{C}}$ onto itself preserving the point at infinity, which is conformal in the disk $\Delta^* = \{z \in \mathbb{H}^1 : |z| > 1\}$ and carries out the infinitesimal circles from the disk $\Delta = \{|z| < 1\}$ onto the infinitesimal ellipses with the axes ratio equal to $K > 1$ and such that the direction of their great axes coincides with the horizontal trajectories $Q(w)dw^2 \geq 0$ of a given quadratic differential with rational Q . Suppose also that the image $w(\Delta)$ can contain at most the simple poles of Q . The problem consists of restoring this homeomorphism and its representation.

A deep analysis carried out in [Ku10] provides the following general result.

THEOREM 6.5. *The desired map $w = w(z)$ is determined by equality*

$$\int \sqrt{Q(w)} dw = \begin{cases} F(z) & \text{if } |z| \geq 1, \\ (1 - k^2)^{-1} [\overline{G(1/\bar{z})} + G(1/\bar{z})] & \text{if } |z| \leq 1. \end{cases} \quad (6.12)$$

Here $k = (K - 1)/(K + 1)$ and the functions F and G are defined by

$$F(z) = -\frac{1}{1 - k^2} \int \sqrt{h} \sin I dz, \quad G(z) = \int \sqrt{h} \cos I dz \quad (6.13)$$

with

$$I(z) = \sqrt{1 - k^2} \int \frac{g}{h} dz, \quad (6.14)$$

where g and h are well-defined rational functions of a special form.

Kühnau discovered many interesting features of these rational quadratic differentials.

6.4.2. Theorem 6.5 provides various important distortion estimates. Let us start with the following strengthening of the classical Koebe one-quarter theorem for univalent functions.

THEOREM 6.6. *For every map from $S(k)$, the image of the unit circle $S^1 = \partial \Delta$ is located in the circular annulus*

$$\mathcal{A}(k) = \{w : m(k) \leq |w| \leq M(k)\}, \quad (6.15)$$

where

$$m(k) = \exp \left\{ -2C - 6 \log 2 - 2\psi \left(\frac{1}{2} - \frac{1}{2\pi} \arccos k \right) - \pi \sqrt{\frac{1}{K}} \right\}$$

and

$$M(k) = \exp \left\{ -2C - 6 \log 2 - 2\psi \left(\frac{1}{2} - \frac{1}{2\pi} \arccos k \right) - \pi \sqrt{K} \right\}.$$

Here ψ denotes the Euler psi-function ($\psi = \Gamma'/\Gamma$, where Γ is the gamma-function), $C = 0.577 \dots$ is the Euler constant and $K = (1+k)/(1-k)$. These bounds are sharp.

The proof of this theorem in [Ku11] reveals explicitly the extremal maps. In somewhat another form, the bounds $m(k)$ and $M(k)$ were established also in [Gu3,Gu4] and [ScSc1].

For small k , we have the following asymptotic representations

$$m(k) = 1 - \frac{8}{\pi} Gk + o(k^2), \quad M(k) = 1 + \frac{8}{\pi} Gk + o(k^2),$$

where $G = 0.915 \dots$ is Catalan's constant. First those bounds were established (in a different form and by another method) in [Kru3,Kru4].

Letting $1-k$ be small (i.e., for large values of K), one obtains

$$m(k) = \frac{1}{4} [1 + (1-k)O(\sqrt{1-k})],$$

$$M(k) = \exp \left\{ \pi \sqrt{\frac{2}{1-k}} + O(\sqrt{1-k}) \right\},$$

which implies as $k \rightarrow 0$ the one-quarter theorem of Koebe.

Theorem 6.6 is a consequence of the following theorem obtained in [Ku11].

THEOREM 6.7. *For the maps $f \in \Sigma(k)$, we have the sharp estimate*

$$\frac{2}{\sqrt{M(k)}} \leq |f(1) - f(-1)| \leq \frac{2}{\sqrt{m(k)}}. \quad (6.16)$$

As other illustrations of Theorem 6.5, let us mention the following results of Kühnau.

THEOREM 6.8 [Ku13]. *The exact range domain of the point $w(1)$, where w runs over the class $\Sigma(k)$, is the closed disk determined by the inequality*

$$\left| w(1) - 1 + \frac{4}{\pi^2} \arcsin^2 k \right| \leq \frac{4}{\pi} \arcsin k, \quad (6.17)$$

with equality

$$w(1) = 1 - \frac{4}{\pi^2} \arcsin^2 k + \frac{4}{\pi} (\arcsin k) e^{it}$$

only when w has the form

$$w(z) = w(1) + \frac{(z-1)(z-A^2)}{z} \cosh^2 \left[\frac{1}{2} \log \frac{\sqrt{z}-A}{\sqrt{z}+A} - \frac{1}{\pi} (\arcsin k) \frac{\sqrt{z}-1}{\sqrt{z}+1} \right] \quad (6.18)$$

for $|z| \geq 1$, and

$$w(z) = w(1) + \left[\frac{\overline{G(1/\bar{z})} - kAG(1/\bar{z})}{1-k^2} \right]^2 \quad (6.19)$$

for $|z| \leq 1$, with

$$G(z) = i\sqrt{1-k^2A}\sqrt{z-1-A^2+A^2z^{-1}} \sinh(\dots). \quad (6.20)$$

The values of $\sinh$ and $\cosh$ must be chosen from the same values of arguments. This map carries out the infinitesimal circles from the disk $\{|z| < 1\}$ onto the infinitesimal ellipses with the axes ratio equal to K whose great axes are going along confocal parabolas with the focus $w(1)$ and the focal axis inclined to the real axis under the angle $\pi + t$ ($A = e^{it}$).

Composing w with fractional linear transformations, one obtains the Schiffer–Schober estimate (6.9)–(6.10). The following result relates to their bound (6.11).

THEOREM 6.9 [Ku13]. *In the class of conformal maps w of the upper half-plane $\{z: \operatorname{Im} z > 0\}$ extending to k -quasiconformal homeomorphisms of $\mathbb{C}$ with $w(\infty) = \infty$, the exact range domain of the ratio $w''(z)/w'(z)$ is for any fixed z and k the closed disk defined by the inequality*

$$\left| i \frac{w''(z)}{w'(z)} \operatorname{Im} z + \frac{4}{\pi^2} \arcsin^2 k \right| \leq \frac{4}{\pi} \arcsin k.$$

6.5. Variations of Gutlyansky

6.5.1. A somewhat related but different variational approach was developed by Gutlyansky in [Gu2–Gu4]. It combines Schiffer’s approach to quasiconformal maps with classical Schiffer–Golusin’s method of variations in the theory of conformal maps.

We present here briefly the main ideas restricting ourselves to the class $S(k)$ for arbitrary $k \in (0, 1)$. The arguments work well also for the classes $\Sigma(k)$. The approach of Gutlyansky relies on two basic lemmas ensuring the existence of an ample family of appropriate admissible variations in this class of maps.

LEMMA 6.10. *Given a map $f = f^\mu \in S(k)$, then, for any $v \in L_\infty(\mathbb{C})$ with $\|v\| \leq k$ and sufficiently small $\varepsilon > 0$, the map*

$$f^*(z) = f(z) + \varepsilon \frac{f(z)^2}{\pi} \iint_{|\zeta|>1} \frac{(\mu(\zeta) - v(\zeta))(\partial_\zeta f)^2}{(f(\zeta) - f(z))f(\zeta)^2} d\xi d\eta + o(\varepsilon) \quad (6.21)$$

belongs to the same class $S(k)$. The remainder is estimated by $o(\varepsilon)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ locally uniformly in $\mathbb{C}$.

The proof of this variational lemma follows the standard line using the integral representation of a quasiconformal automorphism w of $\widehat{\mathbb{C}}$ with the Beltrami coefficient $(1 - \varepsilon)\mu + \varepsilon v$ with small $\varepsilon > 0$.

The proof of the following fundamental lemma combines the arguments giving the Schiffer–Golusin variational formula for the class S with the technique of the theory of quasiconformal maps.

LEMMA 6.11. *Let a function $f \in S$ be a restriction to Δ of a quasiconformal homeomorphism $\hat{f}^\mu$, which belongs to $S(k)$. Then, for an arbitrary compact set $e \in \Delta$ and for small $\varepsilon > 0$, the function*

$$\begin{aligned} f^*(z) = f(z) + \varepsilon \frac{f(z)^2}{\pi} \iint_e \left\{ \frac{A(\zeta)f(\zeta)^2}{(f(z) - f(\zeta))} \frac{\zeta^2 f'(\zeta)^2}{f(\zeta)^2} \right. \\ \left. + A(\zeta)\psi(z, \zeta) + \overline{A(\zeta)}\psi(z, 1/\bar{\zeta}) \right\} d\xi d\eta + o(\varepsilon), \end{aligned} \quad (6.22)$$

where

$$A(z) = \frac{|\mu(1/\bar{z})|}{\mu(1/\bar{z})} \frac{e^{i\alpha(z)}}{\bar{z}^2}, \quad \psi(z, \zeta) = \frac{zf'(z)}{2} \frac{\zeta + z}{\zeta - z} - \frac{f(z)}{2}, \quad (6.23)$$

and $\alpha(z)$ is an arbitrary measurable function on e satisfying $\|\alpha\|_\infty < \pi/2$, also belongs to the class S and extends to a quasiconformal homeomorphism from $S(k)$. Here $o(\varepsilon)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly on compact sets in Δ .

This variational formula has an independent interest.

6.5.2. There is an intrinsic connection between the conformal and quasiconformal variations. For example, the general Schiffer's interior variation, by attaching a cell to a Riemann surface X of finite analytic type (see [ScSp]), can be obtained by varying a suitable Beltrami differential on X . This was first observed by Gardiner (see, e.g., [GaL, Chapter 13]) and independently by Gutlyansky for a more special case.

6.5.3. Let now $J = J(f|\Delta)$ be a continuous real Gateaux differentiable functional on the class S for which we shall use the notations (6.1)–(6.4). We are concerned with estimating J on $S(k)$. Assume again that the function $J'(h(w; f))$ does not vanish identically on $f(\Delta^*)$ on any extremal f of J .

Applying variations (6.21), (6.22), one obtains the following result related to Theorems 6.2 and 6.5, which provides the necessary conditions for extremals.

THEOREM 6.12 [Gu2,Gu4]. *Any function f_0 maximizing the functional $J(f|\Delta)$ on $S(k)$, with*

$$J'(h(w; f_0)) = J'\left(\frac{f_0^2}{(f_0 - w)w^2}\right) \neq 0 \quad \text{on } f_0(\Delta),$$

is holomorphic on the closed disk $\bar{\Delta}$ with the possible exception of points on the boundary $\partial\Delta$ at which $J'(h(f_0(z); f_0)) = 0$ and satisfies the equations

$$\begin{aligned} J(h(f(z); f))z^2 f'(z)^2 + \psi(\zeta, z) + \overline{J(\psi(\zeta, 1/\bar{z}))} \\ + (1 - k^2)\overline{J(f(\tau); f)\tau^2(\partial_z f(\tau))^2} = 0 \end{aligned} \quad (6.24)$$

on the disk Δ , where $\tau = 1/\bar{z}$,

$$\begin{aligned} \operatorname{Re}\{J'(h(f(z); f))z^2 f'(z)^2\} + k|J'(h(f(z); f))z^2 f'(z)^2| \\ + (1 - k^2)\operatorname{Re} J'(\psi(\zeta, z)) = 0 \end{aligned} \quad (6.25)$$

on $\partial\Delta$, and

$$\partial_z f = k \frac{|J'(h(f(z); f))|}{J'(h(f(z); f))} \partial_z f \quad (6.26)$$

on Δ^* (outside the critical point set $\{z: J'(h(f_0(z); f_0)) = 0\}$).

These necessary conditions for extremal functions can be rewritten in somewhat different form involving (similar to Theorem 6.5) a holomorphic function $p(z)$ on $\bar{\Delta}$ with a single-valued $z^2 p'(z)$ outside the critical points of p (see [Gu4]).

6.5.4. Applying this method, one can obtain, for example, the following distortion theorem.

Let us introduce the elliptic integrals

$$\begin{aligned} u(z, r) &= \int_0^z \frac{dt}{\sqrt{t(t-r)(1-rt)}}, \\ w(z, r, x) &= \int_0^z \frac{t dt}{(1-2xt+t^2)\sqrt{t(t-r)(1-rt)}}, \end{aligned}$$

choosing the continuous branch of the square root with nonnegative imaginary part on the real axis, and define

$$\Phi(z, r, x) = u(r, r)w(z, r, x) - u(z, r)w(r, r, x).$$

THEOREM 6.13 [Gu3]. *For any $f \in S(k)$, we have the sharp bounds*

$$\frac{r}{4} \exp \int_0^r \frac{h(\zeta, \gamma_1) - 1}{\zeta} d\zeta \leq |f(z)| \leq \frac{r}{4} \exp \int_0^r \frac{h(\zeta, \gamma_2) - 1}{\zeta} d\zeta, \quad (6.27)$$

where $r = |z| < 1$,

$$h(z, \gamma) = \left(\frac{r(1 - 2zx + z^2)}{(r - z)(1 - rz)} \right)^{1/2} \cos \frac{\gamma \Phi(z, r, x)}{\Phi(-1, r, x)},$$

$\gamma_1 = \arccos k$, $\gamma_2 = \pi/2 - \gamma_1$ and $x = x(r, \gamma)$ is the unique root of the equation

$$\frac{\Phi(-1, r, x)}{u(r, r)} = \frac{\gamma}{\sqrt{(1 - x^2)(1 - 2rx + x^2)}}$$

on the interval $(-1, 1)$. The extremal functions are of the form $f(z) = \epsilon f_0(\epsilon^{-1}z)$, $|\epsilon| = 1$, with

$$f_0(z) = \begin{cases} \varphi(z) \left[1 + \frac{\varphi(z)}{\varphi(r)} \right]^{-2} & \text{if } |z| \leq 1, \\ -\varphi(-r)g(z)\overline{g(z)}^{k\eta} [1 - g(z)\overline{g(z)}^{k\eta}]^{-2} & \text{if } |z| \geq 1. \end{cases}$$

Here

$$\varphi(z) = z \exp \int_0^r \frac{h(\zeta, \gamma) - 1}{\zeta} d\zeta, \quad g(z) = \exp \int_\infty^z \frac{q(\zeta, \gamma)}{\sqrt{(1 - k^2)}} \frac{d\zeta}{\zeta}$$

and

$$q(z, \gamma) = \left(\frac{r(1 - 2zx + z^2)}{(r - z)(1 - rz)} \right)^{1/2} \sin \frac{\gamma \Phi(z, r, x)}{\Phi(-1, r, x)},$$

with $\eta = 1$ in the case of a maximum and $\eta = -1$ in the case of a minimum.

As $r \rightarrow 1$, the function $h(z, \gamma)$ assumes the form

$$h(z, \gamma) = \frac{1}{2} \left(\frac{1 + z^{1/2}}{1 - z^{1/2}} \right)^{2\gamma/\pi i} + \frac{1}{2} \left(\frac{1 - z^{1/2}}{1 + z^{1/2}} \right)^{2\gamma/\pi i},$$

and the bounds (6.27) turn into (6.15).

6.6. Applications of the Dirichlet principle and of Fredholm eigenvalues. Kühnau's method. Applications

6.6.1. Let us now consider a somewhat different variational approach concerning general quasiconformal maps of finitely connected domains and reveal the extremal properties of the maps onto domains obtained from the sphere by parallel linear cuts. This was established in [Ku5] by extending the strip method of Grötzsch and the contour integration as well as in [Ku14] by minimization of a modified Dirichlet integral. Here we touch on the last method.

Let $G \subseteq \mathbb{C}$ be a finitely-connected domain containing the point at infinity with the boundary $C = \partial G$ possessing application of Green's integral formula. Consider a function $p_0(z) \geq 1$ having in G piecewise Hölder continuous partial derivatives (hence $\|p_0\|_\infty \leq \infty$) and assume that $p_0(z) \equiv 0$ in a neighborhood of infinity.

There exists a quasiconformal homeomorphism g_0 with Lavrentiev's dilatation

$$p_w(z) = \frac{|\partial_z w| + |\partial_{\bar{z}} w|}{|\partial_z w| - |\partial_{\bar{z}} w|}$$

equal to $p_0(z)$ and hydrodynamical normalization

$$g_0(z) = z + A_{1,0}z^{-1} + \dots,$$

mapping the domain G onto a domain $g_0(G)$ whose boundary components are the straight cuts parallel to the real axes $\mathbb{R}$. Consider also the conformal map

$$\omega_0(z) = z + \mathcal{A}_{1,0}z^{-1} + \dots$$

of G onto a domain bounded by straight cuts parallel to $\mathbb{R}$ and put

$$\Phi = \operatorname{Re} \omega_0, \quad \Phi^* = \operatorname{Re} g_0 = \Phi + \varphi^*. \quad (6.28)$$

Then, due to [Ku17] (see also [KK1, Part 2]), the function Φ^* is a solution of the differential equation

$$\operatorname{div} \left(\frac{1}{p_0} \operatorname{grad} \Phi^* \right) = 0. \quad (6.29)$$

The *admissible comparison functions* φ on G are those for which

$$|\operatorname{grad} \psi(z)| \leq c|z|^{-2} \quad \text{as } z \rightarrow \infty \quad (6.30)$$

with a constant c .

Then one obtains the following extremal principle, which is a generalization of the Diaz–Weinstein principle for conformal maps (cf., e.g., [Ku17]).

THEOREM 6.14. *For all nonconstant admissible ψ , we have*

$$\frac{[\iint_G (1 - \frac{1}{p_0}) \operatorname{grad} \Phi \operatorname{grad} \psi \, dx \, dy]^2}{\iint_G \frac{1}{p_0} \operatorname{grad}^2 \psi \, dx \, dy} \leq 2\pi \operatorname{Re}(A_{1,0} - \mathcal{A}_{1,0}) - \iint_G \left(1 - \frac{1}{p_0}\right) \operatorname{grad}^2 \Phi \, dx \, dy. \quad (6.31)$$

The equality in (6.28) occurs only for $\psi = \alpha\varphi + \beta$, where α and β are constant.

The most interesting, though the simplest case, occurs when $G = \mathbb{C}$ and $p_0(z) \equiv K$ in a union of a finite number of distinct simply-connected domains G_j bounded by nonintersecting analytic curves $C_k \subset \mathbb{C}$, and $p_0(z) \equiv 1$ in the complement of this union containing the point at infinity.

6.6.2. The above variational principle provides various sharp quantitative estimates. We restrict ourselves to three Kühnau's theorems, referring to [Ku17] and to his Part 2 of the joint book [KK1] (cf. [McL]).

THEOREM 6.15. *The exact range domain of the Grunsky functional $\sum_{l,s=1}^n c_{ls} x_l x_s$ on the family $\mathcal{F}(p_0)$ of all $p_0(z)$ -quasiconformal maps $w(z) = z + a_1 z^{-1} + \dots$ of $\mathbb{C}$ with $p_0(z) \equiv 1$ in a neighborhood of infinity is the closed disk whose boundary circle is located in the open annulus centered at the origin, with radii*

$$\frac{1}{2\pi} \sum_{l,s=1}^n x_l \bar{x}_s \iint_{\mathbb{C}} \left(1 - \frac{1}{p_0}\right) z^{l-1} \bar{z}^{s-1} \, dx \, dy$$

and

$$\frac{1}{2\pi} \sum_{l,s=1}^n x_l \bar{x}_s \iint_{\mathbb{C}} (1 - p_0) z^{l-1} \bar{z}^{s-1} \, dx \, dy,$$

provided $p_0(z) \not\equiv 1$.

THEOREM 6.16. *The exact range domain of the functional*

$$\log \frac{w(z_1) - w(z_2)}{z_1 - z_2}$$

for two fixed distinct points z_1 and z_2 on the family $\mathcal{F}(p_0)$ is the closed disk whose boundary circle is located in the open annulus centered at the origin, with radii

$$\frac{1}{2\pi} \iint_{\mathbb{C}} \left(1 - \frac{1}{p_0}\right) z^{l-1} \bar{z}^{s-1} \frac{dx \, dy}{|z - z_1| |z - z_2|}$$

and

$$\frac{1}{2\pi} \iint_{\mathbb{C}} (1 - p_0) z^{l-1} \bar{z}^{s-1} \frac{dx dy}{|z - z_1||z - z_2|}$$

provided $p_0(z) \not\equiv 1$.

Note that one does not require here that $p_0(z)$ be equal 1 near the fixed points z_1 and z_2 . Kühnau has observed also that in many cases the assumption $p_0(z) \equiv 1$ can be omitted or replaced by a weaker one that p_0 tends to 1 sufficiently fast.

Let us mention here the special cases when $z_1 = 0$ and the class $\mathcal{F}(p_0)$ is either $\Sigma(k)$ or $S(k)$, which concerns Theorem 6.15. The bounds of $\log[w(z)/z]$ on these classes following from Theorem 6.16 can be represented also by means of the complete elliptic integral $\mathbf{K}(\kappa)$ of the first kind. For example, we have the following theorem.

THEOREM 6.17. *The range domain of $\log[w(z)/z]$ with a fixed $z \in \mathbb{C}$ on the maps from $S(k)$ for each $k \in (0, 1)$ (i.e., for $K = (1+k)/(1-k) > 1$) is a closed disk whose boundary circle is located in the open annulus centered at the origin, with radii*

$$\frac{1}{2\pi} \left(1 - \frac{1}{K}\right) \int_0^{|z|} \mathbf{K}(\kappa) d\kappa \quad \text{and} \quad \frac{1}{2\pi} (K - 1) \int_0^{|z|} \mathbf{K}(\kappa) d\kappa \quad (6.32)$$

for $|z| \leq 1$, and

$$\begin{aligned} & \frac{1}{2\pi} \left(1 - \frac{1}{K}\right) \left\{ 2G + \int_0^{|z|} \mathbf{K}\left(\frac{1}{\kappa}\right) \frac{d\kappa}{\kappa} \right\} \\ & \text{and} \\ & \frac{1}{2\pi} (K - 1) \left\{ 2G + \int_0^{|z|} \mathbf{K}\left(\frac{1}{\kappa}\right) \frac{d\kappa}{\kappa} \right\} \end{aligned} \quad (6.33)$$

for $|z| > 1$. Here G denotes the Catalan constant.

The bounds (6.33) follows also from Theorem 6.13.

6.6.3. The general Theorem 6.14 can be combined with the properties of the Fredholm eigenvalues λ_C of a finite union of Jordan curves $C = \bigcup_j C_j$ (cf. Section 2.5). This provides, for example, the following result.

Assume that a domain G is of the same type as in Theorem 6.14 and that its boundary curves are analytic. Let $I = I(G^*)$ denote the (finite) area of the complement domain $G^* = \widehat{\mathbb{C}} \setminus \overline{G}$. Consider the class $\mathcal{F}(K)$ of univalent $\mathbb{C}$ -holomorphic functions $f(z) = z + b_1 z^{-1} + \dots$ on G having K -quasiconformal extensions to $\widehat{\mathbb{C}}$. Put

$$\Lambda_C = (\lambda_C + 1)(\lambda_C - 1) > 1.$$

THEOREM 6.18 [Ku17]. *The range domain of the coefficient b_1 on $\mathcal{F}(K)$ is the disk whose boundary circle is located in the open annulus centered at the origin, with radii*

$$\frac{I(K-1)}{2\pi} \left(1 - \frac{K-1}{1/\Lambda_C + K} \right) \quad \text{and} \quad \frac{I(K-1)}{2\pi} \left(1 - \frac{K-1}{\Lambda_C + K} \right). \quad (6.34)$$

Both quantities in (6.34) coincide only if $\Lambda_C = 1$, i.e., $\lambda_C = \infty$, which occur when C consists of one curve which is a circle. Then $\mathcal{F}(K) = \Sigma(k)$ and (6.34) is reduced to the well-known bound $|b_1| \leq k$; the equality holds only for the function

$$f(z) = \begin{cases} z + tz^{-1} & \text{for } |z| \geq 1, \\ z + t\bar{z} & \text{for } |z| < 1 \end{cases} \quad (6.35)$$

with $|t| = 1$. This was first established in [Ku7].

6.7. The Dirichlet principle and the area method

6.7.1. As is well known, the Dirichlet integral remains K -quasiinvariant (i.e., up to factor K) under K -quasiconformal homeomorphisms, in particular, under K -quasireflections. This property was first applied to univalent functions with quasiconformal extensions in [Ah5].

In fact, the Dirichlet principle is equivalent to the area theorems, because for k -quasiconformal maps (where $k = (K-1)/(K+1) < 1$) we have inequality

$$|\partial_z f|^2 + |\partial_{\bar{z}} z f|^2 \leq \frac{1+k^2}{1-k^2} (|\partial_z f|^2 - |\partial_{\bar{z}} z f|^2). \quad (6.36)$$

The area method is one of the basics tools in the theory of univalent functions (see, e.g., [Leb,Mil]). A crucial step in its extension to functions with quasiconformal extensions was made in Lehto's paper [Leh1]. A further development was given by Gutlyansky [Gu1], using this idea. He gave the following strengthening of the general Lebedev–Milim area theorem:

THEOREM 6.19. *Let $w = f(z) \in \Sigma(k)$, and let $Q(w)$ be an arbitrary nonconstant holomorphic function on the image $f(\Delta_{R_0})$ of a disk $\Delta_{R_0} = \{|z| < R_0\}$ with $1 < R_0 < \infty$. Suppose that the Laurent expansion of the composed map $f_Q = Q \circ f$ in the annulus $\{1 < |z| < R_0\}$ has the form*

$$f_Q(z) = \sum_{n=0}^{\infty} \eta_n z^n + \sum_{n=1}^{\infty} \omega_n z^{-n}.$$

Then

$$\sum_{n=1}^{\infty} n |\omega_n|^2 \leq k^2 \sum_{n=1}^{\infty} n |\eta_n|^2,$$

with equality only for those functions $f \in \Sigma(k)$ for which

$$f_Q(z) = \begin{cases} \sum_{n=0}^{\infty} \eta_n z^n + k e^{i\theta} \sum_{n=1}^{\infty} \bar{\eta}_n z^{-n}, & 1 < |z| < R_0, \\ \sum_{n=0}^{\infty} \eta_n z^n + k e^{i\theta} \sum_{n=1}^{\infty} \bar{\eta}_n \bar{z}^n, & |z| \leq 1, \end{cases} \quad (6.37)$$

where θ is a real constant.

In particular, for $Q(w) = w$, one obtains that all $f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma(k)$ satisfy

$$\sum_{n=1}^{\infty} n |b_n|^2 \leq k^2;$$

the equality occurs here only for the functions (6.35). This strengthening of the classical Gronwall area theorem was obtained by Kühnau [Ku7] and Lehto [Leh1], using different methods.

Theorem 6.19 allows us to obtain for the classes $\Sigma(k)$ and $S(k)$ various distortion theorems which improve the corresponding results for general conformal maps (see, e.g., [Leb, Mil]). Let us illustrate this by two results following [Gu1, Kru5] (noting that these results have been obtained also by different methods; see, e.g., [Ku5, Ku6]).

THEOREM 6.20. *If $f \in \Sigma(k)$, then*

$$\left(1 - \frac{1}{|\xi|^2}\right)^k \leq |f'(\xi)| \leq \left(\frac{|\xi|^2}{|\xi|^2 - 1}\right)^k. \quad (6.38)$$

This estimate is sharp. The equality in the first of (6.38) for a finite ξ holds only for the functions

$$f(z) = \begin{cases} (z - \xi) \left(1 - \frac{1}{\xi z}\right)^{k e^{i\theta}} + c, & |z| > 1, \\ (z - \bar{\xi}) \left(1 - \frac{\bar{z}}{\bar{\xi}}\right)^{k e^{i\theta}} + c, & |z| \leq 1, \end{cases}$$

and in the second part for the functions

$$f(z) = \begin{cases} (z - \xi) \left(1 - \frac{1}{\xi z}\right)^{-k e^{i\theta}} + c, & |z| > 1, \\ (z - \bar{\xi}) \left(1 - \frac{\bar{z}}{\bar{\xi}}\right)^{-k e^{i\theta}} + c, & |z| \leq 1, \end{cases}$$

where θ is a real constant and c is a complex constant.

THEOREM 6.21. *For any function $f \in \Sigma(k)$, we have the sharp bound for its Schwarzian*

$$S_f(z) \leq 6k(|z|^2 - 1)^{-2}, \quad |z| > 1.$$

The area method was further extended by several authors.

Sheretov proved, using the Dirichlet principle, a variant of the area theorem for univalent functions $f \in \Sigma$ admitting homeomorphic extensions, which are quasiconformal in the mean (see [She1]). Another his extension relies on applying the covering maps $Q(w)$ of more general type (see [She3, She4]), which provides somewhat new inequalities ensuring global univalence.

Grinshpan [Gri1] improved the results of Gutlyansky and Sheretov by involving homeomorphic extensions of more general type, with finite Dirichlet integral. His area theorem has allowed him to obtain various quantitative estimates for coefficient functionals and to strengthen many classical estimates for those classes.

Hoy [Ho1] gave an extension of the area theorem to univalent functions with $p_0(z)$ -quasiconformal extensions.

An alternative development of the area method providing several important consequences was given by Pommerenke (see [Po1, pp. 289–294]).

6.7.2. Quasiconformal maps have a deep intrinsic connection with the generalized Dirichlet principle, which concerns the minimums of the energy integral corresponding to the Riemannian metrics on Riemann surfaces. The extremals are harmonic maps (satisfying the corresponding Beltrami–Laplace equation). This approach has a long history beginning from the 1950s and was studied by many authors.

For example, consider a smooth Riemannian metric $ds = \lambda(w)|dw|$ on a Riemann surface X' of a finite-analytic type, with $\|\lambda(w)^2\|_{L_2(X')} = 1$. Let X be another Riemann surface of the same analytic type. Fix a homotopy class of homeomorphisms $f : X \rightarrow X'$ and minimize in this class the energy integral

$$E_\lambda(f) = \frac{1}{2} \iint_{X'} (|f_z|^2 + |f_{\bar{z}}|^2) \lambda \circ f(z) |dz \wedge d\bar{z}|$$

(determined for homeomorphisms with square integrable distributional derivatives $f_z = \partial_z f$ and $f_{\bar{z}} = \partial_{\bar{z}} f$).

The Euler–Lagrange equation for this functional assumes the form

$$\lambda \circ f(z) f_{z\bar{z}} + 2\lambda'_w \circ f(z) f_z f_{\bar{z}} = 0. \quad (6.39)$$

This equation is quasilinear. C^2 -smooth solutions of (6.39) (the extremals of $E_\lambda(f)$) are called λ -harmonic maps. It is a map of Teichmüller type with Beltrami coefficient $\mu_f(z) = k(z)|\varphi|/\varphi$ defined by holomorphic quadratic differential $\varphi = \lambda^2 \circ f(z) f_z \bar{f}_{\bar{z}} dz^2$ on X . Equation (6.39) has sense also for the Riemannian metrics with isolated singularities. Then its solutions are determined in the domain, where the defining metric λ is smooth. In particular, the Teichmüller map $f : X \rightarrow X'$ with defining quadratic differentials φ on X and ψ on X' is harmonic in the metric $\lambda(w) = \sqrt{|\psi(w)|} |dw|$, which produces a harmonic flow on the corresponding Teichmüller space.

This approach provides many beautiful results in the theory of quasiconformal maps, Teichmüller space theory, real geometry and in other fields. We cannot go into details,

because these results are outside the framework of our survey. We refer, e.g., to [AnMM, BLMM, GoS, KP2, Ma1, Ma2, Min, Pa, ScY, She4, Wo1, Wo2].

On the other hand, (6.39) in the case of the Euclidean metric $\lambda = |dw|$ gives the Laplace equation $f_{z\bar{z}} = 0$, and one obtains the usual harmonic maps f . Locally, $f = F_1 + \bar{F}_2$, where F_1 and F_2 are holomorphic functions. The theory of harmonic maps is now being intensively developed as a natural extension of the classical Geometric Function Theory (see, e.g., [BsH, Du]), and one can expect its deep interactions with quasiconformal maps.

6.8. Other methods and results

6.8.1. The classical theory of conformal maps uses different basic methods. We have already described in the previous sections the extensions of certain basic methods to univalent holomorphic functions having quasiconformal extensions to the whole plane, even $p_0(z)$ -extensions with nonconstant $p_0(z)$, and presented some fundamental results obtained by these methods. Let us now mention other important methods.

- The method of contour integration introduced by Grunsky (see [Gru2, Je, Ne3]).
- The method of strips introduced by Grötzsch and the closely related method of extremal length developed later by Beurling and Ahlfors (see, e.g., [Ah1, Ah2, Du, Po1, Va3]).
- Löwner's method based on his differential equation and the closely related method of parametric representations (see, e.g., [Al, Du, Gol]).
- The extreme point method.

The first two methods were strengthened for the maps with $p_0(z)$ -extensions by Kühnau starting with the papers [Ku1–Ku3] (see also [KK1, Part 2]). He obtained various qualitative results and distortion theorems. Many of those results can be reproved by other methods already presented in the preceding sections.

6.8.2. Löwner's method plays a fundamental role in different directions though it was introduced originally for solving extremal problems for conformal maps. This aspect closely relates to variational methods.

The geometric aspect of Löwner's equation, i.e., the description of the families of conformal maps of the unit disk onto the expanding domains was initiated independently by Kufarev and Schiffer in the 1940s (see [Kf1, Kf2, Sch1]) and developed by many authors. A deep contribution of Becker (see [Bec1, Bec2, Po1]) shed light on the intrinsic features of Löwner's chains.

Other aspects of Löwner's equation concern, beginning from [Lo], the semigroups of conformal maps and applications of semigroups of analytic functions to branching processes. These questions are also investigated by many authors. Such an approach was systematically applied to solving the problems in all the directions mentioned above by Goryainov (see, e.g., [Gor1–Gor4]).

Löwner's chains can be applied to quasiconformal extensions of holomorphic functions. Certain problems involving the parametric representations of quasiconformal homeomorphisms of the plane were treated in [Kru1, KL1, KL2, Re2, Sha, ShF, Sh2].

Recently Earle and Epstein [EE] have applied Löwner's equation to solving a problem of Gaier on real analytic dependence of conformal radii of the slit domains on the slit length. These are the simply connected domains $D_t = D \setminus \Gamma_t$ obtained from Jordan domains $D \subset \mathbb{C}$ by cutting along a variable subarc $\Gamma_t = \gamma([0, t])$ of a Jordan arc $\gamma: [0, T) \rightarrow D$ in D , with $0 < T \leq \infty$. The classical Löwner's equation connected the derivatives $\partial f(z, t)/\partial z$ and $\partial f(z, t)/\partial t$ of the normalized conformal maps $z \mapsto f(z, t)$ of the disk onto D_t . Using quasiconformal extensions of holomorphic functions and the technique of holomorphic motions, Earle and Epstein established an essential improvement of Löwner's theorem and gave a complete description of the smoothness order of conformal map $f(z, t)$ with respect to parameterization of the slit.

Semigroups of continuous maps (deformations) are essentially applied in various questions of Geometric Function Theory. Such an approach provides, in fact, the existence theorems, even when these cannot be obtained by other methods.

Semigroups of quasiconformal maps in $\mathbb{R}^n$, $n \geq 2$, were considered in the works of Reshetnyak, Reimann and Semenov (see, e.g., [Res1, Rei, Se1–Se3]). They established many interesting results.

No special applications of this method to univalent functions with quasiconformal extensions have been given.

6.8.3. The method of extreme and support points arose in mathematics from the famous Krein–Milman theorem and provides quite a power tool. It was applied also to maximization of linear functionals on some compact families of holomorphic function, for example, on Σ . The results obtained are presented in the books of Duren [Du], Schober [Scho1] and Hallenbeck–McGregor [HM].

Until now, there are no wide applications of this method to classes of holomorphic functions with quasiconformal extensions.

6.8.4. Milin's approach to the classical coefficient problems relies on the inequalities of exponential and logarithmic types for univalent functions and involves in an essential way the Grunsky functional.

Grinshpan and Pommerenke have provided an extension of this fruitful method to holomorphic functions with quasiconformal extensions whose dilatation is measured by the Grunsky norm (see [Gri2–Gri4, GrP1, GrP2]).

Let $\tilde{S}(k)$ denote the class of univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ with the Grunsky constant $\kappa(f) \leq k < 1$. Due to examples of Kühnau and Krushkal mentioned in Section 2, this class is much wider than $S(k)$. We are concerned with the growth order in n of the functionals

$$I_n(f) = |a_{n+1}| - |a_n|$$

on the class $\tilde{S}(k)$. Estimating this important functional in the whole class of the normalized univalent functions has a rather long history going back to Golusin, Hayman and their followers (see, e.g., [Du, Hay1, Mil, GrP1]).

THEOREM 6.22 [GrP1]. *Let f be a function in $\tilde{S}(k)$, $k \in (0, 1)$. Then for $n > 1$,*

$$|I_n(f)| < \begin{cases} L(k)n^{-\delta-1/2}, & k \in (0, 1/2), \\ L(k)n^{-\delta} \log n, & k = 1/2, \\ L(k)n^{k-1}, & k \in (1/2, 1), \end{cases}$$

where $\delta > 0$ and L depends only on k .

6.8.5. An important place in the theory of conformal maps is traditionally occupied by extremal problems on various special compact subclasses of normalized univalent functions. Among those, the class S_M of bounded univalent functions f plays an important role. The extremal domains in this class arise by slitting the disk. The main problems and results obtained here are well described, for example, in [Goo] and [Pr2].

It would be interesting to discover the features of approximating (in the topology the space $\mathbb{B}$ of Schwarzians or in weaker topology of local uniform convergence) maps having quasiconformal extensions. No results have been established in this direction.

6.9. Multivalent functions

Another important direction in Geometric Function Theory is provided by the geometry of multivalued holomorphic functions, which goes back to the basic works of Hayman, Schiffer and other authors (see, e.g., [Hay2, Schi1]). Its generalization to functions with quasiconformal extensions looks very interesting because it closely relates to the geometry of finite-to-one covering maps. Only a few results have been obtained in this way (see [V1]).

7. Univalent functions and universal Teichmüller space

7.1. The Bers embedding of universal Teichmüller space

7.1.1. The universal Teichmüller space $\mathbb{T}$ is the space of quasisymmetric homeomorphisms h of the unit circle factorized by Möbius transformations. Its topology and real geometry are determined by the Teichmüller metric which naturally arises from extensions of those h to the unit disk. As was mentioned in Section 2.2, this space admits also the complex structure of a complex Banach manifold by means of the Bers embedding as a bounded subdomain of the Banach space $\mathbb{B}$ of holomorphic functions φ in the disk Δ^* with the norm $\|\varphi\| = \sup_{\Delta^*} (|z|^2 - 1)^2 |\varphi(z)|$. Note that $\varphi(z) = O(|z|^{-4})$ as $z \rightarrow \infty$.

We shall identify the space $\mathbb{T}$ with this domain. In this model the points $\psi \in \mathbb{T}$ represent the Schwarzian derivatives S_f of univalent holomorphic functions f in Δ^* , which have quasiconformal extensions to the whole sphere $\hat{\mathbb{C}}$.

Recall that the universal Teichmüller space $\mathbb{T}$ is obtained from the Banach ball

$$\text{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}): \mu|_{\Delta^*} = 0, \|\mu\| < 1\}$$

of conformal structures on $\widehat{\mathbb{C}}$ by the natural identification, letting μ and ν in $\text{Belt}(\Delta)_1$ be equivalent if $w^\mu|S^1 = w^\nu|S^1$, $S^1 = \partial\Delta$. We denote the equivalence classes by $[\mu]$.

For an arbitrary (finitely or infinitely generated) Fuchsian group G with invariant unit circle $\partial\Delta$ we set

$$\mathbb{B}(\Gamma) = \{\varphi \in \mathbb{B}: (\varphi \circ \gamma)\gamma'^2 = \varphi \text{ for all } \gamma \in \Gamma\},$$

which is the space of hyperbolically bounded Γ -automorphic 2-forms. This yields that $\mathbb{T}$ contains the copies of Teichmüller spaces $\mathbb{T}(\Gamma)$ of arbitrary Riemann surfaces and of uniformizing Fuchsian groups. These spaces are isometrically embedded into $\mathbb{T}$. It is established that $\mathbb{T}(\Gamma) = \mathbb{T} \cap \mathbb{B}(\Gamma)$ (see, e.g., [Leh2]). The spaces $\mathbb{T}(\Gamma)$ involve univalent holomorphic functions with quasiconformal extensions compatible with the Fuchsian and quasi-Fuchsian groups.

Let us introduce also the sets

$$\mathbb{S} = \{\varphi = S_f: f \text{ univalent in } \Delta^*\}, \quad \mathbb{S}(\Gamma) = \mathbb{S} \cap \mathbb{B}(\Gamma).$$

We consider on $\mathbb{S}(\Gamma)$ the topology induced by the norm in $\mathbb{B}$; the convergence in this topology is invariant with respect to Möbius transformations of $\widehat{\mathbb{C}}$. The Schwarzian S_f can be regarded as a measure for deviation of the mapping f from a Möbius one.

In some instances in the sequel, it would be more convenient to consider the functions holomorphic in the disk Δ , instead of Δ^* . We shall keep for this case the above notations.

7.1.2. There are certain natural intrinsic complete metrics on the space $\mathbb{T}$. The first one is the *Teichmüller metric*

$$\tau_{\mathbb{T}}(\phi_{\mathbb{T}}(\mu), \phi_{\mathbb{T}}(\nu)) = \frac{1}{2} \inf \{ \log K(w^{\mu*} \circ (w^{\nu*})^{-1}): \mu_* \in \phi_{\mathbb{T}}(\mu), \nu_* \in \phi_{\mathbb{T}}(\nu) \},$$

where $\phi_{\mathbb{T}}$ is the canonical projection

$$\phi_{\mathbb{T}}(\mu) = [\mu]: \text{Belt}(\Delta)_1 \rightarrow \mathbb{T}.$$

This metric is generated by the *Finsler structure* on $\mathbb{T}$ (in fact, on the tangent bundle $\mathcal{T}(\mathbb{T}) = \mathbb{T} \times \mathbb{B}(\mathbb{T})$; this structure is defined by

$$F_{\mathbb{T}}(\phi_{\mathbb{T}}(\mu), \phi'_{\mathbb{T}}(\mu)v) = \inf \{ \|v_*(1 - |\mu|^2)^{-1}\|_{\infty}: \phi'_{\mathbb{T}}(\mu)v_* = \phi'_{\mathbb{T}}(\mu)v; \\ \mu \in \text{Belt}(\Delta)_1; v, v_* \in L_{\infty}(\mathbb{C}) \}. \quad (7.1)$$

On the other hand, the universal Teichmüller space like complex Banach manifolds admits the invariant metrics and holomorphic contractions, plurisubharmonic functions and related pluricomplex potentials.

The Carathéodory and Kobayashi metrics on $\mathbb{T}$ are, as usual, the smallest and the largest semimetrics d on $\mathbb{T}$, which are contracted by holomorphic maps $h: \Delta \rightarrow \mathbb{T}$. Denote these metrics by $c_{\mathbb{T}}$ and $d_{\mathbb{T}}$, respectively. Then

$$c_{\mathbb{T}}(\psi_1, \psi_2) = \sup \{ d_{\Delta}(h(\psi_1), h(\psi_2)): h \in \text{Hol}(\mathbb{T}, \Delta) \},$$

while $d_{\mathbb{T}}(\psi_1, \psi_2)$ is the largest pseudometric d on $\mathbb{T}$ satisfying

$$d(\psi_1, \psi_2) \leq \inf \{d_{\Delta}(0, t): h(0) = \psi_1 \text{ and } h(t) = \psi_2, h \in \text{Hol}(\Delta, \mathbb{T})\},$$

where d_{Δ} is the hyperbolic Poincaré metric on Δ of Gaussian curvature -4 .

The fundamental Royden–Gardiner theorem states that Teichmüller and Kobayashi metrics coincide on every Teichmüller space (see, e.g., [EKK, GaL, Roy2]).

7.2. Holomorphic curves in the set of Schwarzian derivatives of univalent functions

7.2.1. Bers posed several important problems concerning geometrical features of Teichmüller spaces (see, e.g., [Ber5]). Let us start with his question whether the closure of $\mathbb{T}$ in $\mathbb{B}$ coincides with $\mathbb{S}$, i.e., with the set of all Schwarzian derivatives of univalent functions in the disk.

Gehring [Ge1] established that if the Schwarzian S_f of a univalent function $f: \Delta \rightarrow \widehat{\mathbb{C}}$ belongs to $\mathbb{S}$ with a whole neighborhood in $\mathbb{B}$, then $S_f \in \mathbb{T}$. Such a result can be obtained also by applying the lambda-lemma by Mañé, Sad and Sullivan on holomorphic motions (see [MSS]). We shall touch on holomorphic motions and their applications in the consequent paper [Kru24].

Applying the Ahlfors–Bers theorem that $\mathbb{T}$ is an open subset of $\mathbb{B}$, one obtains that $\mathbb{T}$ must coincide with the interior of $\mathbb{S}$. This result of Gehring was extended to arbitrary Fuchsian groups Γ by Zhuravlev and other authors.

The proof of many results concerning the geometric features of Teichmüller spaces relies on the properties of holomorphic curves in the sets $S(\Gamma)$. These properties are revealed by the following remarkable theorem of Zhuravlev:

THEOREM 7.1 [Zh1, Zh2]. *Let a function $F: \overline{\Delta} \rightarrow \mathbb{B}$ be holomorphic in Δ , continuous in $\overline{\Delta}$ and such that $F(\partial \Delta) \subset S(\Gamma)$. Then the following hold:*

- (a) $F(\Delta) \subset S(\Gamma)$;
- (b) if additionally, $F(\overline{\Delta}) \cap \mathbb{T} \neq \emptyset$, then $F(\Delta) \subset \mathbb{T}$;
- (c) if $F(\overline{\Delta}) \cap \mathbb{T}(\Gamma) \neq \emptyset$, then $F(\Delta) \subset T(\Gamma)$.

The proof of this theorem involves the technique based on the Grunsky inequalities, in particular, Pommerenke’s theorem that every map $f \in \Sigma$ with the Grunsky constant $\kappa(f) = k < 1$ has k' -quasiconformal extension to $\widehat{\mathbb{C}}$ with some $k' \geq k$ ([Po1]; see also [Zh1], [KK1, Part 1]).

7.2.2. Gehring established also that the closure of $\mathbb{T}$ in $\mathbb{B}$ does not coincide with $\mathbb{S}$, i.e., $\mathbb{S} \setminus \mathbb{T} \neq \emptyset$ (see [Ge2]). Actually, he proved that $\mathbb{S} \setminus \mathbb{J} \neq \emptyset$, where

$$\mathbb{J} = \{S_f: f(\Delta^*) \text{ is a Jordan domain}\}.$$

Later Thurston [Th1] proved the much stronger result that $\mathbb{S} \setminus \mathbb{T}$ contains a noncountable set of isolated components. Thurston established the existence of so-called *conformally rigid* simply-connected domains $D \subset \widehat{\mathbb{C}}$, having the property that there is a constant $\varepsilon_0(D) > 0$ such that any injective holomorphic map $h : D \rightarrow \widehat{\mathbb{C}}$ whose Schwarzian derivative has norm $\sup_D \lambda^{-2} |S_h| < \varepsilon_0$ must reduce to a linear fractional transformation (here λ is the hyperbolic density of D). A conformal map f^* of Δ^* onto a rigid domain D determines a point $S_f \in \mathbb{B}$ that is isolated in $\mathbb{S}$ and exterior to $\mathbb{T}$. Later Astala [As1] made variations on Thurston's examples.

Additional improvements of Gehring's result were made by Flinn [Fl] and Sugawa [Su2]. It was established in [Fl] that $\mathbb{J} \setminus \mathbb{T} \neq \emptyset$. Sugawa extended the Gehring–Flinn constructions to arbitrary Fuchsian groups of the second kind and showed that for any such group Γ ,

$$\mathbb{S}(\Gamma) \setminus \overline{\mathbb{T}(\Gamma)} \neq \emptyset, \quad \mathbb{J}(\Gamma) \setminus \overline{\mathbb{T}(\Gamma)} \neq \emptyset,$$

where $\mathbb{J}(\Gamma) = \mathbb{J} \cap \mathbb{B}(\Gamma)$. For the Fuchsian group Γ of the first kind the question remains open.

7.2.3. The problems related to complex geometry of universal Teichmüller space and applications of the Grunsky inequalities were treated also by Shiga, Tanigawa, Shen Yuliang and other authors (see, e.g., [Shi1, Shi2, ShT] as well as Section 7.5).

7.3. Some topological properties

One of the open problems in Teichmüller space theory is to describe the boundary properties of these spaces in Bers' embedding.

In the finite-dimensional case, that is, for Teichmüller spaces of finitely generated Fuchsian groups of the first kind, the structure of Bers' boundary in terms of Kleinian groups was established by Bers [Ber4] and Maskit [Mas1], see also [Ab1].

Another notion of the boundary for (finite-dimensional) Teichmüller space was introduced by Thurston [Th2] using his measured laminations and general convergence of Kleinian groups. Kerckhoff [Ke] proved that Bers' and Thurston's boundaries coincide almost everywhere.

The following Abikov–Bers–Zhuravlev theorem reveals another property of the boundaries of Teichmüller spaces.

THEOREM 7.2. *For any Fuchsian group Γ the domains $\mathbb{T}(\Gamma)$ and $\mathbb{B} \setminus \overline{\mathbb{T}(\Gamma)}$ have a common boundary.*

This was first established in [Ab3] for finite-dimensional spaces and extended by different methods in [Ber10, Zh3] to arbitrary Teichmüller spaces. Zhuravlev's proof involves Theorem 7.1, while in [Ber10] the improved lambda-lemma on holomorphic motions was applied (see [BerR, MSS]).

7.4. Conformally rigid domains and shape of Teichmüller spaces

7.4.1. The following question concerns complex geometry of Teichmüller spaces and was stated in [BerK] in a collection of unsolved problems for Teichmüller spaces and Kleinian groups:

For an arbitrary finitely or infinitely generated Fuchsian group Γ is the Bers embedding of its Teichmüller space $\mathbb{T}(\Gamma)$ starlike?

Intuitively, it seems that $\mathbb{T}(G)$ cannot be starlike, but must have a considerably more complicated structure; yet, on the other hand, the assumption of starlikeness for these spaces does not contradict the known results in the theory of univalent functions.

It was shown in [Kru12] that universal Teichmüller space $\mathbb{T}$ has points which cannot be joined to a distinguished point even by curves of a considerably general form, in particular, by polygonal lines with the same finite number of rectilinear segments. The proof relies on the existence of conformally rigid domains.

THEOREM 7.3 [Kru12]. *Let a function $\gamma(z, t): \bar{\Delta}^* \times [0, 1] \rightarrow \mathbb{C}$ be jointly continuous in (z, t) , holomorphic in z for each t and satisfy:*

- (i) $\gamma(z, 0) = 0, \gamma(z, 1) = 1$ for all $z \in \Delta^*$;
- (ii) *there exists $\delta > 0$ such that $\gamma(z, t_1) \neq \gamma(z, t_2)$ for all $t_1, t_2 \in (1 - \delta, 1)$.*

Then there exist points $\varphi \in \mathbb{T}$ such that every curve $t \mapsto \gamma(\cdot, t)\varphi: [0, 1] \rightarrow \mathbb{B}$ does not lie entirely in $\mathbb{S}$.

SKETCH OF THE PROOF. Assume, on the contrary, that for each $\varphi \in \mathbb{T}$ the corresponding curve $t \mapsto \gamma(\cdot, t)\varphi$ lies entirely in $\mathbb{S}$. Let us take a function $f^* \in \Sigma$ such that the domain $f^*(\Delta^*)$ is conformally rigid and consider for this function the family $f_r^*(z) = rf^*(z/r), 0 \leq r \leq 1$. Each $S_{f_r^*} \in \mathbb{T}$. Due to our assumption, all the points $\gamma S_{f_r^*}$ must belong to $\mathbb{S}$ and, therefore, must be the Schwarzian derivatives of some functions $f_{r,t}(z) = z + a_1(r, t)z^{-1} + \dots$ univalent in Δ^* . Then $S_{f_{r,t}}(z) = \gamma(z, t)S_{f_r^*}(z)$. We now fix $t \in (0, 1)$ and consider for it the corresponding family $\{f_{r,t}\}$.

Using compactness of the class Σ in the topology of locally uniform convergence in Δ^* , one obtains that there exists a limit function $f_{1,t} = \lim_{r \rightarrow 1} f_{r,t}$, and in the same topology,

$$S_{f_{1,t}}(z) = \lim_{r \rightarrow 1} S_{f_{r,t}}(z) = \gamma(z, t)S_{f^*}(z).$$

Thus the curve $t \mapsto S_{f_{1,t}} = \gamma(\cdot, t)S_{f^*}, 0 \leq t \leq 1$, must lie entirely in $\mathbb{S}$.

However the property (i) of γ provides the inequality

$$\|S_{f_{1,t}} - S_{f^*}\|_{\mathbb{B}_2} \leq 6 \max_{z \in \bar{\Delta}^*, t_0 \leq t \leq 1} |\gamma(z, t) - 1| < \varepsilon_0$$

for $1 - \delta_0(\varepsilon) \leq t \leq 1$. Since $f^*(\Delta^*)$ is rigid, it must be $S_{f_{1,t}} = S_{f^*}$ for all $t \in [1 - \delta_0, 1]$, which contradicts (ii). $\square$

Similar arguments imply the following result:

THEOREM 7.4. *The universal Teichmüller space $\mathbb{T}$ is not starlike with respect to any of its points. Moreover, there exist points $\varphi \in \mathbb{T}$ for which the line interval $\{t\varphi: 0 < t < 1\}$ contains the points from $\mathbb{B} \setminus \mathbb{S}$.*

7.4.2. Toki [To] extended the result on the nonstarlikeness of the space $\mathbb{T}$ to Teichmüller spaces of Riemann surfaces that contain hyperbolic disks of arbitrary large radius, in particular, for the spaces corresponding to Fuchsian groups of second kind. The crucial point in the proof of [To] is the same as in Theorem 7.3.

On the other hand, it was established in [Kru16] that all finite-dimensional Teichmüller spaces $\mathbb{T}(\Gamma)$ of high enough dimensions are not starlike.

Recall that a Riemann surface X has finite *conformal type* (g, n) if X is conformally equivalent to a closed surface of genus g with n punctures.

Let us assume that $2g - 2 + n > 0$, i.e., that X is hyperbolic. Then the corresponding Teichmüller space $\mathbb{T}(g, n)$ of such surfaces has complex dimension $m = 3g - 3 + n$.

The surface X is represented as Δ/Γ by a finitely-generated Fuchsian group Γ of the first kind, without torsion, and its Teichmüller space $\mathbb{T}(\Gamma)$ can be regarded as a model of $\mathbb{T}(g, n)$ with distinguished base point X .

THEOREM 7.5 [Kru16]. *There is an integer $m_0 > 1$ such that all the spaces $\mathbb{T}(g, n) = \mathbb{T}(\Gamma)$ of dimension $m \geq m_0$ are not starlike (in the Bers embedding).*

The idea of the proof is as follows. By Thurston's theorem there exists an isolated point $\varphi_0 \in \mathbb{S}$. Therefore, there is an open neighborhood V of φ_0 in the topology of uniform convergence on compact subsets of the disk Δ^* , such that for any $\varphi \in V$, the ray $[0, 1]\varphi$ is not contained entirely in $\mathbb{S}$. (Otherwise, φ_0 would not be isolated, since $\mathbb{S}$ is closed in the topology of uniform convergence on compact sets.) Thus the proof reduces to showing that V meets a given finite-dimensional Teichmüller space: once we have $\varphi \in \mathbb{T}(\Gamma) \cap V$, we have that $\mathbb{T}(\Gamma)$ is not starlike with respect to the origin. This reduces the proof to a suitable approximation of the space $\mathbb{B}$ by finite-dimensional spaces $\mathbb{B}(\Gamma)$.

It seems likely that no Teichmüller space can be starlike, i.e., $m_0 = 1$.

7.5. Remarks on other holomorphic embeddings of universal Teichmüller space

7.5.1. There are some other holomorphic embeddings of the universal Teichmüller space. They are also not starlike.

Under Becker's embedding, $\mathbb{T}$ is modeled by a bounded domain $\mathbf{b}(\mathbb{T})$ in the Banach space $\mathbb{B}_1(\Delta^*)$ of holomorphic functions ψ on Δ^* with the norm

$$\|\psi\| = \sup_{\Delta^*} |(|z|^2 - 1)z\psi(z)|;$$

the functions $\psi_f = f''/f'$ for $f \in \Sigma$ are the points of this domain. It was investigated by Becker [Bec1, Bec4], Astala and Gehring [AsG1, AsG2], Hamilton [H2] and Pommerenke [Po1].

Then $S_f = \psi'_f - \psi_f^2/2$, and there are established the inequalities

$$\|\psi'_f\|_{\mathbb{B}} \leq \text{const} \cdot \|\psi_f\|_{\mathbb{B}_1},$$

and for any pair $f_1, f_2 \in \Sigma$,

$$\|S_{f_1} - S_{f_2}\|_{\mathbb{B}} \leq \text{const} \cdot \|\psi_{f_1} - \psi_{f_2}\|_{\mathbb{B}_1}, \quad (7.2)$$

with some absolute constant (which is not greater than 41), see, e.g., [Bec2].

Applying the last inequality to the functions f^* and $f_{0,t}$ considered in the proof of Theorem 7.3 now yields that *for each of the functions $\gamma(z, t)$, with the same properties as in Theorem 7.3, there are points ψ in $\mathbf{b}(\mathbb{T})$ for which the curve*

$$t \rightarrow \gamma(\cdot, t)\psi : [0, 1] \rightarrow \mathbb{B}_1(\Delta^*)$$

does not lie entirely in $\mathbf{b}(\mathbb{T})$; and, in particular, the domain $\mathbf{b}(\mathbb{T})$ is not starlike with respect to the zero point.

Under Zhuravlev's embedding [Zh4], $\mathbb{T}$ is modeled by a bounded open subset $\mathbf{j}(\mathbb{T})$ of the Banach space $\mathbb{B}_1(\Delta)$ of the Bloch functions, i.e., holomorphic functions F in the unit disk with $F(0) = 0$ and with finite norm

$$\|F\| = \sup_{\Delta} (1 - |z|^2) |F'(z)|.$$

This set is filled by the quantities $F_f = \log f'(z)$, where now $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ are univalent in Δ and extend quasiconformally to $\widehat{\mathbb{C}}$; an inequality analogous to (7.2) holds for them. The connected component of $\mathbf{j}(\mathbb{T})$ containing the zero point corresponds to the functions f bounded on Δ . An assertion analogous to Theorem 7.3 is obtained for this component.

7.5.2. Theorem 7.3 yields that there are no criteria in terms of the Taylor coefficients of the quantities S_f , ψ_f and F_f for global univalence of a holomorphic function f on the disk.

7.5.3. Using universal Teichmüller space $\mathbb{T}$, one can apply the methods of complex differential geometry involving complex Finsler metrics, their holomorphic curvatures, etc. to solving the problems of Geometric Function Theory. This concerns a matter outside of the framework of this paper and will not be considered here.

References

- [Ab1] W. Abikoff, *On boundaries of Teichmüller spaces and on Kleinian groups, III*, Acta Math. **134** (1975), 211–237.
- [Ab2] W. Abikoff, *Real Analytic Theory of Teichmüller Spaces*, Lecture Notes in Math., Vol. 820, Springer-Verlag, Berlin (1980).

- [Ab3] W. Abikoff, *A geometric property of Bers' embedding of the Teichmüller space*, Riemann Surfaces and Related Topics: Proc. 1978 Stony Brook Conf., Ann. of Math. Stud., Vol. 97, Princeton University Press, Princeton, NJ (1981), 3–5.
- [Ag] S. Agard, *Distortion theorems for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **413** (1968), 1–12.
- [Ah1] L.V. Ahlfors, *Remarks on the Neumann–Poincaré equation*, Pacific J. Math. **2** (1952), 271–280.
- [Ah2] L.V. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. **3** (1953–1954), 1–58.
- [Ah3] L.V. Ahlfors, *Conformality with respect to Riemannian metrics*, Ann. Acad. Sci. Fenn. Ser. A I Math. **206** (1955), 1–22.
- [Ah4] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton (1966).
- [Ah5] L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [Ah6] L.V. Ahlfors, *A remark on schlicht functions with quasiconformal extensions*, Collected Papers, Vol. 2, Birkhäuser, Basel (1982), 438–441.
- [AB] L.V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 383–404.
- [AW] L.V. Ahlfors and G. Weill, *A uniqueness theorem for the Beltrami equation*, Proc. Amer. Math. Soc. **13** (1962), 975–978.
- [Al] I.A. Aleksandrov, *Parametric continuations in the theory of univalent functions*, Nauka, Moscow (1976) (in Russian).
- [AVV] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Mappings*, Wiley, New York (1997).
- [An] J.M. Anderson, *The Grunsky inequalities*, Inequalities. Fifty years on from Hardy, Littlewood and Polya, Proc. Int. Conf. Birmingham, UK, 1987, Lecture Notes Pure Appl. Math. **129** (1991), 21–28.
- [AC1] C. Andreian Cazacu, *Problèmes extrémaux des représentations quasiconformes*, Rev. Roumaine Math. Pures Appl. **10** (1965), 409–429.
- [AC2] C. Andreian Cazacu, *On extremal quasiconformal mappings*, Rev. Roumaine Math. Pures Appl. **22** (1977), 1359–1365.
- [AnMM] I. Anić, V. Marković and M. Mateljević, *Uniformly bounded maximal Φ -disks, Bers spaces and harmonic maps*, Proc. Amer. Math. Soc. **128** (2000), 2947–2956.
- [As1] K. Astala, *Selfsimilar zippers*, Holomorphic Functions and Moduli, Vol. I, D. Drasin et al., eds, Springer-Verlag, New York (1988), 61–73.
- [As2] K. Astala, *Area distortion for quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [AsG1] K. Astala and F.W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Anal. Math. **46** (1986), 16–57.
- [AsG2] K. Astala and F.W. Gehring, *Crickets, zippers, and the universal Teichmüller space*, Proc. Amer. Math. Soc. **110** (1990), 675–687.
- [AsIM] K. Astala, T. Iwaniec and G. Martin, *Elliptic Equations and Quasiconformal Mappings in the Plane*, Syracuse Univ. (2002).
- [AsM] K. Astala and G. Martin, *Holomorphic motions*, Papers on Analysis, Rep. Univ. Jyväskylä, Dept. Math. Statist. 83, Univ. of Jyväskylä (2001), 27–40.
- [BG] A.F. Beardon and F.W. Gehring, *Schwarzian derivatives, Poincaré metric and the kernel function*, Comment. Math. Helv. **55** (1980), 50–64.
- [BP] A.F. Beardon and Ch. Pommerenke, *The Poincaré metric of plane domains*, J. London Math. Soc. **18** (2) (1978), 475–483.
- [Bec1] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **225** (1972), 23–43.
- [Bec2] J. Becker, *Löwnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. **202** (1973), 321–335.
- [Bec3] J. Becker, *Über eine Golusinsche Ungleichung für quasikonform fortsetzbare schlichte Funktionen*, Math. Z. **131** (1973), 177–182.
- [Bec4] J. Becker, *Conformal mappings with quasiconformal extensions*, Aspects of Contemporary Complex Analysis, Proc. Confer. Durham 1979, D.A. Brannan and J.G. Clunie, eds, Academic Press, New York (1980), 37–77.

- [BeP] J. Becker und Ch. Pommerenke, *Über die quasikonforme Fortsetzung schlichter Funktionen*, Math. Z. **161** (1978), 69–80.
- [Bel1] P.P. Belinskii, *On the solution of extremal problems of quasiconformal mappings by the method of variations*, Dokl. Akad. Nauk SSSR **121** (1958), 199–201 (in Russian).
- [Bel2] P.P. Belinskii, *Solution of extremal problems in the theory of quasiconformal mappings by the variational method*, Sibirsk. Mat. Zh. **1** (1960), 303–330 (in Russian).
- [Bel3] P.P. Belinskii, *General Properties of Quasiconformal Mappings*, Nauka, Novosibirsk (1974) (in Russian).
- [Ber1] L. Bers, *Quasiconformal mappings and Teichmüller theorem*, Analytic Functions, L. Ahlfors, H. Behnke, H. Grauert, L. Bers et al., eds, Princeton Univ. Press, Princeton (1960), 89–119.
- [Ber2] L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
- [Ber4] L. Bers, *On boundaries of Teichmüller spaces and on Kleinian groups. I*, Ann. of Math. **91** (1970), 570–600.
- [Ber3] L. Bers, *Extremal quasiconformal mappings*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud., Vol. 66, Princeton Univ. Press, Princeton (1971), 27–52.
- [Ber5] L. Bers, *Uniformization, modules and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [Ber6] L. Bers, *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
- [Ber7] L. Bers, *An extremal problem for quasiconformal mappings and a theorem of Thurston*, Acta Math. **141** (1978), 73–98.
- [Ber8] L. Bers, *A new proof of fundamental inequality for quasiconformal mappings*, J. Anal. Math. **36** (1979), 15–30.
- [Ber9] L. Bers, *Finitely dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. **5** (1981), 131–172.
- [Ber10] L. Bers, *On a theorem of Abikoff*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 83–87.
- [BerK] L. Bers and I. Kra (eds), *A Crash Course on Kleinian Groups*, Lecture Notes in Math., Vol. 400, Springer-Verlag, Berlin (1974).
- [BerR] L. Bers and H.L. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.
- [Bi1] P. Biluta, *The solution of extremal problems for a certain class of quasiconformal mappings*, Siberian Math. J. **10** (1969), 533–540.
- [Bi2] P. Biluta, *An extremal problem for quasiconformal mappings of regions of finite connectivity*, Siberian Math. J. **13** (1972), 16–22.
- [BiK] P. Biluta and S.L. Krushkal, *On the question of extremal quasiconformal mappings*, Soviet Math. Dokl. **11** (1971), 76–79.
- [BLMM] V. Božin, N. Lakic, V. Marković and M. Mateljević, *Unique extremality*, J. Anal. Math. **75** (1998), 299–338.
- [BsH] D. Bshouty and W. Hengartner, *Univalent harmonic mappings in the plane*, Workshop on Complex Analysis (Lublin, 1994), Ann. Univ. Mariae Curie-Skłodowska Sect. A **48** (1994), 12–42.
- [Bu] J. Burbea, *Grunsky inequalities and Fredholm spectrum in general domains*, J. Anal. Math. **45** (1987), 1–36.
- [Ca] L. Carleson, *On mappings conformal at the boundary*, J. Anal. Math. **19** (1967), 1–13.
- [CHMG] J.G. Clunie, D.J. Hallenbeck and T.H. MacGregor, *A peaking and interpolation problem for univalent functions*, J. Math. Anal. Appl. **111** (1985), 559–570.
- [Da] V.A. Danilov, *Estimates of distortion of quasiconformal mapping in space C_α^m* , Siberian Math. J. **14** (1973), 362–369.
- [DB] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.
- [Din] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford (1989).
- [Dit] B. Dittmar, *Extremalprobleme quasikonformer Abbildungen der Ebene als Steuerungsprobleme*, Z. Anal. Anwendungen **5** (1986), 563–573.
- [DE] A. Douady and C.J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.
- [Du] P. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [DL] P. Duren and Y.J. Leung, *Generalized support points in the set of univalent functions*, J. Anal. Math. **46** (1986), 94–108.

- [Ea1] C.J. Earle, *Teichmüller theory*, Discrete Groups and Automorphic Function, Proc. Conf. Cambridge, 1975, Academic Press, London (1977), 143–162.
- [EE] C.J. Earle and A.L. Epstein, *Quasiconformal variation of slit domains*, Proc. Amer. Math. Soc. **129** (2001), 3363–3372.
- [EGL] C.J. Earle, F.P. Gardiner and N. Lakic, *Vector fields for holomorphic motions of closed sets*, Contemp. Math. **211** (1997), 193–225.
- [EK] C.J. Earle and I. Kra, *On sections of some holomorphic families of closed Riemann surfaces*, Acta Math. **137** (1976), 49–79.
- [EKK] C.J. Earle, I. Kra and S.L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Trans. Amer. Math. Soc. **343** (1994), 927–948.
- [EL1] C.J. Earle and Li Zhong, *Extremal quasiconformal mappings in plane domains*, Quasiconformal Mappings and Analysis: A Collection of Papers Honoring F.W. Gehring, P. Duren et al., eds, Springer-Verlag, New York (1997), 141–157.
- [EL2] C.J. Earle and Li Zhong, *Isometrically embedded polydisks in infinite dimensional Teichmüller spaces*, J. Geom. Anal. **9** (1999), 51–71.
- [EM] C.J. Earle and S. Mitra, *Analytic dependence of conformal invariants on parameters*, Contemporary Math. **256** (2000).
- [Fe1] R. Fehlmann, *Über extremale quasikonforme Abbildungen*, Comment. Math. Helv. **56** (1981), 1558–1580.
- [Fl] B.B. Flinn, *Jordan domains and the universal Teichmüller space*, Trans. Amer. Math. Soc. **282** (1984), 603–610.
- [FV] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam (1980).
- [Ga1] F.P. Gardiner, *An analysis of the group operation in universal Teichmüller space*, Trans. Amer. Math. Soc. **132** (1968), 471–486.
- [Ga2] F.P. Gardiner, *Schiffer's interior variation and quasiconformal mappings*, Duke Math. J. **42** (1975), 371–380.
- [Ga3] F.P. Gardiner, *On partially Teichmüller–Beltrami differentials*, Michigan Math. J. **29** (1982), 237–242.
- [Ga4] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
- [GaL] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Math. Surveys Monogr., Vol. 76, Amer. Math. Soc., Providence, RI (2000).
- [Ge1] F.W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment Math. Helv. **52** (1977), 561–572.
- [Ge2] F.W. Gehring, *Spirals and the universal Teichmüller space*, Acta Math. **141** (1978), 99–113.
- [Ge3] F.W. Gehring, *Characteristic Properties of Quasidisks*, Les Presses de l'Université de Montréal (1982).
- [Go] Z. Göktürk, *Estimates for univalent functions with quasiconformal extensions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **589** (1974), 1–21.
- [GoS] A.A. Golubev and V.G. Sheretov, *Quasiconformal extremals of energy integral*, Math. Notes **55** (1994), 580–585.
- [Gol] G.M. Goluzin, *Geometric Theory of Functions of Complex Variables*, Transl. Math. Monogr., Vol. 26, Amer. Math. Soc., Providence, RI (1969).
- [Goo] A.W. Goodman, *Univalent Functions*, Vols I and II, Polygonal Publishing Co., Washington (1983).
- [Gor1] V.V. Goryainov, *The parametric method and extremal conformal mappings*, Soviet Math. Dokl. **28** (1983), 205–208.
- [Gor2] V.V. Goryainov, *General uniqueness theorem and geometry of extremal conformal mappings in distortion and rotation problems*, Izv. Vyssh. Uchebn. Zaved. Mat. **10** (1986), 40–47; English transl.: Soviet Math. **30** (1986), 54–63.
- [Gor3] V.V. Goryainov, *Semigroups of probability generating functions, and infinitely splittable random variables*, Theory of Random Processes **1** (17) (1995), 2–9.
- [Gor4] V.V. Goryainov, *Some analytic properties of time inhomogeneous Markov branching processes*, Z. Angew. Math. Mech. **76** (1996), 290–291.
- [Gri1] A.Z. Grinshpan, *On the growth of coefficients of univalent functions with quasiconformal extension*, Sibirsk. Math. Zh. **23** (1982), 208–211 (in Russian).

- [Gri2] A.Z. Grinshpan, *Coefficient inequalities for conformal mappings with homeomorphic extension*, Siberian. Math. J. **26** (1985), 37–50.
- [Gri4] A.Z. Grinshpan, *Univalent functions and measurable mappings*, Siberian Math. J. **27** (1986), 825–837.
- [Gri3] A.Z. Grinshpan, *Univalent functions with logarithmic restrictions*, Ann. Polon. Math. **55** (1991), 117–138.
- [Gri5] A.Z. Grinshpan, *The Bieberbach conjecture and Milin's functionals*, Amer. Math. Monthly **106** (1999), 203–214.
- [GrM] A.Z. Grinshpan and I.M. Milin, *Simply connected domains with finite logarithmic area and Riemann mapping functions*, Constantin Carathéodory: An International Tribute, Th.M. Rassias, ed., World Scientific, Singapore (1991), 381–403.
- [GrP1] A.Z. Grinshpan and Ch. Pommerenke, *The Grunsky operator and coefficient difference*, Complex Var. **33** (1997), 113–127.
- [GrP2] A.Z. Grinshpan and Ch. Pommerenke, *The Grunsky norm and coefficient estimates for bounded functions*, Bull. London Math. Soc. **29** (1977), 705–712.
- [Gro1] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **80** (1928), 367–376.
- [Gro2] H. Grötzsch, *Über ein Variationsproblem der konformen Abbildung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **82** (1930), 251–263.
- [Gro3] H. Grötzsch, *Über die Verschiebung bei schlichter konformer Abbildung schlichter Bereiche*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **83** (1931), 254–279.
- [Gro4] H. Grötzsch, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **84** (1932), 114–120.
- [Gro5] H. Grötzsch, *Über die Geometrie der schlichten konformen Abbildung* (2 Mitt.), Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. (1933), 893–908.
- [Gru1] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45** (1939), 29–61.
- [Gru2] H. Grunsky, *Lectures on Theory of Functions in Multiply Connected Domains*, Vandenhoeck & Ruprecht, Göttingen (1978).
- [Gu1] V.Ya. Gutlyanskii, *On the area method for a class quasiconformal mappings*, Soviet Math. Dokl. **14** (1973), 1401–1406.
- [Gu2] V.Ya. Gutlyanskii, *On the method of variations for univalent analytic functions with quasiconformal extension*, Soviet Math. Dokl. **18** (1977), 1298–1301.
- [Gu3] V.Ya. Gutlyanskii, *On distortion theorems for univalent analytic functions with a quasiconformal continuation*, Soviet Math. Dokl. **19** (1978), 617–620.
- [Gu4] V.Ya. Gutlyanskii, *On the method of variations for univalent analytic functions with quasiconformal extension*, Siberian Math. J. **21** (1980), 190–204.
- [Gu5] V.Ya. Gutlyanskii, *On linear extremal problems for univalent analytic functions with quasiconformal extension*, Constructive Function Theory and Theory of Mappings, Naukova Dumka, Kiev (1982), 89–102.
- [GuR] V.Ya. Gutlyanskii and V.I. Ryazanov, *On the variational method for quasiconformal mappings*, Siberian Math. J. **28** (1987), 59–81.
- [HM] D.J. Hallenbeck and T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Monogr. Studies in Math., Vol. 22, Pitman, Boston, MA (1984).
- [H1] D. Hamilton, *The extreme points of Σ* , Proc. Amer. Math. Soc. **85** (1982), 393–396.
- [H2] D. Hamilton, *BMO and Teichmüller space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 213–224.
- [Ha] R.S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. **138** (1969), 399–406.
- [Har] R. Harmelin, *Generalized Grunsky coefficients and inequalities*, Israel J. Math. **57** (1987), 347–364.
- [Hay1] W.K. Hayman, *On successive coefficients of univalent functions*, J. London Math. Soc. **38** (1963), 228–243.
- [Hay2] W.K. Hayman, *Multivalent Functions*, 2nd edn, Oxford Univ. Press (1998).
- [HiF] E. Hille and R.S. Fillips, *Functional Analysis and Semigroups*, Colloq. Publ. Ser., Vol. 31, Amer. Math. Soc., Providence, RI (1957).

- [Ho1] E. Hoy, *Flächensätze für quasikonform fortsetzbare Abbildungen mit ortsabhängiger Dilatationsbeschränkung*, Math. Nachr. **121** (1985), 147–161.
- [Ho2] E. Hoy, *Variationscharakterisierung für gewisse quasikonforme Abbildungen*, Z. Anal. Anwendungen **8** (1989), 463–472.
- [Ho3] E. Hoy, *On Bernstein's theorem for quasiminimal surfaces*, Part II, J. Anal. Appl. **16** (1997), 1027–1032.
- [Hu1] J.A. Hummel, *The Grunsky coefficients of a schlicht function*, Proc. Amer. Math. Soc. **15** (1964), 142–150.
- [Hu2] J.A. Hummel, *Lectures on variational methods in the theory of univalent functions*, Univ. of Maryland Lecture Notes (1970).
- [IT] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo (1992).
- [IM] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press (2001).
- [Je] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin (1965).
- [Ke] S. Kerckhoff, *The asymptotic geometry of Teichmüller space*, Topology **19** (1980), 23–41.
- [Ki] S. Kirsch, *Lageabschätzung für einen Kondensator minimaler Kapazität*, Z. Anal. Anwendungen **3** (1984), 119–131.
- [Ko1] S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.
- [Ko2] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer-Verlag, New York (1997).
- [Kr1] I. Kra, *Automorphic Forms and Kleinian Groups*, Math. Lecture Notes Ser., Benjamin, Reading, MA (1972).
- [Kr3] I. Kra, *On the Carathéodory metric on the universal Teichmüller space*, Complex Analysis, Joensuu, 1978, Lecture Notes in Math., Vol. 747, Springer-Verlag (1979), 230–241.
- [Kr2] I. Kra, *On Teichmüller's theorem on the quasi-invariance of cross ratios*, Israel J. Math. **30** (1981), 152–158.
- [Kr4] I. Kra, *The Carathéodory metric on Abelian Teichmüller disks*, J. Anal. Math. **40** (1981), 129–143.
- [Kr5] I. Kra, *Quadratic differentials*, Rev. Romain Math. Pure Appl. **39** (1994), 751–787.
- [Kru1] S.L. Krushkal, *The method of variations in the theory of quasiconformal mappings of closed Riemann surfaces*, Dokl. Akad. Nauk SSSR **157** (1964), 781–784 (in Russian).
- [Kru2] S.L. Krushkal, *On the theory of extremal problems for quasiconformal mappings of closed Riemann surfaces*, Soviet Math. Dokl. **7** (1966), 1541–1544.
- [Kru3] S.L. Krushkal, *Some extremal problems for univalent analytic functions*, Soviet Math. Dokl. **9** (1968), 1191–1194.
- [Kru4] S.L. Krushkal, *Some extremal problems for conformal and quasiconformal mappings*, Siberian Math. J. **12** (1971), 541–559.
- [Kru5] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Wiley, New York (1979).
- [Kru6] S.L. Krushkal, *On the Grunsky coefficient conditions*, Siberian Math. J. **28** (1987), 104–110.
- [Kru7] S.L. Krushkal, *A new approach to variational problems in the theory of quasiconformal mappings*, Soviet Math. Dokl. **35** (1987), 219–222.
- [Kru8] S.L. Krushkal, *A new method of solving variational problems in the theory of quasiconformal mappings*, Siberian Math. J. **29** (1988), 245–252.
- [Kru9] S.L. Krushkal, *The coefficient problem for univalent functions with quasiconformal extension*, Holomorphic Functions and Moduli I, Math. Sci. Res. Inst. Publ., Vol. 10, D. Drasin et al., eds, Springer-Verlag, New York (1988), 155–161.
- [Kru10] S.L. Krushkal, *Extension of conformal mappings and hyperbolic metrics*, Siberian Math. J. **30** (1989), 730–744.
- [Kru11] S.L. Krushkal, *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650–660.
- [Kru12] S.L. Krushkal, *On the question of the structure of the universal Teichmüller space*, Soviet Math. Dokl. **38** (1989), 435–437.
- [Kru13] S.L. Krushkal, *New developments in the theory of quasiconformal mappings*, Geometric Function Theory and Applications of Complex Analysis to Mechanics: Studies in Complex Analysis and Its Appli-

- cations to Partial Differential Equations, Vol. 2, R. Kühnau and W. Tutschke, eds, Pitman Research Notes in Mathematical Series, Longman, Harlow, UK (1991), 3–26.
- [Kru14] S.L. Krushkal, *Quasiconformal extremals of non-regular functionals*, Ann. Acad. Sci. Fenn. Ser. A I Math. **17** (1992), 295–306.
- [Kru15] S.L. Krushkal, *Exact coefficient estimates for univalent functions with quasiconformal extension*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), 349–357.
- [Kru16] S.L. Krushkal, *Teichmüller spaces are not starlike*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), 167–173.
- [Kru17] S.L. Krushkal, *On Grunsky conditions, Fredholm eigenvalues and asymptotically conformal curves*, Mitt. Math. Sem. Gissen **228** (1996), 17–23.
- [Kru18] S.L. Krushkal, *Quasiconformal mirrors*, Siberian Math. J. **40** (1999), 742–753.
- [Kru19] S.L. Krushkal, *Quasiconformal maps decreasing L_p norm*, Siberian Math. J. **41** (2000), 884–888.
- [Kru20] S.L. Krushkal, *On the Teichmüller–Kühnau extension of univalent functions*, Georgian Math. J. **7** (2000), 723–729.
- [Kru21] S.L. Krushkal, *On the best approximation and univalence of holomorphic functions*, Complex Analysis in Contemporary Mathematics. In honor of 80th Birthday of Boris Vladimirovich Shabat, E.M. Chirka, ed., Fasis, Moscow (2001), 153–166 (in Russian).
- [Kru22] S.L. Krushkal, *Polygonal quasiconformal maps and Grunsky inequalities*, J. Anal. Math. **90** (2003), 175–196.
- [Kru23] S.L. Krushkal, *Variational principles in the theory of quasiconformal maps*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier Science, Amsterdam (2005), 31–98 (this volume).
- [Kru24] S.L. Krushkal, *Quasiconformal extensions and reflections*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier Science, Amsterdam (2005), 507–553 (this volume).
- [KG] S.L. Krushkal and V.D. Golovan, *Approximation of analytic functions and Teichmüller spaces*, Siberian Math. J. **31** (1990), 82–88.
- [KK1] S.L. Krushkal und R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner-Texte Math., Vol. 54, Teubner, Leipzig (1983).
- [KK2] S.L. Krushkal and R. Kühnau, *A quasiconformal dynamic property of the disk*, J. Anal. Math. **72** (1997), 93–103.
- [KL1] J. Krzyz and J. Lawrynowicz, *Quasiconformal mappings of the unit disc with two invariant points*, Michigan Math. J. **14** (1967), 487–492.
- [KL2] J. Krzyz and J. Lawrynowicz, *Quasiconformal Mappings in the Plane: Parametrical Methods*, Lecture Notes in Math., Vol. 978, Springer-Verlag, Berlin (1983).
- [KP1] J. Krzyz and D. Partyka, *Generalized Neumann–Poincaré operator, chord-arc curves and Fredholm eigenvalues*, Complex Var. Theory Appl. **21** (1993), 253–263.
- [KP2] J. Krzyz and D. Partyka, *Harmonic extensions of quasisymmetric mappings*, Complex Var. Theory Appl. **33** (1997), 159–176.
- [Kf1] P.P. Kufarev, *On one-parametrical families of analytic functions*, Mat. Sb. **13** (55) (1943) (in Russian).
- [Kf2] P.P. Kufarev, *On an investigation method for univalent functions*, Dokl. Akad. Nauk SSSR **107** (1956), 633–635 (in Russian).
- [Ku1] R. Kühnau, *Über gewisse Extremalprobleme der quasikonformen Abbildung*, Wiss. Z. d. Martin-Luther-Univ. Halle-Wittenberg, Math.-Nat. Reihe **13** (1964), 35–40.
- [Ku2] R. Kühnau, *Einige Extremalprobleme bei differentialgeometrischen und quasikonformen Abbildungen*, Math. Z. **94** (1966), 178–192.
- [Ku3] R. Kühnau, *Über die analytische Darstellung von Abbildungsfunktionen, insbesondere von Extremalfunktionen der Theorie der konformen Abbildung*, J. Reine Angew. Math. **228** (1967), 93–132.
- [Ku4] R. Kühnau, *Herleitung einiger Verzerrungseigenschaften konformer und allgemeinerer Abbildungen mit Hilfe des Argumentprinzips*, Math. Nachr. **39** (1969), 249–275.
- [Ku5] R. Kühnau, *Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung*, Math. Nachr. **40** (1969), 1–11.
- [Ku6] R. Kühnau, *Koeffizientenbedingungen bei quasikonformen Abbildungen*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **22–24** (1968–1970), 105–111.

- [Ku7] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77–105.
- [Ku8] R. Kühnau, *Eine funktionentheoretische Randwertaufgabe in der Theorie der quasikonformen Abbildungen*, Indiana Univ. Math. J. **21** (1971), 1–10.
- [Ku9] R. Kühnau, *Zum Koeffizientenproblem bei den quasikonform fortsetzbaren schlichten konformen Abbildungen*, Math. Nachr. **55** (1973), 225–231.
- [Ku10] R. Kühnau, *Zur analytischen Darstellung gewisser Extremalfunktionen der quasikonformen Abbildung*, Math. Nachr. **60** (1974), 53–62.
- [Ku11] R. Kühnau, *Eine Verschärfung des Koebeschen Viertsatzes für quasikonform fortsetzbare Abbildungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 77–83.
- [Ku12] R. Kühnau, *Extremalprobleme bei quasikonformen Abbildungen mit kreisringweise konstanter Dilatationsbeschränkung*, Math. Nachr. **66** (1975), 269–282.
- [Ku13] R. Kühnau, *Zur quasikonformen Fortsetzbarkeit schlichter konformer Abbildungen*, Bull. Soc. Sci. Lett. Łódź **24** (1975), 1–4.
- [Ku14] R. Kühnau, *Zur Methode der Randintegration bei quasikonformen Abbildungen*, Ann. Polon. Math. **31** (1976), 269–289.
- [Ku15] R. Kühnau, *Über die Werte des Doppelverhältnisses bei quasikonformer Abbildung*, Math. Nachr. **95** (1980), 237–251.
- [Ku16] R. Kühnau, *Eine Kernfunktion zur Konstruktion gewisser quasikonformen Normalabbildungen*, Math. Nachr. **95** (1980), 229–235.
- [Ku17] R. Kühnau, *Funktionalabschätzungen bei quasikonformen Abbildungen mit Fredholmschen Eigenwerten*, Comment. Math. Helv. **56** (1981), 297–206.
- [Ku18] R. Kühnau, *Zu den Grunskyschen Koeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 125–130.
- [Ku19] R. Kühnau, *Quasikonforme Fortsetzbarkeit, Fredholmsche Eigenwerte und Grunskysche Koeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 383–391.
- [Ku20] R. Kühnau, *Zur Berechnung der Fredholmschen Eigenwerte ebener Kurven*, Z. Angew. Math. Mech. **66** (1986), 193–201.
- [Ku21] R. Kühnau, *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q -quasikonforme Fortsetzbarkeit?* Comment. Math. Helv. **61** (1986), 290–307.
- [Ku22] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jahresber. Deutsch. Math. Verein. **90** (1988), 90–109.
- [Ku23] R. Kühnau, *Interpolation by extremal quasiconformal Jordan curves*, Siberian Math. J. **32** (1991), 257–264.
- [Ku24] R. Kühnau, *Einige neuere Entwicklungen bei quasikonformen Abbildungen*, Jahresber. Deutsch. Math. Verein. **94** (1992), 141–169.
- [Ku25] R. Kühnau, *Ersetzungssätze bei quasikonformen Abbildungen*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **52** (1998).
- [Ku26] R. Kühnau, *Über die Grunskyschen Koeffizientenbedingungen*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **54** (2000), 53–60.
- [Ku27] R. Kühnau, *Drei Funktionale eines Quasikreises*, Ann. Acad. Sci. Fenn. Ser. A I Math. **25** (2000), 413–415.
- [KuD] R. Kühnau und B. Dittmar, *Einige Folgerungen aus den Koeffizientenbedingungen vom Grunskyschen Typ für schlichte quasikonform fortsetzbare Abbildungen*, Math. Nachr. **66** (1975), 5–16.
- [KuN] R. Kühnau und W. Niske, *Abschätzung des dritten Koeffizienten bei den quasikonform fortsetzbaren schlichten Funktionen der Klasse S* , Math. Nachr. **78** (1977), 185–192.
- [KuTh] R. Kühnau und B. Thürling, *Berechnung einer quasikonformen Extremalfunktion*, Math. Nachr. **79** (1977), 99–113.
- [KuT] R. Kühnau und J. Timmel, *Asymptotische Koeffizientenabschätzungen für die quasikonform fortsetzbaren Abbildungen der Klasse S bzw. Σ* , Math. Nachr. **91** (1979), 357–372.
- [Kuz1] G.V. Kuz'mina, *Methods of geometric function theory. I*, St. Petersburg Math. J. **9** (1998), 455–507.
- [Kuz2] G.V. Kuz'mina, *Methods of geometric function theory. II*, St. Petersburg Math. J. **9** (1998), 889–930.
- [La1] M.A. Lavrentieff, *Sur une classe de représentations continues*, Mat. Sb. **42** (1935), 407–424 (in Russian).

- [La2] M.A. Lavrentiev, *Variational Method for Boundary Value Problems for Systems of Elliptic equations*, English transl. from the Russian: Noordhoff, Groningen (1963).
- [Leb] N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Nauka, Moscow (1975) (in Russian).
- [Leh1] O. Lehto, *Schlicht functions with a quasiconformal extension*, Ann. Acad. Sci. Fenn. Ser. A I Math. **500** (1971), 3–10.
- [Leh2] O. Lehto, *Quasiconformal mappings and singular integrals*, Instit. Naz. Alta Mat., Simposia Matematica, Vol. 18, Academic Press, London (1976), 429–453.
- [Leh3] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987).
- [LV] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin (1965).
- [LeS] Y.J. Leung and G. Schober, *On the structure of support points in the class Σ* , J. Anal. Math. **46** (1986), 176–193.
- [Li1] Li Zhong, *Non-uniqueness of geodesics in infinite-dimensional Teichmüller spaces*, Complex Var. **16** (1991), 261–272.
- [Li2] Li Zhong, *Non-uniqueness of geodesics in infinite-dimensional Teichmüller spaces (II)*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 355–367.
- [Lo] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103–121.
- [LS] L.A. Lusternik and V.I. Sobolev, *Elementy funktsional'nogo analiza*, Nauka, Moscow (1965) (in Russian); English transl.: *Elements of Functional Analysis*, Hindustan Publ. Corp., Dehli; Halsted Press, New York (1974).
- [MGT] T.H. MacGregor and D.E. Tepper, *Finite boundary interpolation by univalent functions*, J. Approx. Theory **52** (1988), 315–321.
- [MSS] R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. **16** (1983), 193–217.
- [Ma1] V. Marković, *Harmonic diffeomorphisms of noncompact surfaces and Teichmüller spaces*, J. London Math. Soc. (2) **65** (2002), 103–114.
- [Ma2] V. Marković, *Harmonic diffeomorphisms and conformal distortion of Riemann surface*, Comm. Anal. Geom. **10** (2002), 847–876.
- [Mar] G. Martin, *The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions*, Proc. Amer. Math. Soc. **125** (1997), 1093–1103.
- [Mas1] B. Maskit, *On boundaries of Teichmüller spaces and on Kleinian groups. II*, Ann. of Math. **91** (1970), 608–638.
- [Mas2] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin (1987).
- [McL] J.O. McLeavey, *Extremal problems in classes of analytic holomorphic functions with quasiconformal extensions*, Trans. Amer. Math. Soc. **195** (1974), 327–343.
- [MSu] C. McMullen and D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1988), 351–395.
- [Mil] I.M. Milin, *Univalent Functions and Orthogonal Systems*, Nauka, Moscow (1971); English transl.: Amer. Math. Soc., Providence, RI (1977).
- [Min] Y. Minsky, *Harmonic maps, length and energy in Teichmüller space*, J. Differential Geom. **35** (1992), 151–217.
- [Mit] S. Mitra, *Teichmüller spaces and holomorphic motions*, J. Anal. Math. **81** (2000), 1–33.
- [Na] S. Nag, *Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).
- [NS] S. Nag and D. Sullivan, *Teichmüller theory and the universal period mappings via quantum calculus and the $H^{1/2}$ space on the circle*, Osaka J. Math. **32** (1995), 1–34.
- [Nak] T. Nakanishi, *The inner radii of finite-dimensional Teichmüller spaces*, Tôhoku Math. J. **41** (1989), 679–688.
- [NaVe] T. Nakanishi and J.A. Velling, *A sufficient condition for Teichmüller spaces to have smallest possible inner radii*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 13–21.
- [NaY] T. Nakanishi and H. Yamamoto, *On the outradius of the Teichmüller space*, Comment. Math. Helv. **64** (1989), 288–299.
- [NapY] V.V. Napalkov, Jr. and R.S. Yulmukhametov, *On the Hilbert transform in Bergman space*, Math. Notes **70** (2001), 68–78.

- [Ne1] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
- [Ne3] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York (1952).
- [Ne2] Z. Nehari, *Some criteria of univalence*, Proc. Amer. Math. Soc. **5** (1954), 700–704.
- [Oh1] H. Ohtake, *On the deformation of Fuchsian groups by quasiconformal mappings with partially vanishing Beltrami coefficients*, J. Math. Kyoto Univ. **29** (1989), 69–90.
- [Oh2] H. Ohtake, *Partially conformal quasiconformal mappings and the universal Teichmüller space*, J. Math. Kyoto Univ. **31** (1991), 171–180.
- [Os] B. Osgood, *Some properties of f''/f' and the Poincaré metric*, Indiana Univ. Math. J. **31** (1982), 449–461.
- [Pa] D. Partyka, *Generalized harmonic conjugation operator*, Proc. Fourth Finnish–Polish Summer School in Complex Analysis at Jyväskylä, Ber. Univ. Jyväskylä Math. Inst. **55** (1993), 143–155.
- [Pf] A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **489** (1971), 1–32.
- [Po1] Chr. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
- [Po2] Chr. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin (1992).
- [PR] Chr. Pommerenke and B. Rodin, *Holomorphic families of Riemann mapping functions*, J. Math. Kyoto Univ. **26** (1) (1986), 13–22.
- [Pr1] D.V. Prokhorov, *Coefficients of functions close to the identity function*, Complex Var. Theory Appl. **33** (2000), 255–263.
- [Pr2] D.V. Prokhorov, *Bounded univalent functions*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, R. Kühnau, ed., Elsevier, Amsterdam (2002), 207–228.
- [Re1] E. Reich, *Quasiconformal mappings of the disk with given boundary values*, Advances in Complex Function Theory, W.E. Kirwan and L. Zalcman, eds, Lecture Notes in Math., Vol. 505, Springer-Verlag, Berlin (1976), 101–137.
- [Re2] E. Reich, *A quasiconformal extension using the parametric representations*, J. Anal. Math. **54** (1990), 246–258.
- [Re3] E. Reich, *An approximation condition and extremal quasiconformal extension*, Proc. Amer. Math. Soc. **125** (1997), 1479–1481.
- [Re4] E. Reich, *The unique extremality counterexample*, J. Anal. Math. **75** (1998), 339–347.
- [Re5] E. Reich, *Extremal quasiconformal mappings of the disk*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, R. Kühnau, ed., Elsevier, Amsterdam, (2002), 75–136.
- [RC] E. Reich and J.X. Chen, *Extensions with bounded $\bar{\partial}$ -derivative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 377–389.
- [RS] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, L.V. Ahlfors et al., eds, Academic Press, New York (1974), 375–391.
- [Rei] H.M. Reimann, *Ordinary differential equations and quasiconformal mappings*, Invent. Math. **33** (1976), 247–270.
- [Ren1] H. Renelt, *Modifizierung und Erweiterung einer Schifferschen Variationsmethode für quasikonforme Abbildungen*, Math. Nachr. **55** (1973), 353–379.
- [Ren2] H. Renelt, *Extremalprobleme bei quasikonformen Abbildungen unter höheren Normierungen*, Math. Nachr. **66** (1975), 125–143.
- [Ren3] H. Renelt, *Konstruktion gewisser quadratischer Differentiale mit Hilfe von Dirichletintegralen*, Math. Nachr. **73** (1976), 353–379.
- [Ren4] H. Renelt, *Über Extremalprobleme für schlichte Lösungen elliptischer Differentialgleichungssysteme*, Comment. Math. Helv. **54** (1979), 17–41.
- [Ren5] H. Renelt, *Elliptic Systems and Quasiconformal Mappings*, Wiley, New York (1988).
- [Res2] Yu.G. Reshetnyak, *Stability Theorems in Geometry and Analysis*, Nauka, Novosibirsk (1982) (in Russian).
- [Res1] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monogr., Vol. 73, Amer. Math. Soc., Providence, RI (1989).
- [Ri] B. Riemann, *Collected Works*, Dover, New York (1953).
- [Ro] B. Rodin, *Behavior of the Riemann mapping function under complex analytic deformations of the domain*, Complex Var. Theory Appl. **5** (1986), 189–195.

- [Roy1] H.L. Royden, *A modification of the Neumann–Poincaré method for multiply connected domains*, Pacific J. Math. **2** (1952), 385–394.
- [Roy2] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud., Vol. 66, Princeton University Press, Princeton (1971), 369–383.
- [Ru] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$* , Springer-Verlag, Berlin (1980).
- [Sa] V.V. Savin, *On the moduli of Riemann surfaces*, Doklady Akad. Nauk SSSR **196**, 783–785 (in Russian).
- [Schi1] M. Schiffer, *Variation of the Green function and theory of the p -valued functions*, Amer. J. Math. **65** (1943), 341–360.
- [Schi2] M. Schiffer, *Sur l'équation différentielle de M. Löwner*, C. R. Acad. Sci. Paris **221** (1945), 369–371.
- [Schi3] M. Schiffer, *The Fredholm eigenvalues of plane domains*, Pacific J. Math. **7** (1957), 1187–1225.
- [Schi4] M. Schiffer, *Extremum problems and variational methods in conformal mapping*, Proc. Internat. Congress Mathematicians 1958, Cambridge Univ. Press (1960), 211–231.
- [Schi5] M. Schiffer, *A variational method for univalent quasiconformal mappings*, Duke Math. J. **33** (1966), 395–411.
- [Schi6] M. Schiffer, *Fredholm eigenvalues and Grunsky matrices*, Ann. Polon. Math. **39** (1981), 149–164.
- [ScSc1] M. Schiffer and G. Schober, *Coefficient problems and generalized Grunsky inequalities for schlicht functions with quasiconformal extensions*, Arch. Ration. Mech. Anal. **60** (1976), 205–228.
- [ScSc2] M. Schiffer and G. Schober, *A variational method for generalized families of quasiconformal mappings*, J. Anal. Math. **38** (1978), 240–264.
- [ScSc3] M. Schiffer and G. Schober, *An application of the calculus of variations for generalized families of quasiconformal mappings*, Lecture Notes in Math., Vol. 747, Springer-Verlag, Berlin (1978), 349–357.
- [ScSp] M. Schiffer and D.C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton (1954).
- [Scho1] G. Schober, *Semicontinuity of curve functionals*, Arch. Ration. Mech. Anal. **53** (1969).
- [Scho2] G. Schober, *Estimates for Fredholm eigenvalues based on quasiconformal mapping*, Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen, Lecture Notes in Math., Vol. 333, Springer-Verlag, Berlin (1973), 211–217.
- [Scho3] G. Schober, *Univalent Functions – Selected Topics*, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin (1975).
- [ScY] R. Schoen and S.T. Yau, *On univalent harmonic maps between surfaces*, Invent. Math. **44** (1978), 265–278.
- [Se1] V.I. Semenov, *Semigroups of some classes of transformations*, Sibirsk. Mat. Zh. **18** (1977), 877–889, 958 (in Russian).
- [Se2] V.I. Semenov, *Quasiconformal flows in Möbius spaces*, Mat. Sb. **119** (161) (1982), 325–339 (in Russian).
- [Se3] V.I. Semenov, *One-parameter groups of quasiconformal homeomorphisms in Euclidean space*, Sibirsk. Mat. Zh. **17** (1986), 177–193 (in Russian).
- [Se4] V.I. Semenov, *Certain applications of the quasiconformal and quasi-isometric deformations*, Proc. Colloq. Complex Anal., the Sixth Romanian–Finnish Seminar, Rev. Roumaine Pure Appl. **36** (1991), 503–511.
- [Se5] V.I. Semenov, *An extension principle for quasiconformal deformations of the plane*, Russian Acad. Sci. Sb. Math. **77** (1994), 127–137.
- [Sha] Shah Dao-shing, *Parametric representation of quasiconformal mappings*, Science Record **3** (1959), 400–407 (in Russian).
- [ShF] Shah Dao-shing and Fan Le-Le, *On parametric representation of quasiconformal mappings*, Sci. Sinica **11** (1962), 149–162.
- [Sh1] Y.L. Shen, *On the geometry of infinite dimensional Teichmüller spaces*, Acta Math. Sinica (N.S.) **13** (1997), 413–420.
- [Sh2] Y.L. Shen, *On the extremality of quasiconformal mappings and quasiconformal deformations*, Proc. Amer. Math. Soc. **128** (1999), 135–139.
- [Sh3] Y.L. Shen, *Pull-back operators by quasisymmetric functions and invariant metrics on Teichmüller spaces*, Complex Var. **42** (2000), 289–307.
- [She1] V.G. Sheretov, *On a variant of the area theorem*, Mathematical Analysis, Krasnodar Univ. (1976), 77–80 (in Russian).

- [She2] V.G. Sheretov, *On the theory of the extremal quasiconformal mappings*, Mat. Sb. **107** (1978), 146–158.
- [She3] V.G. Sheretov, *Criteria for extremality in a problem for quasiconformal mappings*, Mat. Zametki **39** (1986), 14–23 (in Russian).
- [She4] V.G. Sheretov, *Quasiconformal extremals of the smooth functionals and of the energy integral on Riemann surfaces*, Siberian Math. J. **29** (1988), 942–952.
- [She5] V.G. Sheretov, *Analytic Functions with Quasiconformal Extensions*, Tver State Univ. (1991).
- [Shi1] H. Shiga, *On analytic and geometric properties of Teichmüller spaces*, J. Math. Kyoto Univ. **24** (1985), 441–452.
- [Shi2] H. Shiga, *Characterization of quasidisks and Teichmüller spaces*, Tôhoku Math. J. **37** (2) (1985), 541–552.
- [ShT] H. Shiga and H. Tanigawa, *Grunsky's inequality and its application to Teichmüller spaces*, Kodai Math. J. **16** (1993), 361–378.
- [Shs] V.A. Shscepets, *The area theorem for a class of quasiconformal mappings*, Extremal Problems of the Function Theory, Tomsk Univ. (1979), 69–85 (in Russian).
- [SI] Z. Ślodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.
- [SoV] A.Yu. Solynin and M. Vuorinen, *Estimates for the hyperbolic metric of the punctured plane and applications*, J. Math. Anal. Appl. **260** (2001), 623–640.
- [Sp] G. Springer, *Fredholm eigenvalues and quasiconformal mapping*, Acta Math. **111** (1963), 121–142.
- [St1] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises*, Comment. Math. Helv. **36** (1962), 306–323.
- [St2] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Anal. Math. **30** (1976), 464–480.
- [St4] K. Strebel, *Extremal quasiconformal mappings*, Results Math. **10** (1986), 168–210.
- [St5] K. Strebel, *On certain extremal quasiconformal mappings*, Mitt. Math. Sem. Giessen **228** (1996), 39–50.
- [St6] K. Strebel, *On the dilatation of extremal quasiconformal mappings of polygons*, Comment. Math. Helv. **74** (1999), 143–149.
- [Su1] T. Sugawa, *On the Bers conjecture for Fuchsian groups of second kind*, J. Math. Kyoto Univ. **32** (1992), 45–52.
- [Su2] T. Sugawa, *The Bers projection and the λ -lemma*, J. Math. Kyoto Univ. **32** (1992), 701–713.
- [Su3] T. Sugawa, *A class on the space of Schwarzian derivatives and its applications*, Proc. Japan Acad. Ser. A **69** (1993), 211–216.
- [Su4] T. Sugawa, *Holomorphic motions and quasiconformal extensions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 239–252.
- [Su5] T. Sugawa, *A remark on the Ahlfors–Lehto univalence criterion*, Ann. Acad. Sci. Fenn. Ser. A I Math. **27** (2002), 151–161.
- [ST] D. Sullivan and W.P. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.
- [T] P.M. Tamrazov, *Moduli and extremal metrics in nonorientable and twisted Riemannian manifolds*, Ukrainian Math. J. **50** (1998), 1586–1598.
- [Ta] H. Tanigawa, *Holomorphic families of geodesic discs in infinite dimensional Teichmüller spaces*, Nagoya Math. J. **127** (1992), 117–128.
- [Tan] M. Taniguchi, *Boundary variation and quasiconformal maps of Riemann surfaces*, J. Math. Kyoto Univ. **32** (4) (1992), 957–966.
- [Te1] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildungen*, Deutsche Math. **3** (1938), 621–678.
- [Te2] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Math. Naturw. Kl. **22** (1940), 1–197.
- [Te3] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von den Parametern*, Deutsche Math. **7** (1944), 309–336.
- [Te4] O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Math. **7** (1944), 336–343.
- [Th1] W.P. Thurston, *Zipper and univalent functions*, The Bieberbach Conjecture. Proc. Symp. Occasion of Its Proof, A. Baernstein II et al., eds, Amer. Math. Soc., Providence, RI (1986), 185–197.

- [Th2] W.P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1, Princeton Math. Series, Vol. 35, Princeton, NJ (1997).
- [To] M. Toki, *On non-starlikeness of Teichmüller spaces*, Proc. Japan Acad. Ser. A **69** (1993), 58–60.
- [Ts] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959).
- [Tu2] P. Tukia, *Quasiconformal extension of quasisymmetric mappings compatible with a Fuchsian group*, Acta Math. **154** (1985), 153–193.
- [V2] A.N. Vagin, *A class of spaces of quasiconformal mappings of the plane*, Sibirsk. Mat. Zh. **27** (5) (1986), 13–23 (in Russian).
- [V1] A.N. Vagin, *Distortion theory for multivalent quasiconformal mappings in the plane*, Soviet Math. Dokl. **36** (1988), 553–556.
- [Vai] J. Väisälä, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag, Berlin (1971).
- [Va1] A. Vasil'ev, *Isoperimetric growth theorems for quasiconformal automorphisms of the disk*, Russian Math. Izv. VUZ **41** (1997), 12–21.
- [Va2] A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings*, J. Math. Sci. **106** (2001), 3487–3517; Transl. from Itogi Nauki i Tekhniki, Seria Sovremennaya Matematika i Ee Prilozheniya, Tematicheskie Obzory, Vol. 71, Complex Analysis and Representation Theory-2 (2000).
- [Va3] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Lecture Notes in Math., Vol. 1788, Springer-Verlag, Berlin (2002).
- [Ve] I.N. Vekua, *Generalized Analytic Functions*, Fizmatgiz, Moscow, 1959 (in Russian); English transl.: *Generalized Analytic Functions*, Internat. Ser. Pure Appl. Math., Vol. 25, Pergamon Press, New York (1962).
- [Vel1] J.A. Velling, *Degeneration of quasicircles: Inner and outer radii of Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 191–202.
- [Vel2] J.A. Velling, *Mapping the disk to convex subregions*, Analysis and Topology, C. Andreian Cazacu, Olli Lehto and Th.M. Rassias, eds, World Scientific, Singapore (1998), 719–723.
- [Vol1] I.A. Volynets, *A certain conjecture in the theory of analytic functions*, Dokl. Akad. Nauk SSSR **209** (1973), 1261–1263.
- [Vol2] I.A. Volynets, *Some extremal problems for quasiconformal mappings*, Metric Questions in the Theory of Functions and Mappings, Vol. 5, Naukova Dumka, Kiev (1974), 26–39 (in Russian).
- [Vol3] I.A. Volynets, *Becker's problem on metrics in the class Σ* , Sibirsk. Mat. Zh. **26** (2) (1985), 194–198.
- [Vu] M. Vuorinen, *Conformally invariant extremal problems and quasiconformal maps*, Quart. J. Math. Oxford **43** (2) (1992), 501–514.
- [Wo1] M. Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **33** (1991), 487–539.
- [Wo2] M. Wolf, *On the existence of Jenkins–Strebel differentials using harmonic maps from surfaces to graphs*, Ann. Acad. Sci. Fenn. Ser. A I Math. **20** (1995), 269–278.
- [Zh1] I.V. Zhuravlev, *Univalent functions with quasiconformal extension and Teichmüller spaces*, Inst. Mathematics, Novosibirsk, Preprint (1979), 1–23 (in Russian).
- [Zh2] I.V. Zhuravlev, *Univalent functions and Teichmüller spaces*, Soviet Math. Dokl. **21** (1980), 252–255.
- [Zh3] I.V. Zhuravlev, *A topological property of Teichmüller space*, Math. Notes **38** (1985), 803–804.
- [Zh4] I.V. Zhuravlev, *On a model of the universal Teichmüller space*, Siber. Math. J. **27** (1986), 691–697.

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CHAPTER 6

Transfinite Diameter, Chebyshev Constant and Capacity

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Abstract

The aim of the present chapter is to survey alternate descriptions of the classical transfinite diameter due to Fekete and to review several generalizations of it. Here we lay emphasis mainly on the case of one complex variable. We shall generalize this notion in analogy to several situations in plane electrostatics. These include, among others, the Euclidean, hyperbolic and elliptic complex plane as support of a homogeneous medium or, more general an inhomogeneous isotropic medium in the presence of an external field. Throughout the chapter we shall outline the close connection of transfinite diameter and its generalizations with the theory of conformal and quasiconformal mappings and its applications in complex analysis.

1. Introduction

To describe the main aspects of the present chapter we use the electrostatic interpretation of the underlying basic extremal problem. The fundamental electrostatic problem concerns the equilibrium of a discrete unit charge μ on a conductor embedded in a homogeneous medium. If the conductor is regarded as a compact subset E of the complex plane $\mathbb{C}$ and n point charges δ_{z_k} , i.e., positive measures with unit mass concentrated at the points z_k ($k = 1, \dots, n$) on E repel each other according to an inverse distance law, then in the absence of an external field, equilibrium will be reached when the total logarithmic energy

$$\frac{1}{\binom{n}{2}} \sum_{1 \leq j < k \leq n} \log \frac{1}{|z_j - z_k|}$$

attains its minimum, say $\log(1/d_n)$, among all such discrete unit charges $\mu = \frac{1}{n} \sum_{k=1}^n \delta_{z_k}$, freely distributed on E . For every given $n \geq 2$, there exists a minimizing discrete unit charge μ_n which is actually supported on the outer boundary of E . In 1923 Fekete showed that d_n converges to a limit $d = d(E)$, the so-called transfinite diameter of E . The points which support μ_n are called n th Fekete points.

In Section 2 several equivalent definitions for the transfinite diameter are reviewed. This quantity is intimately connected with polynomials of the form $p_n(z) = \prod_{k=1}^n (z - z_k)$, $z_k \in E$, particularly with the so-called Chebyshev polynomial which minimizes the maximum norm $\|p_n\|_E$ among all such polynomials. This is motivated by the fact that $\log(1/\sqrt[n]{|p_n(z)|})$ is the potential of the discrete unit charge μ on the zeros of $p_n(z)$. Further, alternate descriptions of the transfinite diameter are given through potential theory in terms of Green's function, and by a minimum energy principle where the asymptotic equilibrium distribution $\mu^* := \lim_{n \rightarrow \infty} \mu_n$ (in the sense of weak* topology, see (13)) and its potential are the main objects of study here. Yet another portrayal comes through extremal length, or more specifically, reduced extremal distance. In addition, the transfinite diameter has a basic connection with conformal mapping. Its importance in function theory results from the fact that sets of transfinite diameter zero are removable for harmonic functions.

In Section 3 several estimates of the transfinite diameter of a compact set E are given in terms of geometric quantities connected with E . Further we formulate some results concerning with boundary behavior of conformal mapping, transfinite diameter and its change under conformal mapping.

Section 4 concentrates on estimates for the asymptotic distribution of Fekete points and related extremal points due to Fejér, Leja and Menke. All these extremal points provide nearly optimal choices for points of polynomial interpolation. In particular, they can be used for determining the Riemann mapping function of a simply connected domain. They also play an important role by several summation methods in numerical linear algebra.

Logarithmic capacity is closely related with bounded analytic functions not necessarily single valued. The consideration of single-valued bounded analytic functions leads to the notion of analytic capacity. Its significance in function theory is rooted in the fact that sets of analytic capacity zero are removable for bounded analytic functions. Moreover, since analytic capacity of a compact set measures its thinness it is useful in studying problems of rational approximation.

The last section is focused on several generalizations of logarithmic capacity. The underlying electrostatic situations are the Euclidean, the hyperbolic and the elliptic complex plane as support of

- (i) a homogeneous medium, i.e., the dielectric constant is equal to one,
- (ii) an inhomogeneous isotropic medium, i.e., the dielectric constant is a function depending on $z \in \mathbb{C}$,
- (iii) an inhomogeneous nonisotropic medium, i.e., the dielectric constant is a given tensor depending on $z \in \mathbb{C}$,

in the presence of an external field $Q(z)$ and its influence on the equilibrium distribution of positive or signed charges on a condenser.

The introduction of an external field $Q(z)$ in the case (i) creates some significant differences in the fundamental theory, but opens much wider doors to applications. The problem now becomes that of minimizing the weighted energy

$$\iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta) + 2 \int Q(z) d\mu(z)$$

over all positive unit charges μ on E . This is a variational problem that goes back to Gauss [42]. After the pioneering works of Frostman [36] on equilibrium measures, the emphasis shifted to more general kernels, although logarithmic potentials kept their importance because of their close relationship to polynomials and holomorphic functions. Frostman [37] himself considered the weighted energy problem for the case when $Q(z)$ is continuous and superharmonic. Starting from the 1930s the Polish school headed by Leja investigated logarithmic potentials with continuous external fields because of their connections with the solutions to certain Dirichlet problems. A new impulse came in the 1980s when Rakhmanov [156] used potentials with external fields to study orthogonal polynomials with respect to exponential weights. The essential distinction between earlier works (say of the Polish school) and the newer treatment of Mhaskar, Saff, Totik and Rakhmanov [133–135, 156, 160, 179] lies not only in its greater generality, but in its emphasis on determining the support of the equilibrium distribution. One of the glaring differences with the classical electrostatic problem ($Q(z) \equiv 0$) is that the support of the equilibrium distribution need not coincide with the outer boundary of E and, in fact, can be quite an arbitrarily subset E depending on $Q(z)$, possibly with positive area. There are several important aspects of the external field problem and its extensions to signed charges. The most striking is that it provides a unified approach to several problems in constructive analysis. These include, among others, the asymptotic analysis of orthogonal polynomials, the study of incomplete polynomials, the mathematical modeling of elasticity problems where the shape of the elastic medium is distorted by the insertion of an object under pressure, and numerical conformal mapping. In addition, the external field problem provides a rather natural setting for several important concepts in potential theory itself. These include solving Dirichlet's problem, the balayage (sweeping) of a measure on a compact set and solving constrained minimal energy problems. The external field problem can be viewed as a special case of the potential theory developed for energy integrals having symmetric, lower-semicontinuous kernels in locally compact spaces, see the papers of Ninomiya [139] and Ohtsuka [140, 141]. But in this generality many of the features of the external field problem, as well as its concrete applications to constructive analysis, remain hidden.

The weighted energy problem can be generalized to the case of signed charges on a condenser. In the case $Q(z) \equiv 0$ and when the condenser consists of two plates on which the charge is positive and negative with prescribed total charge 1 and -1 , respectively, the extremal point method yields an algorithm for determining the conformal map of a doubly-connected domain onto an annulus. This method is based on the observation that the corresponding minimal energy is a conformal invariant. Considering signed charges on a condenser with total charge zero on each of its plates under the influence of an external field induced, say by a dipole at infinity then the corresponding weighted energy problem provides an extremal characterization of the conformal parallel slit mapping.

We shall outline how potentials corresponding to the electrostatic situations (ii) and (iii) are related with the theory of quasiconformal mappings. As in the conformal case the minimum energy problem can be used to give extremal characterizations for certain canonical quasiconformal mappings. On the other hand, such quasiconformal mappings are solutions of certain function-theoretic extremal problems which are closely related to generalized Cauchy–Riemann differential equations. In the presence of an external field “imaginary” charges may arise particularly on those arcs along the dielectric constant (as function of z) has jumps. These charges can also be characterized by a principle of minimal weighted energy.

The final topic is devoted to capacity in $\mathbb{C}^N$, $N \geq 2$. The starting point for potential theory in one complex variable and in $\mathbb{R}^N$ is Laplace’s equation. For multidimensional generalizations pluripotential theory is used. Here the corresponding role is played by the complex Monge–Ampère equation. This equation is nonlinear while Laplace’s equation is linear. This explains why the corresponding developments in $\mathbb{R}^N$ and $\mathbb{C}^N$ differ considerably.

2. Alternate descriptions of transfinite diameter

2.1. Transfinite diameter

Let E be a compact subset of the complex plane $\mathbb{C}$. Choose a system of n points $z_1, \dots, z_n \in E$ and form the homogenized product

$$\left\{ \prod_{1 \leq j < k \leq n} |z_j - z_k| \right\}^{1/\binom{n}{2}} \quad (1)$$

of distances between the $\binom{n}{2}$ pairs of points z_k . Let $d_n(E)$ be the maximum of these products as the points z_k range over E . Note that $d_2(E)$ is the diameter of E , while $d_3(E)$ measures its “spread”. The quantity $d_n(E)$ is called the n th transfinite diameter of E . Those points $z_k = z_{nk}$ for which the maximum $d_n(E)$ of (1) is attained are called n th Fekete points of E . Obviously, all such points are distinct and “as far as possible apart from each other” on E . For fixed n , the set of n th Fekete points need not be unique. For example the

set of all n th Fekete points of the unit circle $E = \{z \in \mathbb{C}: |z| = 1\}$ are the n th unit roots and its rotations, and it holds $d_n(E) = n^{1/(n-1)}$. The zeros of the polynomial

$$p_n(x) = \frac{d^{n-2}}{dx^{n-2}}(x^2 - 1)^{n-1}$$

are the uniquely determined Fekete points of the segment $E = [-1, 1]$, and it holds [165]

$$d_n(E)^{n(n-1)} = \frac{2^2 3^3 \cdots n^n \cdot 2^2 3^3 \cdots (n-2)^{n-2}}{3^3 5^5 \cdots (2n-3)^{2n-3}}, \quad n > 2, \quad d_2(E) = 2.$$

For numerical computation of Fekete points of some ellipses and squares see [151] and [84].

It can be easily shown that $d_n(E)$ decreases to a limit $d(E) \geq 0$. The quantity $d(E)$ is called *transfinite diameter* of E which was introduced by Fekete [33].

Let G be the *outer domain* of E , that is, the unbounded component of $\mathbb{C} \setminus E$. Since the Fekete points lie on the boundary ∂G of G by the maximum principle for analytic functions, we have $d(E) = d(\partial G)$.

The following properties of transfinite diameter are easily derived from its definition:

Monotonicity: If $E \subset F$, then $d(E) \leq d(F)$.

Homogenicity: If $z^* = az + b$ maps E onto E^* , then $d(E^*) = |a|d(E)$.

Contraction property: Let $\phi: E \rightarrow \mathbb{C}$ be a mapping satisfying $|\phi(z) - \phi(z')| \leq |z - z'|$ for $z, z' \in E$. Then $d(\phi(E)) \leq d(E)$.

We define the inner and outer transfinite diameter of an arbitrary set $E \subset \mathbb{C}$ by

$$d_*(E) = \sup\{d(A): A \subset E, A \text{ compact}\},$$

$$d^*(E) = \inf\{d_*(H): E \subset H, H \text{ open}\}.$$

It is clear that $d_*(E) \leq d^*(E)$. A theorem of Choquet [19] says that for every Borel set the inner and outer transfinite diameter are the same.

The transfinite diameter has a subadditivity property [138] in the following sense: If $E = \bigcup_{k=1}^{\infty} E_k$ has diameter $< d$, then

$$1 / \log \frac{d}{d^*(E)} \leq \sum_{k=1}^{\infty} 1 / \log \frac{d}{d^*(E_k)}. \quad (2)$$

If $d^*(E_k) = 0$, $k = 1, 2, \dots$, then from (1) it follows $d^*(E) = 0$. In particular, a countable set has outer transfinite diameter zero.

A property is said to hold *quasi-everywhere* (q.e.) on a set if the set of exceptional points is of outer transfinite diameter zero.

2.2. Chebyshev constant

There is a close connection between the transfinite diameter of a compact set and polynomials, see [181, p. 71]. We consider polynomials of the form

$$p_n(z) = \prod_{k=1}^n (z - z_k), \quad z_k \in \mathbb{C}, \quad (3)$$

and put

$$\tau_n(E) := \inf \max_{z \in E} |p_n(z)|,$$

where the infimum is taken over all polynomials of the form (3). Then there exists a uniquely determined polynomial t_n of the form (3), such that $\max_{z \in E} |t_n(z)| = \tau_n(E)$. We call t_n the *Chebyshev polynomial* of E of order n . From the definition of t_n , one can prove easily that all zero points of t_n lie in the smallest convex set which contains E . In only a very few cases [46, 165] one exactly knows the Chebyshev polynomials, such as in the case of the unit circle for which $t_n(z) = z^n$ and in the case of the interval $[-1, 1]$ for which

$$t_n(x) = \frac{1}{2^n} \left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right], \quad n \in \mathbb{N}.$$

Fekete [34] proved that the limit $\tau(E) = \lim_{n \rightarrow \infty} \tau_n(E)^{1/n}$ exists. This quantity $\tau(E)$ is called the *Chebyshev constant* of E .

Let z_{nk} ($k = 1, 2, \dots, n$) be a system of n th Fekete points of E and set

$$q_n(z) = \prod_{k=1}^n (z - z_{nk}), \quad M_n(E) = \max_{z \in E} |q_n(z)|.$$

Then, for $n = 1, 2, \dots$,

$$d(E) \leq \tau_n(E)^{1/n} \leq M_n(E)^{1/n} \rightarrow d(E), \quad n \rightarrow \infty.$$

Hence, we get [33]

$$d(E) = \tau(E). \quad (4)$$

Note that (4) remains valid if $\tau(E)$ is defined by polynomials of the form (3) having zeros in E only.

2.3. Green function and Robin constant

We construct now the Green function by the method of Fekete points [112, 136]. Let $E \subset \mathbb{C}$ be compact and let G be the outer domain of E . If $d(E) > 0$, then

$$g(z, \infty) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{|q_n(z)|}{M_n(E)}$$

exists locally uniformly in $G \setminus \{\infty\}$ and represents a positive harmonic function that satisfies

$$g(z, \infty) = \log |z| - \log d(E) + O(1/|z|), \quad |z| \rightarrow \infty. \quad (5)$$

If $u(z)$ is positive and harmonic in $G \setminus \{\infty\}$ except for logarithmic poles and satisfies $u(z) = \log |z| + O(1)$ ($|z| \rightarrow \infty$), then

$$u(z) \geq g(z, \infty) > 0, \quad z \in G \setminus \{\infty\}. \quad (6)$$

On the other hand, if $d(E) = 0$, then there exists no such function $u(z)$. We call $g(z, \infty)$ the *Green function* of G with respect to ∞ , and the limit

$$\lim_{z \rightarrow \infty} (g(z, \infty) - \log |z|) = -\log d(E)$$

is called the *Robin constant* of G , see [175]. We can define the Green function as the smallest positive harmonic function with the development $\log |z| + O(1)$ at ∞ . It exists if and only if $d(E) > 0$, and we see from (6) that it is uniquely determined. It turns out that

$$\lim_{z \rightarrow \zeta} g(z, \infty) = 0 \quad \text{for q.e. } \zeta \in \partial G. \quad (7)$$

We call $\zeta \in \partial G$ a regular point with respect to the Green function $g(z, \infty)$ if (7) holds; otherwise it is called irregular. The set of irregular points of $g(z, \infty)$ has then the outer transfinite diameter zero. Let $0 < \lambda < 1$ and set

$$A_n(\zeta) := \{z: z \notin G, \lambda^n \leq |\zeta - z| < \lambda^{n-1}\}.$$

Wiener proved that $\zeta \in \partial G$ is regular with respect to the Green function if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/d(A_n(\zeta)))} = \infty. \quad (8)$$

The regularity of points play an important role in solving the Dirichlet problem: Let $E \subset \mathbb{C}$ be a compact set of positive transfinite diameter and let G be the outer domain

of E . Suppose that f is a bounded Borel measurable function defined on ∂G . The upper and lower classes of functions corresponding to f and G are defined as

$$H_f^{u,G} := \left\{ g: g \text{ superharmonic and bounded below on } G, \right. \\ \left. \liminf_{z \rightarrow \zeta} g(z) \geq f(\zeta) \text{ for } \zeta \in \partial G \right\}$$

and

$$H_f^{l,G} := \left\{ g: g \text{ subharmonic and bounded above on } G, \right. \\ \left. \limsup_{z \rightarrow \zeta} g(z) \leq f(\zeta) \text{ for } \zeta \in \partial G \right\}.$$

The upper and lower solutions of the Dirichlet problem for the boundary function are given by

$$\bar{H}_f^G(z) := \inf \{ g(z): g \in H_f^{u,G} \}, \quad z \in G,$$

and

$$\underline{H}_f^G(z) := \sup \{ g(z): g \in H_f^{l,G} \}, \quad z \in G.$$

These functions are harmonic in G and it always holds $\bar{H}_f^G(z) \leq \underline{H}_f^G(z)$ on G . If $\bar{H}_f^G(z) \equiv \underline{H}_f^G(z)$, then the function $H_f^G := \bar{H}_f^G(z)$ is called the *Perron–Wiener–Brelot solution* of the *Dirichlet problem* on G for the boundary function f . Suppose f is a continuous function, then the Perron–Wiener–Brelot solution H_f^G exists, is harmonic in G and it satisfies

$$\lim_{z \rightarrow \zeta} H_f^G(z) = f(\zeta) \quad \text{for q.e. } \zeta \in \partial G. \quad (9)$$

It turns out that $\zeta \in \partial G$ is regular with respect to the Green function $g(z, \infty)$ if and only if (9) holds for every continuous f on ∂G .

2.4. Logarithmic capacity

A physical interpretation of the transfinite diameter of a compact set E proceeds as follows [181]. Let $\mathcal{M}(E)$ denote the set of all positive unit Borel measures μ with support $\text{supp}(\mu) \subset E$. Recall that the *support* of a positive measure μ consists of all points z such that $\mu(D_r(z)) > 0$ for every open disk $D_r(z)$ of radius $r > 0$ and with center at z . Imagine that E is a conducting plate and that a charge distribution $\mu \in \mathcal{M}(E)$ is placed on it. Consider the *logarithmic energy*

$$I(\mu) = \iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta) \quad (10)$$

which measures the potential energy of μ in the presence of the *logarithmic potential*

$$u_\mu(z) = \int \log \frac{1}{|z - \zeta|} d\mu(\zeta).$$

The charge will distribute itself so as to minimize the energy integral. If E has several components, we can imagine that they are connected by thin wires to allow a free flow of charge. Let $V := \inf\{I(\mu) : \mu \in \mathcal{M}(E)\}$ be the minimum energy. The *logarithmic capacity* of E is defined as $\text{cap}(E) = e^{-V}$ if $V < \infty$ and $\text{cap}(E) = 0$ if $V = \infty$.

In the case $\text{cap}(E) > 0$ there exists a unique minimizing measure μ , called the *equilibrium measure* of E . The corresponding logarithmic potential $u_\mu(z) = \log|z| + O(1/z)$ is called the *conductor potential* of E . Note that the unicity of the equilibrium measure is a consequence of the fact that the logarithmic kernel $\log \frac{1}{|z - \zeta|}$ satisfies the following *energy principle* [72]: Let $\mu = \mu^+ - \mu^-$ be a signed Borel measure with compact support and total mass $\mu(\mathbb{C}) = 0$. Further suppose that the positive measures $\mu^\pm$ have finite logarithmic energy $I(\mu^\pm)$. Then the logarithmic energy of μ

$$\iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta) \geq 0, \quad (11)$$

and it is zero if and only if $\mu = 0$.

By using potential theoretic principles and variational techniques one can prove that the conductor potential $u_\mu(z) \leq V$ on $\mathbb{C}$ and $u_\mu(z) = V$ q.e. on E , see [36,37]. Thus, $g(z, \infty) = V - u_\mu(z) = \log|z| + V + O(1/z)$ is the Green function of G with respect to ∞ , and so, by (5) we have [175]

$$d(E) = \text{cap}(E). \quad (12)$$

Let $d(E) > 0$. The asymptotic distribution of the n th Fekete points z_{nk} ($k = 1, 2, \dots, n$; $n \in \mathbb{N}$) of E is the equilibrium measure μ of E in the following sense: Let denote the Dirac measure by δ_z with the unit mass at the point z . Define

$$\mu_n := \frac{1}{n} \sum_{k=1}^n \delta_{z_{nk}}.$$

Then $\mu_n \xrightarrow{*} \mu$ in the weak* topology, that means

$$\int f d\mu_n \rightarrow \int f d\mu \quad (13)$$

for all continuous functions $f : E \rightarrow \mathbb{R}$. Hence, μ_n could be used as a discrete substitute for the equilibrium measure μ . Since the Fekete points are distributed on the boundary ∂G , from (13) it follows $\mu(\overset{\circ}{E}) = 0$, where $\overset{\circ}{E}$ denotes the interior of E . In particular, if ∂G consists of a finite number of smooth Jordan curves then $d\mu(z) = \frac{1}{2\pi} \frac{\partial}{\partial n} g(z, \infty) |dz|$, where $|dz|$ denotes arclength measure and n is the inner normal on ∂G .

2.5. Extremal length

Alternate descriptions of transfinite diameter can be given in terms of extremal length of a family of curves. Let Γ be a family of locally rectifiable curves in a domain Ω . An *admissible metric* of Γ is a Borel-measurable function $\rho(z) \geq 0$ with the property

$$\int_{\gamma} \rho(z) |dz| \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

The *extremal length* $\lambda(\Gamma)$ is then defined by

$$\frac{1}{\lambda(\Gamma)} = \inf \iint_{\Omega} \rho(z)^2 dx dy, \quad z = x + iy, \quad (14)$$

where the infimum is taken over all admissible metrics ρ . The concept of extremal length has its root in the length–area method invented and developed by Grötzsch around 1928. It was introduced by Beurling and Ahlfors [3] around 1950 and has many applications in conformal mapping, in particular in connection with quadratic differentials and plays a key role in the theory of quasiconformal mappings. The importance of extremal length in function theory is due to mainly its conformal invariance.

Now let A and B be two disjoint subsets of the boundary of a domain Ω and let Γ be the family of curves in Ω which connect A to B . Then the extremal length $\lambda(\Gamma)$ is called the *extremal distance* from A to B and is denoted by $\lambda_{\Omega}(A, B)$. Let G be the outer domain of the compact set E and suppose that $A = \partial G$ consists of a finite number of components. Let C_r be a circle of radius r centered at the origin and large enough to enclose the set A . Further let G_r be the part of G which lies inside C_r . Then the limit $m(E) := \lim_{r \rightarrow \infty} \{2\pi \lambda_{G_r}(A, C_r) - \log r\}$ exists and it holds

$$d(E) = e^{-m(E)}. \quad (15)$$

The quantity $\lambda_{G_r}(A, C_r)$ can also be characterized as follows [3]

$$\frac{1}{\lambda_{G_r}(A, C_r)} = \inf \iint_{\mathbb{C}} |\nabla u|^2 dx dy, \quad z = x + iy, \quad (16)$$

where the infimum is taken over all Lipschitz-continuous functions $u: \mathbb{C} \rightarrow \mathbb{R}$ satisfying $u = 0$ on $\{z: |z| \geq r\}$ and $u = 1$ on E .

2.6. Conformal mapping radius

The transfinite diameter has a basic connection with conformal mapping. Let the outer domain G of the compact set E be simply connected. By the Riemann mapping theorem there exists a unique function

$$f(z) = z + a_0 + a_{-1}z^{-1} + \dots \quad (17)$$

that maps G conformally onto $|w| > r$ with $f(\infty) = \infty$. The quantity $r = r(G)$ is uniquely determined and is called the *conformal mapping radius* of G with respect to ∞ . Let $u_\mu(z)$ be the conductor potential of E . The function

$$u(z) := u_\mu(z) - V + \log \frac{|f(z)|}{r}, \quad z \in G,$$

is bounded and harmonic in G , and $u(z) \rightarrow 0$ as $z \rightarrow z_0 \in \partial G$ for q.e. $z_0 \in \partial G$. By the maximum principle, $u(z)$ vanishes identically and together with $u(\infty) = -V - \log r$ and (12) we get [175]

$$d(E) = r(G). \quad (18)$$

The fact that $u \equiv 0$ and that the Fekete points are distributed according to the equilibrium measure of E leads to a numerical method for determining the conformal mapping f , see (31).

By means of (18), the transfinite diameter of many sets can be found by conformal mapping. For example, the transfinite diameter of an ellipse with half axes a and b is found to be $\frac{1}{2}(a+b)$. In particular, the disk $|z| \leq r$ has the transfinite diameter r , while the transfinite diameter of a line segment is one-quarter of its length.

Let $p_n(z) = z^n + \dots$ be a polynomial and $E' = \{z \in \mathbb{C}: p_n(z) \in E\}$, where E is a given compact set. Applying the concept of Chebyshev polynomials on E and E' , by (4) we get

$$d(E') = \sqrt[n]{d(E)}. \quad (19)$$

In particular, if E is a disk $|z| \leq r$, then $d(E') = \sqrt[n]{r}$, see [34]. If E is the segment $[0, L^n]$, $p_n(z) = z^n$, then, by (19) the transfinite diameter of the set E' which consists of n equal segments of the length L meeting at the origin under angles of $2\pi/n$ is found to be $L/\sqrt[n]{4}$.

3. Estimates of transfinite diameter

The following application to conformal mapping is due to Fekete [33] (without proof), Hayman [68] (for a special case) and Pommerenke [154]: Let $f(z)$ be a meromorphic function in the outer domain G of the compact set E having expansion (17), and let F be a compact given set. If $f(G) \subset \mathbb{C} \setminus F$, then

$$d(F) \leq d(E). \quad (20)$$

If all limit points of $f(z)$ as $z \rightarrow \partial G$ lie in F and if $f(z)$ has no poles in $G \setminus \{\infty\}$, then

$$d(F) \geq d(E).$$

If the univalent function $f(z) = z + a_0 + a_1 z^{-1} + \dots$ maps the outer domain of E conformally onto the outer domain of F then we have

$$d(E) = d(F),$$

that is, the transfinite diameter of a compact set E is invariant under all such mappings.

Let E be a compact set, and $f(z)$ be a single-valued and analytic function in G having expansion

$$f(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \cdots$$

near infinity. Form the Hankel determinant

$$A_n := \det(a_{k+j})_{k,j=0}^{n-1}.$$

Pólya [146] proved

$$\limsup_{n \rightarrow \infty} |A_n|^{1/n^2} \leq d(E).$$

In the following we give some estimates for the transfinite diameter of a compact set in terms of geometric quantities connected with it. If γ is a rectifiable curve of length $l(\gamma)$, then [145]

$$l(\gamma) \geq 4d(\gamma),$$

where equality holds if γ is a line segment.

Let E be a compact set. If $P \subset l$ is the orthogonal projection of E onto an arbitrary line l then, by using the contraction property of the transfinite diameter, we get

$$d(E) \geq d(P) \geq \frac{1}{4} \text{mes}(P), \quad (21)$$

where equality holds if E is a segment on l , see [146].

By using separating transformation techniques, the inequality (21) can be generalized as follows [22]: Let $z_1, z_2, \dots, z_n$ ($n \geq 2$) be arbitrary points of a continuum E lying respectively on n rays going out from a point z_0 at equal angles. Then

$$d(E) \geq \sqrt[n]{\frac{1}{4} \prod_{k=1}^n |z_k - z_0|},$$

where equality holds only for continua consisting of n straight-line segments of equal length joining z_0 and z_k ($k = 1, \dots, n$).

Grötzsch [60] solved the problem to determine among all plane continua containing $n \geq 2$ given distinct points $z_1, \dots, z_n$ that continuum of minimal transfinite diameter. Such an extremal continuum E is uniquely determined and consists of at most $2n - 3$ analytic arcs meeting at most $n - 2$ branching points. In [95] it was shown, that the extremal continuum E is contained in the closed convex hull h of the points $z_1, \dots, z_n$. This is a simple consequence of the unicity of E and the property that the n th transfinite diameter and

therefore the transfinite diameter of E does not increase under the following contraction ϕ of $\mathbb{C}$: Let g be a line containing any side of h which divides $\mathbb{C}$ into two open half-planes H^+ and $H^- \supset \overset{\circ}{h}$. Define $\phi: \mathbb{C} \rightarrow \mathbb{C}$ by $\phi(z) = z$ for $z \in H^- \cup g$ and $\phi(z) =$ orthogonal projection of the point $z \in H^+$ onto g . In the case $n = 3$, Pirl [143] investigated the extremal continuum in more detail. For the analogous problem in the hyperbolic and elliptic geometry see [95, 124].

Let $D(E)$ denote the diameter of the set E . If E is a continuum then [12]

$$\frac{1}{4}D(E) \leq d(E) \leq \frac{1}{2}D(E). \quad (22)$$

Here equality holds on the left-hand side for a line segment and on the right-hand side for a disk. The left-hand side of (22) which is a consequence of (21) shows that the diameter of a continuum of the transfinite diameter 1 is bounded by 4. If E is not connected this is not true: Considering the set $E_n = \{z \in \mathbb{C}: |(z-n)(z-1/n)| \leq 1\}$, $n \in \mathbb{N}$, that consists of two ovals for $n > 2$, it holds $d(E_n) = 1$ and $n + 1/n \leq D(E_n) \rightarrow \infty$, $n \rightarrow \infty$.

Let $E \subset \{z \in \mathbb{C}: |z| \geq 1\}$ be compact. We denote the radial projection of E onto $|z| = 1$ by A and its linear measure by $\text{mes}(A)$. Then Ahlfors and Beurling [4] proved that

$$d(E) \geq d(A) \geq \sin\left(\frac{1}{4} \text{mes}(A)\right), \quad (23)$$

where equality holds if E is an arc on $|z| = 1$. The inequality (23) follows from the contraction property of the transfinite diameter and (20) by making an appropriate choice of f . This result was known to experts as long ago as the thirties. What can be upper bound for $d(A)$? By (19), the transfinite diameter of the set $A_n^* := \{z \in \mathbb{C}: |z| = 1, |\arg z^n| < (\text{mes}(A))/2\}$ with $\text{mes}(A_n^*) = \text{mes}(A) > 0$ is equal to

$$d(A_n^*) = \left[\sin\left(\frac{1}{4} \text{mes}(A)\right) \right]^{1/n} \rightarrow 1 = d(\{|z| = 1\}), \quad n \rightarrow \infty.$$

Therefore, without additional restrictions on A , there can be only the trivial upper bound. Let A be the union of n closed arcs of the unit circle $\Gamma = \{z \in \mathbb{C}: |z| = 1\}$ of total linear measure $\text{mes}(A) > 0$. Then Haliste [66] showed by means of dissymmetrization that

$$d(A) \leq d(A_n^*) = \left[\sin\left(\frac{1}{4} \text{mes}(A)\right) \right]^{1/n}, \quad (24)$$

where equality holds only in the case when A coincides with A_n^* up to rotation around the origin. The estimate (24) shows that a larger transfinite diameter corresponds to a set that is symmetric and “maximally dispersed on the circle Γ ”. In this connection, there arises the question about a lower bound for a “sufficiently dispersed” set which would be more precise than (23). Let A be an arbitrary closed subset of the unit circle Γ , and let l_k be the linear measure of the intersection of A with the angle

$$\{z = re^{i\theta}: 0 < r < \infty, |\theta - 2\pi k/n| < \pi/n\}, \quad k = 1, 2, \dots, n.$$

Then

$$d(A) \geq \prod_{k=1}^n [\sin(nl_k/4)]^{2/n^2},$$

where equality holds only in the case of even n and A consisting of $n/2$ equal arcs with centers either at the points $e^{i(\pi/n+4\pi k/n)}$, $k = 1, \dots, n/2$, or at the points $e^{i(-\pi/n+4\pi k/n)}$, $k = 1, \dots, n/2$. The proof of this assertion was given by Klein [79] by means of an ingenious application of the well-known Renggli inequality [158].

Let denote the two-dimensional Lebesgue measure of the compact set E by $\text{mes}(E)$. By applying the concept of Chebyshev polynomials and (4), Pólya [145] proved the inequality

$$\text{mes}(E) \leq \pi d(E)^2, \quad (25)$$

where equality holds for a disk. If $d(E) = 0$, then the measure of E is necessarily equal to zero. Adding all domains of the complement of E which do not contain ∞ to E , the measure of it remains zero because the transfinite diameter does not change by this process. In particular, in the case $d(E) = 0$ the complement of E is a domain containing ∞ , and every point of the plane is an inner point or a boundary point of this domain. Further, from (21) it follows that a compact set of transfinite diameter zero does not contain a continuum. These conditions are necessary for a compact set to be of transfinite diameter zero but not sufficient as the ordinary Cantor set shows. It is defined as follows. Let $I = [0, 1]$, $I_1 = [0, x_1]$, $I_2 = [x_2, 1]$, $0 < x_1 < x_2 < 1$, such that $|I_1| = |I_2| = |I|/(2p)$, $x_2 - x_1 = (1 - 1/p)|I|$, $p > 1$. Performing the similar operations on I_1 and I_2 and proceeding similarly, then after n steps, one obtain 2^n intervals $I_{i_1, \dots, i_n}$ ($i_1, \dots, i_n \in \{1, 2\}$). Then $E = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \dots, i_n}^{1,2} I_{i_1, \dots, i_n}$ is called the ordinary Cantor set. Although this set E does not contain a continuum it holds $\text{mes}(E) = 0$ and $d(E) \geq (1 - 1/p)/(2p) > 0$, see [138].

Let $h(t) > 0$ be an increasing function of t ($0 < t \leq 1$), $h(0) = 0$, and $e \subset \mathbb{C}$ be a bounded set. We cover e by at most a countable number of squares of sides $d_i < \rho$, whose sides are parallel to the coordinates axis and put

$$H(\rho) := \inf \sum_i h(d_i),$$

then $H(\rho)$ increases, if ρ decreases, so that $\lim_{\rho \rightarrow 0} H(\rho) =: h^*(e)$ ($0 \leq h^*(e) \leq \infty$) exists. This limit is called the Hausdorff measure of e . Suppose that $\int_0^1 h(t)/t dt < \infty$. If E is a compact set satisfying $h^*(E) > 0$, then $d(E) > 0$. Hence, if $d(E) = 0$, then $h^*(E) = 0$, see [36]. In the case $\int_0^1 h(t)/t dt = \infty$ it does not follow $d(E) > 0$ from $h^*(E) > 0$ in general. For instance, if $h(t) = 1/\log(1/t)$ and $h^*(E) < \infty$, then $d(E) = 0$, see [30].

The following results deal with the boundary behavior of conformal mapping, transfinite diameter and its change under conformal mapping. Let Σ denote the class of functions

$$g(z) = z + b_0 + b_1 z^{-1} + \dots$$

that are univalent and analytic in $\Delta = \{z \in \mathbb{C}: |z| > 1\}$. Let $g \in \Sigma$ and $\lambda > 0$. Then

$$\int_1^2 |g'(rz)| dr \leq \lambda, \quad z \in \partial\Delta \setminus A, \quad (26)$$

where the exceptional set A satisfies $d^*(A) \leq e^{\lambda - \lambda^2/2}$. Hence, g has a radial limit $\lim_{r \rightarrow 1+0} g(rz)$ q.e. on $\partial\Delta$. This existence result for radial limits is due to Beurling [11]. The proof is based on the fact that $g \in \Sigma$ satisfies the inequality $\sum_{k=1}^{\infty} k|b_k|^2 \leq 1$.

The following inequality [152, 163] goes in the opposite direction. Let $g \in \Sigma$ and $E \subset \mathbb{C}$, $A \subset \partial\Delta$. If the closure of $\{g(rz): 1 < r < \infty\}$ intersects E for each $z \in A$ then

$$d^*(E) \geq d^*(A)^2. \quad (27)$$

The proof of (27) is based on a distortion theorem for functions of the class Σ . We denote by $g(A)$ the set of all radial limits $g(z)$ for $z \in A \subset \partial\Delta$. These radial limits exist q.e. on A . Then, from (27) it follows for all $g \in \Sigma$,

$$d^*(g(A)) \geq d^*(A)^2, \quad (28)$$

where equality holds if A is an arc on $\partial\Delta$ and if $g \in \Sigma$ maps A onto a circle and $\partial\Delta \setminus A$ onto a radial slit. A further consequence of (27) is the following subadditivity property different from (2), see [152, 161]. If E is a continuum and $E = E_1 \cup E_2$ then

$$d(E) \leq d^*(E_1) + d^*(E_2).$$

Now we state some inequalities for functions of the class S analogously to (26) and (28). Let S denote the class of all functions

$$f(z) = z + a_2 z^2 + \dots$$

that are analytic and univalent in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. Let $f \in S$ and $\lambda \geq 6$. Then [154, p. 350]

$$\int_0^1 |f'(rz)| dr \leq 6 + \lambda^5, \quad z \in \partial\mathbb{D} \setminus A,$$

where the exceptional set A satisfies $d^*(A) < \sqrt{2/\lambda}$.

Let $f \in S$, $A \subset \partial\mathbb{D}$, and $f(A)$ be the set of all radial limits. Then [154, p. 351] it holds

$$d^*(f(A)) \geq \frac{1}{16} d^*(A)^2. \quad (29)$$

For the special case $A \subset \partial\mathbb{D}$ is a closed arc with aperture α ($0 < \alpha < 2\pi$), in [78] it was shown

$$d(f(A)) \geq d(A)^2 (1 + \sqrt{1 - d(A)^2})^{-4}$$

with $d(A) = \sin(\alpha/4)$, where equality holds if and only if $f \in S$ maps $\mathbb{D}$ onto the complex plane slit along an arc (with midpoint m) on some circle $|w| = r$ and the ray $\{tm: t > r\}$ which is the image of $\partial\mathbb{D} \setminus A$. The last inequality shows that (29) is asymptotically sharp for small transfinite diameter of A .

Finally we state an analogous inequality (28) for bounded univalent functions. Let $\varphi(z)$ be analytic and univalent in $\mathbb{D}$ satisfying $\varphi(0) = 0$, $|\phi(z)| < 1$. Further let $A \subset \partial\mathbb{D}$, $E = \varphi(A)$ (the set of radial limits) and $\bar{E} = \{1/\bar{w}: w \in E\}$. Then [154, p. 347] it holds

$$d^*(E \cup \bar{E}) \geq \frac{d^*(A)}{\sqrt{|\varphi'(0)|}}.$$

This inequality was first proved in [92] for the special case A is an arc on $\partial\mathbb{D}$ and $E = \varphi(A) \subset \partial\mathbb{D}$ in which case the inequality is sharp.

4. Asymptotic distribution of extremal points and applications

4.1. Fekete points

In the following let E be an arbitrary continuum with $d(E) = 1$. Let z_{nk} ($k = 1, 2, \dots, n$) be the n th Fekete points, and let $f(z) = z + a_0 + a_1 z^{-1} + \dots$ be the function that maps the outer domain G of E conformally onto $|w| > 1$. Leja [112] showed that, for all $l = 1, 2, \dots$, the limit $s_l := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n z_{nk}^l$ exists. Pommerenke [149] estimated the rate of convergence:

$$\left| \frac{1}{n} \sum_{k=1}^n z_{nk}^l - s_l \right| < 28l \cdot 4^l \frac{\log n}{n}, \quad l = 1, 2, \dots$$

Set

$$q_n(z) = \prod_{k=1}^n (z - z_{nk}).$$

Since G is simply connected, the function $q_n(z)^{1/n}$ is single valued in G , and we take the branch of the n th root that is positive on the positive real half-line. Then [149] the n th transfinite diameter of E satisfies

$$d_n(E) = n^{1/(n-1)} \left(1 + O\left(\frac{\log \log n}{n} \right) \right), \quad (30)$$

and it holds [150]

$$|q_n(z)^{1/n} - f(z)| < \frac{24}{1 - |f(z)|^{-1}} \frac{\log(n+1)}{n} \quad (31)$$

for $z \in G$ and $n > 1$. Note that the right-hand side of (31) cannot be replaced by $o(1/n)$.

Now let E be a closed Jordan curve with $d(E) = 1$ and let $z_{nk} = f^{-1}(e^{i\theta_{nk}})$ ($k = 1, 2, \dots, n$) be the n th Fekete points of E where f^{-1} denotes the inverse function of f . Then the θ_{nk} ($k = 1, 2, \dots, n$) are equally distributed on $[0, 2\pi]$ for $n \rightarrow \infty$, see [183]. More precisely, let $N_n(\alpha, \beta)$ be the number of the θ_{nk} ($k = 1, 2, \dots, n$) lying in $[\alpha, \beta]$. Kleiner [84] proved that

$$\left| \frac{N_n(\alpha, \beta)}{n} - \frac{\alpha - \beta}{2\pi} \right| < K(E) \frac{\log n}{\sqrt{n}}, \quad (32)$$

where the constant $K(E)$ depends from E only. In particular, if E is a closed analytic Jordan curve Pommerenke [149] showed that the right-hand side of (32) can be replaced by $K(E) (\log n)^{3/2}/n^{2/3}$. Moreover, he obtained the asymptotic estimates [153]

$$\theta_{nk} = \alpha_n + \frac{2k\pi}{n} + \frac{1}{n} \Phi\left(\alpha_n + \frac{2k\pi}{n}\right) + O\left(\frac{\sqrt{\log n}}{n^2}\right), \quad n \rightarrow \infty,$$

and

$$d_n(E) = n^{1/(n-1)} \left(K_0 + O\left(\frac{1}{n}\right) \right)^{1/(n(n-1))}, \quad n > 1,$$

with suitable constants α_n and $K_0 \geq 1$, where $\Phi(t)$ is an appropriate real analytic function depending on E only.

4.2. Polynomial interpolation

In the following we shall illustrate how the systems of Fekete points can be applied in the theory of approximation by interpolation. If $n+1$ pairs (z_k, w_k) , $k = 1, 2, \dots, n+1$, of complex numbers are given, where the z_k are to be distinct, then there exists exactly one polynomial P of degree at most n such that $P(z_k) = w_k$, $k = 1, 2, \dots, n+1$. One way of obtaining this polynomial is through Lagrange's formula of interpolation. To this end, let

$$\omega(z) := \prod_{k=1}^{n+1} (z - z_k), \quad l_k(z) := \frac{\omega(z)}{\omega'(z_k)(z - z_k)}, \quad k = 1, 2, \dots, n+1.$$

Each of these basic polynomials l_k is exactly of degree n , and we have

$$l_k(z_j) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Hence the n th degree polynomial

$$L_n(z) = \sum_{k=1}^{n+1} w_k l_k(z)$$

satisfies the interpolation requirement $L_n(z_k) = w_k$, $k = 1, 2, \dots, n + 1$.

Now we suppose the compact set E is such that $G = \mathbb{C} \setminus E$ is a domain that has a Green function $g(z; \infty)$ with pole at ∞ . The existence of $g(z; \infty)$ is assured if and only if E has positive transfinite diameter.

The set

$$C_\rho = \{z \in G: g(z; \infty) = \log \rho\}, \quad \rho > 1,$$

is called a level curve corresponding to the parameter ρ . For all $\rho \neq \rho_k$ it consists of finitely many analytic Jordan curves γ_j forming the boundary of some unbounded domain D_ρ such that $E \subset \mathbb{C} \setminus \overline{D_\rho}$. Here the ρ_k are at most countably many exceptional values with $\rho_k \rightarrow 1$, for which C_ρ passes through “critical points” of G , where finitely many γ_j meet, see [183, p. 67]. The set of exceptional values ρ_k is empty if and only if G is simply connected. For increasing ρ , the domains D_ρ decrease monotonically in the obvious way, and for each point $z \in G$ there exists exactly one ρ such that $z \in C_\rho$, namely $\rho = e^{g(z, \infty)}$.

If f is analytic on E (and not an entire function), then there exists a maximal ρ with the property that f has a unique analytic continuation from E to the interior of C_ρ . Note here that in the various components of E completely different analytic functions can be defined, each of which can be continued to the corresponding part of the interior of C_ρ . Consequently, C_ρ does not necessarily contain a singularity of the continuation of f ; in fact, C_ρ need not contain such a singularity if it contains a critical point P of G . In this case two analytic continuations meet at P .

This maximal $\rho > 1$ now plays a crucial role for the rate of convergence of the interpolating polynomials. Suppose $\rho > 1$ is the largest number such that f is analytic inside C_ρ . Let $L_n(z)$ be an interpolating polynomial constructed with $(n + 1)$ st Fekete points $z_{n+1,k}$ on the boundary of E and $w_k = f(z_{n+1,k})$, $k = 1, 2, \dots, n + 1$. Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{\max\{|f(z) - L_n(z)|: z \in E\}} = \frac{1}{\rho}.$$

In this generality, the last assertion was first proved by Walsh and Russell [184]; see also Shen [166] and his generalizations to interpolation by rational functions.

4.3. Fejér points

Now we assume E is a compact set whose unbounded component G of $\mathbb{C} \setminus E$ is a simply connected domain. Let g denote a conformal mapping of $|w| > 1$ onto G with $g(\infty) = \infty$. We require that g has a continuous extension to $|w| \geq 1$; this will be the case, for example, if ∂E is a Jordan curve or a Jordan arc. Then the points

$$z_{nk} := g(e^{2\pi(k-1)/n}), \quad k = 1, \dots, n,$$

are called *n*th Fejér points on E . Form the *Fejér polynomial* of E of order n

$$q_n(z) = \prod_{k=1}^n (z - z_{nk}).$$

Fejér [32] showed that

$$\lim_{n \rightarrow \infty} \left\{ \max_{z \in E} |q_n(z)| \right\}^{1/n} = d(E),$$

where $d(E)$ is the transfinite diameter of E .

4.4. A summation method in numerical linear algebra

The Fejér points play an important role for universal algorithms to solve systems of linear equations given in the fixed point form

$$x = Tx + c, \quad x, c \in \mathbb{C}^N,$$

where T is an $N \times N$ matrix with complex elements. We assume that this equation has a unique solution x . Let denote the set of all eigenvalues $\lambda_1, \dots, \lambda_N$ of T by $\sigma(T)$, called the spectrum of T . The number $\rho(T) := \max\{|\lambda_k|: k = 1, \dots, N\}$ is called the spectral radius of T . It is well known that, if $\rho(T) < 1$, then the vector sequence x_n defined by the iterative process

$$x_0 \in \mathbb{C}^N, \quad x_n = Tx_{n-1} + c, \quad n \geq 1,$$

converges to x for each $x_0 \in \mathbb{C}^N$. In order to enforce the convergence if the iterative sequence x_n diverges or to accelerate its convergence one can apply the following *summation method*. In addition to the assumptions on the compact set E in Section 4.3, we assume $\sigma(T) \subseteq E$ and $1 \in G$. Define

$$p_n(z) := \frac{q_n(z)}{q_n(1)} = \sum_{k=0}^n a_{nk} z^k, \quad n \geq 1,$$

where $q_n(z)$ is a Fejér polynomial of E of order n . Now we transform the iterates $x_0, x_1, \dots$ into the form

$$y_0 = x_0, \quad y_n := \sum_{k=0}^n a_{nk} x_k, \quad n \geq 1.$$

Set $e_n := x - y_n$. Then [183, Chapter 4, Theorem 7] $e_n = p_n(T)e_0$ and

$$\limsup_{n \rightarrow \infty} \left(\sup_{e_0 \neq 0} \frac{\|e_n\|}{\|e_0\|} \right)^{1/n} \leq \frac{1}{|w_1|} (< 1),$$

where $g(w_1) = 1$. Note that this assertion remains true if the Fejér points are replaced by Fekete or Leja points. For further informations we refer to [31] and the paper of Eiermann [29] that gives an overview of the connection between complex analysis and numerical linear algebra for the study of such summation methods.

4.5. Menke points

Menke [128] introduced an extremal point system of E by using intermediate points. Let C be an analytic closed Jordan curve with $d(C) = 1$. Let $R_n(C)$ denote the maximum of the expression

$$\prod_{\mu=1}^n \prod_{v=1}^n |z_\mu - \zeta_v|,$$

where $z_1, \dots, z_n, \zeta_1, \dots, \zeta_n$ are arbitrary points of C such that $z_k \prec \zeta_k \prec z_{k+1}$ for $k = 1, 2, \dots, n-1$ and $z_n \prec \zeta_n \prec z_1$ (the points of the curve C being ordered by its orientation). Let $g(w) = w + b_0 + b_1 w^{-1} + \dots$ maps $|w| > 1$ conformally onto the exterior of C . Let $z_{n1}, \dots, z_{nn}, \zeta_{n1}, \dots, \zeta_{nn}$ be the n th Menke points of C , i.e.,

$$R_n(C) = \prod_{\mu=1}^n \prod_{v=1}^n |z_{n\mu} - \zeta_{nv}|,$$

with $z_{nk} = g(e^{is_{nk}})$ and $\zeta_{nk} = g(e^{it_{nk}})$ for $k = 1, 2, \dots, n$, where $t_{nn} - 2\pi = t_{n0} < s_{n1} < t_{n1} < \dots < s_{nn} < t_{nn}$. Menke showed [128] that

$$s_{nk} = t_{n0} + \frac{2k-1}{n}\pi + O(r^n), \quad t_{nk} = t_{n0} + \frac{2k}{n}\pi + O(r^n)$$

and

$$R_n(C) = 2^n (1 + O(r^n)),$$

where $r \in (0, 1)$ is a constant depending on C only.

Let $H_n(w)$ denote the Lagrange interpolation polynomial of $(2n-1)$ st degree in the variable $1/w$ that maps the $2n$ th unit roots (up to some rotation around the origin) onto the n th Menke points of C . Then [128]

$$\frac{g(w)}{w} = H_n(w) + O(q^n), \quad |w| \geq 1, n \rightarrow \infty,$$

where $q \in (0, 1)$ depends on C only. For the numerical calculation of the n th Menke points of C and the numerical approximation of $g(w)/w$ see [129]. Analogous results for the case of piecewise analytic Jordan curves and smooth Jordan curves are obtained in [130].

4.6. Leja points

However, the determination of an n th Fekete point system is equally hard for it is equivalent to an extremal problem in n variables. As we have seen, Fekete points are of great value in practice; for example, they are almost ideal nodes for interpolation and approximation. Hence it is worth looking for simpler procedures that generates points similar to Fekete points. First Leja [114] considered an associated sequence $\{z_n\}$ that is adaptively generated from earlier points according to the law: Starting with some $z_0 \in E$, z_n is defined as a point maximizing the expression

$$|(z - z_0)(z - z_1) \cdots (z - z_{n-1})|$$

on the continuum E . These so-called *Leja points* of E are again distributed like the equilibrium measure μ of E , i.e.,

$$\mu_n := \frac{1}{n} \sum_{k=1}^n \delta_{z_k} \xrightarrow{*} \mu \quad (33)$$

in the sense of weak* topology, see (13). Thus, one can use them in place of the Fekete points. Set

$$q_n(z) := \prod_{k=0}^{n-1} (z - z_k).$$

Since the outer domain G of the continuum E is simply connected, the function $q_n(z)^{1/n}$ is single valued in G , and in what follows we take the branch of the n th root that is positive on the positive real half-line. Then

$$\lim_{n \rightarrow \infty} |q_n(z_n)|^{1/n} = r(G), \quad f(z) := \lim_{n \rightarrow \infty} q_n(z)^{1/n} \quad (34)$$

where the last limit exists locally uniformly inside G , and $f(z)$ maps G conformally onto $|w| > r(G)$.

In order that this method yields numerically a feasible procedure and that the Leja points be numerically applicable to computing μ , $r(G)$ and f , we now give a discretized version of the aforementioned procedure: Let $\{\varepsilon_n\}$ be a sequence of positive numbers satisfying

$$\varepsilon_n^{1/n} \rightarrow 0, \quad n \rightarrow \infty. \quad (35)$$

Fix a corresponding sequence of discrete subsets $\{S_n\}$ of E in such a way that $S_n \subset S_{n+1}$ and, for each n and $z \in E$, there is a point in S_n whose distance from z is at most ε_n . Let $\hat{z}_0$ be arbitrary, and for each $n \in \mathbb{N}$ define $\hat{z}_n \in S_n$ as a point maximizing the expression

$$|(z - \hat{z}_0)(z - \hat{z}_1) \cdots (z - \hat{z}_{n-1})|$$

on S_n . Substituting z_n by $\hat{z}_n$, the assertions in (34) remain valid.

The assumption (35) is rather strong, even for relatively small n it requires a tremendous amount of computation to determine the points $\hat{z}_n$ successively. Suppose E is a smooth closed Jordan curve. Then (35) can be relaxed to $\varepsilon_n = O(n^{-\alpha})$ for some $\alpha > 0$.

A major part of the work of the Polish school around Leja in the 1950s focused on this extremal point method for determining Green function, conformal mapping etc., and this school developed the relevant theory in detail and generalized it to other kernels or dimensions, see Leja [107–122], Kleiner [80–89], Górski [50–59] and Siciak [167–170].

5. Analytic capacity and rational approximation

There is also a close connection between transfinite diameter and bounded analytic functions. Let E be a compact set. By using the minimal property (6) of the Green function with $u(z) = -\log |f(z)|$, one conclude [187, Appendix I]

$$d(E) = \sup |a_1|, \quad (36)$$

where the supremum is taken over all functions f analytic but not necessarily single valued on the unbounded component G of the complement of E which satisfy

$$|f(z)| \text{ is single valued,} \quad (37)$$

$$|f(z)| \leq 1, \quad z \in G, \quad (38)$$

and have expansion

$$f(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad (39)$$

in a neighborhood of infinity.

The importance of transfinite diameter in function theory results from the fact that sets of transfinite diameter zero are removable sets for harmonic functions: Suppose U is an open set, $E \subset U$ is compact, and $d(E) = 0$. Then any function bounded and harmonic on $U \setminus E$ can be extended harmonically to all of U .

5.1. Analytic capacity

In the theory of bounded single-valued analytic functions, *analytic capacity* of a compact set plays an important role. This quantity is defined by

$$\gamma(E) := \sup |a_1|, \quad (40)$$

where the supremum is taken over all functions f single valued and analytic on G satisfying (38) and (39). A normal families argument shows there is an extremal function f

with $a_1 = \gamma(E)$. It turns out [4,35] that for any compact set E there is a unique such extremal function, called the Ahlfors function, and all its finite zeros lie in the convex hull of E [147]. If E has n components, then the Ahlfors function is an n -fold covering map onto the unit disk [1]. For an arbitrary set F define

$$\gamma(F) := \sup\{\gamma(E) : E \subset F, E \text{ compact}\}.$$

Clearly, the analytic capacity has the monotonicity and homogeneity property as the transfinite diameter. From the definition and (36) it follows

$$d(E) \geq \gamma(E),$$

where equality holds if E is a continuum. A concise treatment of further elementary properties of analytic capacity can be found in [187] and [41].

The actual definition of analytic capacity is due to Ahlfors [1], who was interested in function theoretic extremal problems of finitely-connected domains. The object of this paper is to treat the true equivalent of Schwarz's lemma for single-valued bounded analytic functions in a multiply-connected region. His work was refined by Grunsky [62–64] (see also the survey article [65]) and by Garabedian [40]. A summary of [40] and related work is given in Nehari's survey article [137]. Ahlfors generalized Garabedian's result to regions on Riemann surfaces [2]; see Royden's paper [159] for another treatment as well as further references to the literature.

Havinson [67] has used analytic capacity to considerable advantage in studying problems of function theory. His work is partly an extension of the work of Ahlfors and Garabedian to infinitely-connected domains.

Ahlfors and Beurling [4] were the first to study analytic capacity from a systematic viewpoint. Pommerenke [147] extended their results. He proved that for an arbitrary compact set E

$$\text{mes}(E) \leq \pi \gamma(E)^2.$$

Suppose G is a finitely-connected domain containing the point at infinity. Let

$$\sum_{k=1}^n E_k = \{e_1 + \cdots + e_n : e_1 \in E_1, \dots, e_n \in E_n\}$$

be the Minkowski sum of the boundary continua $E_1, \dots, E_n$ of G that is a continuum again. Then

$$\gamma\left(\sum_{k=1}^n E_k\right) \geq \gamma\left(\bigcup_{k=1}^n E_k\right).$$

With the help of this inequality Pommerenke [147] showed that if E is a compact set on a line with the linear measure $l(E)$, then $\gamma(E) = l(E)/4$. For instance, let

$E(a) = [-a-2, -a] \cup [a, a+2]$, $a > 0$, then $\gamma(E(a)) = 1$ and $d(E(a)) = \sqrt{a+1}$, see (19), such that the transfinite diameter converges to infinity for $a \rightarrow \infty$ while the analytic capacity is constant. Furthermore, Pommerenke [147] showed that, roughly speaking, the analytic capacity is approximatively the sum of the analytic capacities of the components if they lie far apart. More precisely, let $F_1, \dots, F_n$ be given compact sets. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: If E_k is the image of F_k by some translation T_k ($k = 1, 2, \dots, n$) and all distances between E_i and E_j ($i \neq j$) are greater than δ then

$$\left| \gamma\left(\bigcup_{k=1}^n E_k\right) - \sum_{k=1}^n \gamma(E_k) \right| < \varepsilon.$$

The curvature $c(\mu)$ of a positive Borel measure μ on $\mathbb{C}$ is defined by the formula

$$c(\mu)^2 := \iiint R^{-2}(z, \zeta, w) d\mu(z) d\mu(\zeta) d\mu(w),$$

where $R = R(z, \zeta, w)$ is the radius of the circle passing through three different points z, ζ, w ; otherwise we put $R = \infty$. If E is a compact set then [127]

$$\gamma(E) \geq M \sup \{ [\mu(E)]^{3/2} [\mu(E) + c(\mu)^2]^{-1/2} \},$$

where M is an absolute constant and the supremum is taken over all positive Borel measures μ such that $\text{supp } \mu \subset E$ and $\mu(B_r(\zeta)) \leq r$ for every disk $B_r(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < r\}$.

In [127] Melnikov gives also a *discrete approach* to analytic capacity: Given a positive constant r and n complex numbers $z_1, \dots, z_n$ with $|z_i - z_j| > 2r, i \neq j$, and let $A = A(z_1, \dots, z_n, r)$ be the $n \times n$ matrix with entries

$$\alpha_{ij} := r \sum_{k \neq i, k \neq j} [(z_i - z_k)(\overline{z_j - z_k}) - r^2]^{-1}, \quad i, j = 1, 2, \dots, n.$$

Put $\lambda_1(z_1, \dots, z_n, r) := \langle (r^{-1}I + A)^{-1}(\mathbf{1}), \mathbf{1} \rangle$, where I is the $n \times n$ unit matrix, $\mathbf{1} = (1, \dots, 1)$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{C}^n$. If G is a bounded open subset of $\mathbb{C}$, then

$$\gamma(G) = \sup \{ \lambda_1(z_1, \dots, z_n, r) \},$$

where the supremum is taken over all subsets $\{z_1, \dots, z_n\}$ of G and all $r > 0$ such that $|z_i - z_j| > 2r, i \neq j$ and $\text{dist}(z_i, \partial G) > r, i = 1, 2, \dots, n$.

The study of sets of analytic capacity zero goes back all the way to Painlevé [142]. Refer also to the papers of Denjoy [20] and Besicovitch [9]. The significance of analytic capacity in problems of function theory is rooted in the fact that sets of analytic capacity zero are removable for bounded analytic functions. More precisely, let E be a compact set contained in an open set U . Then every function bounded and analytic in $U \setminus E$ can be extended analytically to all of U if and only if $\gamma(E) = 0$.

5.2. Rational approximation

The analytic capacity of a compact set E as a measure of its thinness is useful in studying problems of *rational approximation*. Let $R(E)$ denote the set of all functions on E uniformly approximable by rational functions with poles off E . A point $z \in E$ is called a *peak point* for $R(E)$ if there exists a function $f \in R(E)$ satisfying $f(z) = 1$ and $|f(\zeta)| < 1$ for $\zeta \in E \setminus \{z\}$. Let $E^c := \mathbb{C} \setminus E$ and

$$\gamma_n := \gamma(E^c \cap \{\zeta: 2^{-n-1} \leq |\zeta - z| \leq 2^{-n}\}), \quad n \in \mathbb{N}.$$

Then [125,126] z is a peak point for $R(E)$ if and only if

$$\sum_{n=0}^{\infty} 2^n \gamma_n = \infty. \quad (41)$$

Consider, for example, a compact set E constructed as follows. Delete from the closed unit disk $\bar{D} = \{z \in \mathbb{C}: |z| \leq 1\}$ a sequence of open disks $D_n = \{z \in \mathbb{C}: |z - x_n| \leq r_n\}$, where

- (i) $1 > x_1 > x_2 > \cdots \rightarrow 0$,
- (ii) $x_1 + r_1 < 1$,
- (iii) $x_{n+1} + r_{n+1} < x_n - r_n$ for all $n \in \mathbb{N}$.

Then set $E = \bar{D} \setminus \bigcup_{n=1}^{\infty} D_n$. It is obvious that each point of $\partial E \setminus \{0\}$ is a peak point for $R(E)$. By (41), the point $z = 0$ is a peak point for $R(E)$ if and only if

$$\sum_{n=1}^{\infty} \frac{r_n}{x_n} = \infty.$$

Let $A(E)$ be the set of all functions continuous on E and analytic on $\overset{\circ}{E}$, the interior of E . To study $A(E)$ it is useful to introduce another measure, the AC-capacity, which was introduced by Dolzhenko in 1962. The *AC-capacity* $\alpha(M)$ of a set $M \subset \mathbb{C}$ is defined by

$$\alpha(M) := \sup |a_1|,$$

where the supremum is taken over all functions f that are continuous on $\mathbb{C}$, satisfy

$$|f(z)| \leq 1, \quad z \in \mathbb{C},$$

and are analytic in the complement of a compact subset of M with expansion

$$f(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

in a neighborhood of infinity. This definition differs from that of the analytic capacity γ in that we require not only boundedness of the functions under consideration but also continuity; further we demand analyticity on all of $\mathbb{C} \setminus M$. From definition, it follows $\alpha(E) \leq \gamma(E)$ for every compact set E .

Let E is a compact set. Obviously, $R(E) \subset A(E)$. Vitushkin [182] showed that $R(E) = A(E)$ if and only if

$$\alpha(D \setminus E) = \alpha(D \setminus \overset{\circ}{E}) \quad (42)$$

for each open disk D . Roughly speaking, this condition means that the complements of E and $\overset{\circ}{E}$ must be equally “thick” in the vicinity of each point, as measured by the AC-capacity α . For example, if the complement $\mathbb{C} \setminus E$ of the compact set E consists of a finite number of components, then the condition (42) is fulfilled and it holds $R(E) = A(E)$. For further results on analytic capacity, AC-capacity and rational approximation we refer to [41, 187].

6. Generalizations of logarithmic capacity

6.1. Weighted capacity

Let now $E \subset \mathbb{C}$ be a closed set and $w : E \rightarrow [0, \infty)$. We call such a function w a *weight function* on E . A weight function w is said to be *admissible* if it satisfies the following three conditions:

- (i) w is upper semicontinuous,
- (ii) $E_0 := \{z \in E : w(z) > 0\}$ has positive outer transfinite diameter,
- (iii) if E is unbounded, then $|z|w(z) \rightarrow 0$ as $|z| \rightarrow \infty$, $z \in E$.

Set

$$Q(z) := -\log w(z).$$

Then $Q : E \rightarrow [-\infty, \infty]$ is lower semicontinuous, $Q(z) < \infty$ on a set of positive outer transfinite diameter and if E is unbounded, then $(Q(z) - \log |z|) \rightarrow \infty$ as $|z| \rightarrow \infty$, $z \in E$.

Let $\mathcal{M}(E)$ be the set of all positive unit measures μ with support $\text{supp}(\mu) \subset E$. The *weighted energy integral* is defined by

$$\begin{aligned} I_w(\mu) &:= \iint \log \frac{1}{|z - \zeta|w(z)w(\zeta)} d\mu(z) d\mu(\zeta) \\ &= \iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta) + 2 \int Q(z) d\mu(z), \quad \mu \in \mathcal{M}(E), \end{aligned}$$

where the last representation is valid whenever both integrals exists and are finite. It follow from property (iii) that the first integral is well defined. The classical case corresponds to choosing E to be compact and $w \equiv 1$ on E .

The problem now becomes that of minimizing the weighted energy $I_w(\mu)$ in the class $\mathcal{M}(E)$. Let $V_w := \inf\{I_w(\mu) : \mu \in \mathcal{M}(E)\}$. Then the following properties hold [37, 134]:

- (a) V_w is finite.

- (b) There exists a unique measure $\mu_w \in \mathcal{M}(E)$ such that $I_w(\mu_w) = V_w$, and its logarithmic energy $I(\mu_w)$ is finite.
- (c) The support S_w of μ_w is compact, is contained in $E_\varepsilon := \{z \in E: w(z) \geq \varepsilon\}$ for some $\varepsilon > 0$ and has positive logarithmic capacity.
- (d) Setting $F_w := V_w - \int Q d\mu_w$ and $u_{\mu_w}(z) := \int \log \frac{1}{|z-\zeta|} d\mu_w(\zeta)$, we have

$$u_{\mu_w}(z) + Q(z) \geq F_w \quad \text{q.e. on } E \quad (43)$$

and

$$u_{\mu_w}(z) + Q(z) \leq F_w \quad \text{for all } z \in S_w. \quad (44)$$

- (e) In particular, it holds

$$u_{\mu_w}(z) + Q(z) = F_w \quad \text{q.e. on } S_w. \quad (45)$$

The function

$$U(z, Q) := F_w - u_{\mu_w}(z), \quad z \in \mathbb{C} \quad (46)$$

can be determined by the extremal point method, see the subsection *Weighted transfinite diameter*.

The measure μ_w is called the *equilibrium measure* associated with w . The constant F_w is called the modified *Robin constant* for w . The associated *weighted capacity* is defined as $c_w := e^{-V_w}$.

Suppose that all weight function $w, w_n : E \rightarrow [0, \infty)$, $n = 1, 2, \dots$, below are admissible. If $w_{n+1} \geq w_n$ ($n = 1, 2, \dots$) and $w = \lim_{n \rightarrow \infty} w_n$, then, for $n = 1, 2, \dots$,

$$c_{w_{n+1}} \geq c_{w_n}, \quad \lim_{n \rightarrow \infty} c_{w_n} = c_w, \quad \lim_{n \rightarrow \infty} \mu_{w_n} = \mu_w$$

in the weak* topology of measures, see (13),

$$\lim_{n \rightarrow \infty} F_{w_n} = F_w$$

and

$$\lim_{n \rightarrow \infty} u_{\mu_{w_n}}(z) = u_{\mu_w}(z)$$

for every $z \in \mathbb{C}$; see [160, Chapter I, Theorems 6.2 and 6.5].

The compactness of the support S_w of μ_w is enforced by the assumption that Q increases sufficiently fast around infinity. Remark that the properties (43) and (45) uniquely characterize the extremal measure μ_w in the sense that if $\mu \in \mathcal{M}(E)$ has compact support and finite logarithmic integral $I(\mu)$ and satisfies

$$\int \log \frac{1}{|z-\zeta|} d\mu(\zeta) + Q(z) = c \quad \text{q.e. on } \text{supp}(\mu) \quad (47)$$

and

$$\int \log \frac{1}{|z - \zeta|} d\mu(\zeta) + Q(z) \geq c \quad \text{q.e. on } E \quad (48)$$

then $\mu = \mu_w$ and $c = c_w$. We also mention that there are many other μ 's satisfying (47) alone.

Let w be admissible. Using the maximum principle for logarithmic potentials, from (44) it follows that the equilibrium potential u_{μ_w} is bounded on compact subsets of $\mathbb{C}$. Furthermore, it is continuous at every $z \notin S_w$, and at every $z \in S_w$ where (45) holds; hence u_{μ_w} is continuous q.e. on $\mathbb{C}$. Besides, $u_{\mu_w} + Q$ (considered as a function on S_w) is continuous at $z \in S_w$ if and only if (45) holds. In particular, $u_{\mu_w} + Q$ is continuous q.e. on S_w (considered as a function on S_w) and, as a consequence, Q is continuous q.e. on S_w (considered as a function on S_w); see [160, Chapter I, Theorem 4.4]. The last statement is actually a surprising fact about the positioning of S_w , for Q may have “many” points of discontinuity on E . Note also that a general logarithmic potential can be discontinuous at “many” points of the support of the generating measure. In fact, if μ is a discrete measure with $\text{supp}(\mu) = [-1, 1]$, then u_μ takes the value $+\infty$ on a dense set, so it is discontinuous at every point where u_μ is finite. Thus, this potential is discontinuous q.e. on $\text{supp}(\mu)$.

Dirichlet problem. In the connected components of the complement of the support S_w of μ_w the equilibrium potential $-u_{\mu_w}$ turns out to be the solution of the *Dirichlet problem* (modulo an additive constant) with boundary function Q . More precisely, let w be admissible and $\mathcal{R}$ be a bounded component of $\mathbb{C} \setminus S_w$. Then

$$F_w - u_{\mu_w}(z) = H_Q^{\mathcal{R}}(z), \quad z \in \mathcal{R}.$$

If $\mathcal{R}$ is the unbounded component of $\mathbb{C} \setminus S_w$, then

$$F_w - u_{\mu_w}(z) = H_Q^{\mathcal{R}}(z) + g_{\mathcal{R}}(z, \infty), \quad z \in \mathcal{R},$$

where $g_{\mathcal{R}}(z, \infty)$ denotes the Green function of $\mathcal{R}$ with pole at ∞ and $H_Q^{\mathcal{R}}$ the Perron–Wiener–Brelot solution of the Dirichlet problem on $\mathcal{R}$ for the boundary function Q , see Section 2.3.

If we wish to solve the Dirichlet problem with the boundary function Q on the components of $\mathbb{C} \setminus E$, where E is a given compact set, then we have a way to do it as follows [160, Chapter V, Theorems 2.1, 2.2 and 2.3]. Let E be a compact set of positive capacity and suppose that Q can be extended to a twice-continuously-differentiable function (as a function of two real variables) to a neighborhood E' of E contained in the disk $D_r = \{z \in \mathbb{C}: |z| \leq r\}$. Set

$$q(z) := \log(R^2 + |z|^2), \quad R > r.$$

Then there exists a $\lambda_0 > 0$ such that the function

$$\frac{1}{\lambda} (U(z, q + \lambda Q) - U(z, q)),$$

where U is defined in (46), is the solution of the Dirichlet problem with the boundary function Q on every component of $\mathbb{C} \setminus E$ for $0 < \lambda \leq \lambda_0$. Now let Q be a lower semicontinuous function on E . Then

$$U_Q(z) := \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} (U(z; q + \lambda Q) - U(z, q))$$

coincides with the solution $H_Q^{\mathcal{R}}$ of the Dirichlet problem with the boundary function Q on every component $\mathcal{R}$ of $\mathbb{C} \setminus E$.

Extremal properties of S_w . The equilibrium measures μ_w and their supports S_w have some features that are missing in the classical theory ($w \equiv 1$). Indeed, almost all measures can appear as equilibrium measures with respect to an appropriate external field $Q = -\log w$. Furthermore, the support S_w of μ_w need not coincide with the outer boundary of E and, in fact, can be quite an arbitrary subset of E , possibly with positive area. More precisely, let $S \subset \mathbb{C}$ be compact with the property that the intersection of S with any neighborhood of any point of S is of positive outer transfinite diameter. Then there is an admissible weight w such that $S_w = S$, see [160, Chapter IV, Theorem 1.1].

The most important property of the support S_w of μ_w is that it maximizes the so-called $\mathcal{F}$ -functional

$$\mathcal{F}(K) := \log \text{cap}(K) - \int_K Q d\mu$$

among all compact sets $K \subset E$ of positive capacity, where μ denotes the (uniquely determined) equilibrium measure associated with the set K . This $\mathcal{F}(K)$ -functional of Mhaskar and Saff [134, 135] is one of the most powerful tools in finding S_w and μ_w .

In several important cases S_w is essentially the only compact set K for which the $\mathcal{F}(K)$ -functional attains its maximum, which allows us to transform the problem of determining S_w to the problem of determining the maximizing set K for $\mathcal{F}(K)$. Of course, the determination of the maximizing set for the $\mathcal{F}$ -functional can still be quite complicated, but it turns out that sometimes we know in advance some properties of S_w that allow us to consider the maximum only for a special class of compact sets K . We list some simple but useful geometric a priori properties of S_w [160, Chapter IV, Theorem 1.10]:

If Q is superharmonic in the interior of E , then $S_w \subset \partial E$.

Symmetries of w such as axial or circular symmetry are inherited by S_w .

If Q is convex on $E = \mathbb{R}$, then S_w is an interval.

In the last case, the maximizing problem for the $\mathcal{F}$ -functional becomes a simple maximum problem in two variables (the endpoints of the unknown interval $S_w = [a, b]$):

$$\mathcal{F}([a, b]) = \max_{\alpha, \beta} \mathcal{F}([\alpha, \beta]),$$

where

$$\mathcal{F}([\alpha, \beta]) = \log \left(\frac{\beta - \alpha}{4} \right) - \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{Q(x) dx}{\sqrt{(x - \alpha)(\beta - x)}}, \quad \alpha < \beta.$$

Then the endpoints a, b satisfy the following two conditions [45,48,49]

$$\frac{1}{\pi} \int_a^b Q'(x) \sqrt{\frac{x-a}{b-x}} dx = 1$$

and

$$\frac{1}{\pi} \int_a^b Q'(x) \sqrt{\frac{b-x}{x-a}} dx = -1,$$

where $Q'(x)$ denotes the right derivative of the convex function $Q: \mathbb{R} \rightarrow \mathbb{R}$. These integral equations can be explicitly solved for special weights. For example, if $w(x) = e^{-|x|^\lambda}$ ($x \in \mathbb{R}, \lambda > 0$), then [133,156] $S_w = [-a, a]$ with

$$a = \left(\frac{\sqrt{\pi} \Gamma(\frac{\gamma}{2})}{2 \Gamma(\frac{\gamma+1}{2})} \right)^{1/\lambda},$$

where $\Gamma(\cdot)$ denotes the gamma function.

Let w be an admissible weight on E , $P_n(z)$ a polynomial of degree $\leq n$. We consider *weighted polynomials* of the form

$$w^n(z) P_n(z)$$

which essentially differ from the usual definition of a weighted polynomial because here the weight varies together with the degree. The support S_w of the extremal measure μ_w can be characterized by the supremum norm behavior of weighted polynomials as follows

$$\|w^n P_n\|_E^* = \|w^n P_n\|_{S_w}^*, \quad \deg P_n \leq n, \quad (49)$$

where $\|f\|_K^*$ denotes the smallest number that is an upper bound for $|f|$ q.e. on K , see [134,135,169]. This means, roughly speaking that the supremum norm of every weighted polynomial “lives” on the subset S_w of E that is independent of n and P_n and the behavior outside of S_w is “small”. Let $S \subset E$ be a closed set. If, for every $n = 1, 2, \dots$ and every polynomial P_n of degree less than or equal to n ,

$$\|w^n P_n\|_E^* = \|w^n P_n\|_S^*, \quad (50)$$

then $S_w \subset S$, see [179]. This, together with (49) shows that S_w is the smallest set with the property (50).

Determination of μ_w . The support S_w of the equilibrium measure μ_w corresponding to an admissible weight $w = e^{-Q}$ is one of the most important quantities in determining of μ_w . In fact, suppose the S_w is a nice set, say it is bounded by a finite number of smooth Jordan curves and w is a continuous function. Then inside S_w the potential u_{μ_w} coincides with $-Q$ plus a constant and in the “holes” of S_w the potential u_{μ_w} is the solution of a

certain Dirichlet problem. Thus, S_w gives a way of determining of μ_w : Now inside S_w the measure μ_w can be obtained by taking $-1/2\pi$ times the Laplacian of u_{μ_w} , i.e., that of $-Q$ (understood in the distributional sense), see [160, Chapter II, Theorem 1.3]. Knowing μ_w inside S_w we can subtract from u_{μ_w} the potential of the part of μ_w lying inside S_w and if the difference is denoted by $u_{\mu_w}^*$, then the part of μ_w that is supported on the boundary of S_w can be obtained by taking $-1/2\pi$ times the sum of the directional partial derivatives of $u_{\mu_w}^*$ along the normal and along its opposite, see [160, Chapter II, Theorem 1.5]. Thus, for a given w everything is computable – at least in principle – once S_w is known. Later we give a simple method by which S_w can be numerically determined, see the subsection *Weighted transfinite diameter*.

Constrained weighted energy problem. For the problem to minimize the weighted energy $I_w(\mu)$ in the class of all measures $\mu \in \mathcal{M}(E)$, what happens if we impose a constraint on the measures μ ? More precisely, suppose that σ is a given positive measure with $\text{supp}(\sigma) = E$ and total mass $\|\sigma\| > 1$. Let $\mathcal{M}^\sigma(E)$ be the set of all positive unit Borel measures μ with $\text{supp}(\mu) \subset E$ satisfying $\mu \leq \sigma$. The *constrained weighted energy problem* concerns the minimization

$$V_w^\sigma := \inf\{I_w(\mu): \mu \in \mathcal{M}^\sigma(E)\}.$$

In the unweighted case ($w \equiv 1$), this problem was first introduced by Rakhmanov [157] who used it to deduce the asymptotic zero distribution of certain “ray sequences” of Chebyshev polynomials of a discrete variable. It was shown by Dragnev and Saff [21] that the constrained unweighted energy problem is equivalent to an unconstrained weighted energy problem. More precisely, suppose that E is compact and the logarithmic potential u_σ generated by σ is continuous on E . Let $\mu^\sigma \in \mathcal{M}^\sigma(E)$ be the extremal measure to the constrained unweighted problem. If μ_w is the solution to the unconstrained weighted energy problem on E with respect to the weight $w = \exp(u_\sigma/(\|\sigma\| - 1))$, then $\mu^\sigma = \sigma - (\|\sigma\| - 1)\mu_w$.

Dragnev and Saff also investigate in [21] the more general constrained energy problem in the presence of an external field and obtain an analogue of results (a)–(e) along with other characterizations. For example, they prove that if $w = e^{-Q}$ is an admissible weight on E and the constraint σ satisfies $\sigma(E_0) > 1$ and has finite energy on compact sets, then there exists a unique measure $\mu = \mu_w^\sigma \in \mathcal{M}^\sigma(E)$ such that $I_w(\mu) = V_w^\sigma$. Furthermore, there exists a constant F_w^σ such that $u_\mu + Q \geq F_w^\sigma$ holds $(\sigma - \mu)$ -almost everywhere and $u_\mu + Q \leq F_w^\sigma$ holds for all $z \in \text{supp}(\mu)$.

Weighted transfinite diameter. Now we give a discrete version of the weighted energy problem. Let $w = e^{-Q}$ be an admissible weight on the closed set $E \subset \mathbb{C}$. The points $z_k = z_{nk}$ ($k = 1, 2, \dots, n$) for which the expression

$$\left\{ \prod_{1 \leq j < k \leq n} |z_j - z_k| w(z_k) w(z_k) \right\}^{1/\binom{n}{2}}$$

attains its maximum value $d_{w,n}$ for all possible choices of $z_j, z_k \in E$ are called *weighted n th Fekete points* associated with the weight w . For fixed n , these points need not be unique. The sequence $\{d_{w,n}\}$ decreases to a limit d_w , called the *weighted transfinite diameter* associated with the weight w ; see [107–122], [160, Chapter III, Theorem 1.1].

The asymptotic distribution of the weighted n th Fekete points z_{nk} ($k = 1, 2, \dots, n$; $n \in \mathbb{N}$) of E is the equilibrium measure μ_w of E in the following sense: Let denote the Dirac measure by δ_z with the unit mass at the point z . Then

$$\mu_{w,n} := \frac{1}{n} \sum_{k=1}^n \delta_{z_{nk}} \xrightarrow{*} \mu_w \quad (51)$$

in the weak* topology, see (13). Hence, $\mu_{w,n}$ could be used as a discrete substitute for the equilibrium measure μ_w , while $\log 1/d_{w,n}$ could be used as a discrete substitute for the minimal weighted energy $V_w = \log 1/c_w$.

In [44] the discrepancy between $\mu_{w,n}$ and μ_w is estimated for special classes of weight functions w .

In the classical case ($w \equiv 1$), it easily follows from the maximum principle for analytic functions that Fekete points lie on the outer boundary of E . In the weighted case the situation is more subtle. Let

$$S_w^* := \{z \in E: u_{\mu_w}(z) + Q(z) \leq F_w\}.$$

Obviously, by (44), $S_w \subset S_w^*$. It turns out that S_w^* is a compact set containing all weighted n th Fekete points for all $n \geq 2$. Define

$$R_w := \{z \in E: u_{\mu_w}(z) + Q(z) < F_w\}.$$

This is a bounded set that consists of countably many compact sets having transfinite diameter zero. Then the closure of this set has to be added to the support S_w of the equilibrium measure μ_w to get the smallest closed set that contains weighted n th Fekete points z_{nk} ($k = 1, 2, \dots, n$) for all $n \geq 2$, see [160, Chapter III, Theorem 2.8].

The polynomials

$$q_{w,n}(z) := \prod_{k=1}^n (z - z_{nk})$$

are called Fekete polynomials associated with the weight w . Immediately, from (51) we get

$$\lim_{n \rightarrow \infty} |q_{w,n}(z)|^{1/n} = e^{-u_{\mu_w}(z)}$$

uniformly on compact subsets of $\mathbb{C} \setminus S_w^*$. Furthermore, we have

$$\lim_{n \rightarrow \infty} \|w^n q_{w,n}\|_E^{1/n} = c_w \exp\left(\int Q d\mu_w\right),$$

where $\|\cdot\|_E$ denotes the supremum norm on E , see [160, Chapter III, Theorems 1.8 and 1.9].

Weighted Chebyshev constant. For an admissible weight w on the closed set E , the numbers

$$\tau_{w,n} := \inf\{\|w^n p_n\|_E : \text{polynomials } p_n = z^n + \dots\}$$

are called the weighted n th Chebyshev numbers corresponding to w . It is easily seen that the infimum is attained for a polynomial $t_{w,n}(z)$ which is called an n th Chebyshev polynomial corresponding to w . In the case when the support S_w of the equilibrium measure μ_w has empty interior and connected complement, the asymptotic distribution of the zeros of the n th Chebyshev polynomials is μ_w in the sense of (13). This result is no longer true if S_w has nonempty interior or disconnected complement, see [160, Chapter III, Theorem 3.6, Examples 3.7 and 3.8].

The n th root of $\tau_{w,n}$ tends to a limit τ_w , called the *weighted Chebyshev constant* associated with $w = e^{-Q}$. In the classical setting, i.e., when E is compact and $w \equiv 1$, the three quantities logarithmic capacity, transfinite diameter and Chebyshev constant all coincide. In the weighted case we have the analogous formula

$$c_w = d_w = \tau_w \exp\left(-\int Q d\mu_w\right),$$

which reduces to the classical one if $Q \equiv 0$, see [160, Chapter III, Theorem 3.1].

The exact determination of weighted Fekete points is equally hard. We give a very simple method for numerically determining the support S_w of the equilibrium measure μ_w and μ_w itself. The method originated from Leja [114] (unweighted case) and Górski [50] (three-dimensional case). Let $z_0 \in E$ be an arbitrary point such that $w(z_0) \neq 0$ and define a sequence z_n , $n = 1, 2, \dots$, that is adaptively generated from earlier points according to the law: z_n is a point where the weighted polynomial expression

$$|(z - z_0)(z - z_1) \cdots (z - z_{n-1})w(z)|^n$$

takes its maximum on E . These so-called *weighted Leja points* of E are again asymptotically distributed like the equilibrium measure μ_w of E , so one can use them in place of weighted Fekete points. Weighted Leja points are used as points of interpolation despite the fact that a little is known concerning the norm of such interpolation sequences for compact sets on $\mathbb{C}$; see [123]. However, numerical evidence and the philosophy behind their generating idea suggest that they provide good choices for nodes.

6.2. Hyperbolic capacity

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the hyperbolic plane with the *hyperbolic metric*

$$\{z, \zeta\}_h := \min_{\gamma} \int_{\gamma} \frac{|dz|}{1 - |z|^2} \quad \text{for } z, \zeta \in \mathbb{D},$$

where the minimum is taken over all rectifiable curves γ in $\mathbb{D}$ from z to ζ . It is attained for the hyperbolic segment from z to ζ , that is the arc of the circle through z and ζ orthogonal to $\partial\mathbb{D}$. The *pseudohyperbolic metric* is defined by

$$[z, \zeta]_h := \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|.$$

A simple calculation shows that the pseudohyperbolic metric has the geometric interpretation $[z, \zeta]_h = \tanh\{[z, \zeta]_h\}$. Clearly, this metric is Moebius-invariant; i.e., if φ maps $\mathbb{D}$ conformally onto itself, then $[\varphi(z), \varphi(\zeta)]_h = [z, \zeta]_h$.

Let E be a compact set of $\mathbb{D}$. Define

$$V_h := \inf \iint \log \frac{1}{[z, \zeta]_h} d\mu(z) d\mu(\zeta), \quad (52)$$

where the infimum is taken over all positive unit Borel measures on E . Tsuji [181, p. 94] defined the *hyperbolic capacity* of E as $\text{cap}_h(E) = e^{-V_h}$ if $V_h < \infty$ and $\text{cap}_h(E) = 0$ if $V_h = \infty$. The construction shows that $0 \leq \text{cap}_h(E) < 1$. It is also clear that $\text{cap}_h(E_1) \leq \text{cap}_h(E_2)$ if $E_1 \subset E_2$.

Tsuji [181, p. 95] showed that if $\text{cap}_h(E) > 0$ then there exists a unique minimizing measure μ on E , called the *hyperbolic equilibrium measure*. Its *hyperbolic conductor potential*

$$u_\mu(z) = \int \log \frac{1}{[z, \zeta]_h} d\mu(\zeta)$$

is a harmonic function in $\mathbb{D} \setminus E$ satisfying $u_\mu(z) = 0$ on $\partial\mathbb{D}$ and $u_\mu(z) \leq V_h$ on $\mathbb{D}$, where $u_\mu(z) = V_h$ q.e. on E .

If $E \subset \mathbb{D}$ is a continuum, then the ring domain Ω between E and the unit circle $\partial\mathbb{D}$ can be mapped conformally onto an annulus $\rho < |w| < 1$. Tsuji [180] showed

$$\text{cap}_h(E) = \rho.$$

The hyperbolic capacity can be expressed in terms of *extremal length* as follows. Let $E \subset \mathbb{D}$ be a closed set whose outer boundary C consists of finitely many analytic Jordan curves. Let Ω be the domain “between” C and the unit disk $\partial\mathbb{D}$. Then [26] we have

$$\text{cap}_h(E) = \exp\{-2\pi\lambda_\Omega(C, \partial\mathbb{D})\}, \quad (53)$$

where $\lambda_\Omega(C, \partial\mathbb{D})$ is the extremal distance between C and $\partial\mathbb{D}$ with respect to Ω , see (16). From this one conclude $\text{cap}_h(E) = \text{cap}_h(C)$. The calculation in the proof of (53) actually gives the following connection between hyperbolic capacity and the flux of the harmonic measure $u = u_\mu/V_h$ of C with respect to Ω across the unit circle

$$\frac{1}{\log \text{cap}_h(E)} = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\partial u}{\partial n} ds,$$

where n denotes the outer normal on $\partial\mathbb{D}$.

We note that the hyperbolic capacity is conformally invariant. More precisely, let $\mathcal{F}$ be the family of conformal mappings f of Ω onto some other domain $\tilde{\Omega} = \mathbb{D} \setminus \tilde{E}$ bordering on the unit circle, such that $f(\partial\mathbb{D}) = \partial\mathbb{D}$. Then $\text{cap}_h(\tilde{E}) = \text{cap}_h(E)$ for all $f \in \mathcal{F}$. This follows at once from the conformal invariance of extremal length.

Tsuji [180] defined the *hyperbolic transfinite diameter* and *hyperbolic Chebyshev constant* of a closed set $E \subset \mathbb{D}$ as

$$d_h(E) := \lim_{n \rightarrow \infty} \max_{z_1, \dots, z_n \in E} \left\{ \prod_{1 \leq j < k \leq n} \left| \frac{z_j - z_k}{1 - \bar{z}_k z_j} \right| \right\}^{1/\binom{n}{2}} \quad (54)$$

and

$$\tau_h(E) := \lim_{n \rightarrow \infty} \min_{p_n} \max_{z \in E} |p_n(z)|^{1/n}, \quad (55)$$

respectively, where the minimum is taken over all functions

$$p_n(z) = \prod_{k=1}^n e^{i\alpha_k} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right), \quad z, z_k \in \mathbb{C}, \alpha_k \in \mathbb{R}, k = 1, 2, \dots, n.$$

He showed that, for any closed set $E \subset \mathbb{D}$,

$$\text{cap}_h(E) = d_h(E) = \tau_h(E).$$

For example, the hyperbolic capacity of the disk $E = \{z \in \mathbb{C}: |z| \leq r\} \subset \mathbb{D}$ is found to be $\text{cap}_h(E) = r$, and the closed interval $[0, r] \subset \mathbb{D}$ has the hyperbolic capacity

$$\text{cap}_h([0, r]) = \exp \left\{ -\frac{\pi}{2} \frac{K(\sqrt{1-r^2})}{K(r)} \right\},$$

where

$$K(r) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},$$

see [106, p. 62].

By means of extremal length method Grötzsch proved the following extremal property of the interval $[0, r] \subset \mathbb{D}$. For every continuum $E \subset \mathbb{D}$ containing the points $z = 0$ and $z = r$

$$\text{cap}_h([0, r]) \leq \text{cap}_h(E)$$

where equality holds if and only if $E = [0, r]$. This also follows from the discrete version (54) of hyperbolic capacity in connection with the inequality $[z, \zeta]_h \geq [|z|, |\zeta|]_h$, see [181, p. 97].

The points z_{nk} ($k = 1, 2, \dots, n$) for which the product in (54) attains its maximum for all possible choices of $z_j, z_k \in E$ are called *n th Tsuji points* of E .

Pommerenke [148] obtained the following results. Let $E \subset \mathbb{D}$ be a closed set, and number the $(n+1)$ st Tsuji points $z_0, z_1, \dots, z_n$ on E so that

$$A_n := \prod_{m=1}^n [z_0, z_m]_h = \min_k \prod_{m \neq k} [z_k, z_m]_h,$$

and set

$$p_n(z) = \prod_{m=1}^n \left(\frac{1 - \bar{z}_m}{1 - z_m} \right) \left(\frac{z - z_m}{1 - \bar{z}_m z} \right),$$

then

$$\max_{z \in E} |p_n(z)| = A_n, \quad \lim_{n \rightarrow \infty} A_n^{1/n} = \text{cap}_h(E).$$

This last is an analogue of a result due to Leja [108] for the logarithmic capacity of E . Further let Ω be the component of $\mathbb{D} \setminus E$ which borders on the unit circle $\partial\mathbb{D}$, and let $\rho = \text{cap}_h(E) > 0$. Then $g(z) := \lim_{n \rightarrow \infty} p_n(z)^{1/n}$ exists locally uniformly in $H := \Omega \cup \{z \in \mathbb{C}: 1 \leq |z| < r\}$ for some $r > 1$, and $g(z)$ is the “smallest” function satisfying:

- (i) $g(z)$ is locally analytic and with a single-valued modulus in H ,
- (ii) $|g(z)| = 1$ for $|z| = 1$, and
- (iii) $\rho \leq |g(z)| \leq 1$ for $z \in \Omega$.

Moreover, $g(1) = 1$, $\int_0^{2\pi} d \arg g(e^{i\theta}) = 2\pi$, and if ζ is a boundary point of Ω that lies on a continuum in E , then

$$\lim_{z \rightarrow \zeta, z \in \Omega} |g(z)| = \rho.$$

Menke [131] proved the following results on the asymptotic distribution of the n th Tsuji points z_{nk} ($k = 1, 2, \dots, n$) of an analytic Jordan curve lying in the unit disk $\mathbb{D}$. With the above notions let $g(z_{nk}) =: \rho e^{i t_{nk}}$, $k = 1, 2, \dots, n$; $0 \leq t_{n1} < \dots < t_{nn} < 2\pi$. Then there exists a constant L independent of n , such that

$$\left| t_{nk} - t_{nn} - 2\pi \frac{k-1}{n} \right| \leq L \frac{(\log n)^{3/2}}{n}, \quad k = 1, \dots, n.$$

For R close to 1, the ring domain $R < |z| < 1/R$ determines the Laurent expansion

$$\log \frac{g(z)}{z} = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k z^k - \bar{\alpha}_k z^{-k}),$$

where $\operatorname{Re} \alpha_0 = 0$. Note that g maps the unit circle onto itself. Then [132] there exists a constant L independent of n , such that

$$\left| \bar{\alpha}_l l - \frac{1}{n} \sum_{k=1}^n z_{nk}^l \right| \leq L l \frac{\log n}{n}, \quad l = 1, 2, \dots$$

6.3. Elliptic capacity

Let $\widehat{\mathbb{C}}$ be the Riemann sphere of diameter one, which touches the z -plane $\mathbb{C}$ at $z = 0$. Identifying pairs of points lying diametrical symmetric on $\widehat{\mathbb{C}}$, the z -plane $\mathbb{C}$ (identified with the Riemann sphere under stereographic projection) can be considered as a model for the elliptic plane with the elliptic metric

$$\{z, \zeta\}_e := \min_{\gamma} \int_{\gamma} \frac{|dz|}{1 + |z|^2} \quad \text{for } z, \zeta \in \mathbb{C},$$

where the minimum is taken over all rectifiable curves γ on $\mathbb{C}$ from z to ζ . The quantity $\{z, \zeta\}_e$ is the length less than or equal to $\pi/2$ of the geodesic on the sphere joining the projections of z and ζ on $\widehat{\mathbb{C}}$.

Kühnau [95,97] considered the pseudoelliptic metric¹

$$[z, \zeta]_e := \left| \frac{z - \zeta}{1 + \bar{\zeta}z} \right|,$$

where $[z, \infty]_e = 1/|z|$, $[\infty, \zeta]_e = 1/|\zeta|$. A simple calculation shows that the pseudoelliptic metric has the geometric interpretation $[z, \zeta]_e = \tan\{z, \zeta\}_e$. The points z and $z^* = -1/\bar{z}$ are called antipodal points because they project to diametrically opposite points on $\widehat{\mathbb{C}}$. Observe that $[z, \zeta]_e = \infty$ if and only if $\zeta = z^*$. A straightforward calculation shows that $[z^*, \zeta]_e = 1/[z, \zeta]_e$, such that $[z^*, \zeta^*]_e = [z, \zeta]_e$. Clearly, the pseudoelliptic metric is invariant under rotations of the Riemann sphere, i.e., if

$$\varphi(z) = \frac{az - b}{\bar{b}z - \bar{a}}, \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 > 0, \quad (56)$$

¹Tsuji [180; 181, p. 87] used the chordal metric

$$[z, \zeta]_c := \frac{|z - \zeta|}{\sqrt{(1 + |z|^2)(1 + |\zeta|^2)}}$$

to define elliptic transfinite diameter, elliptic Chebyshev constant and elliptic capacity of a closed set E . He showed that these three quantities coincide. Since the chordal metric is the length of the chord joining the stereographic projections of the two point z and ζ , it has the equivalent expression $[z, \zeta]_c = \sin\{z, \zeta\}_e$. However, the chordal metric turns out to have an unfortunate choice, since it does not lead to further development of the theory as in the Euclidean and hyperbolic settings. Moreover, the analogy to the hyperbolic case gets lost to a great extent.

then $[\varphi(z), \varphi(\zeta)]_e = [z, \zeta]_e$. Although the pseudohyperbolic metric is a true metric, the pseudoelliptic metric is not, since the triangle inequality may fail. Given a set $E \subset \mathbb{C}$, we define the antipodal set $E^* := \{z^*: z \in E\}$. Following Kühnau [95], a set E is said *elliptically schlicht* if $E \cap E^* = \emptyset$. At the other extreme, a set E is said to be *diametrically symmetric* if $E^* = E$. A mapping f is diametrically symmetric if it is defined on a diametrically symmetric set E and it has the property $f(z^*) = f(z)^*$, $z \in E$. The transformations in (56) are examples of diametrically symmetric mappings.

Kühnau [95] defined the *elliptic capacity* $\text{cap}_e(E)$, the *elliptic transfinite diameter* $d_e(E)$ and the *elliptic Chebyshev constant* $\tau_e(E)$ of a closed elliptic schlicht set E as in the hyperbolic case replacing the term

$$\frac{z - \zeta}{1 - \bar{\zeta}z} \quad \text{by} \quad \frac{z - \zeta}{1 + \bar{\zeta}z}$$

in (52), (54) and (55). By this analogy many results from the hyperbolic case can be transferred to the elliptic case almost word-for-word. In particular, for every elliptically schlicht set E it holds

$$\text{cap}_e(E) = d_e(E) = \tau_e(E).$$

It is clear from definition that $\text{cap}_e(E^*) = \text{cap}_e(E)$, and that $\text{cap}_e(E_1) \leq \text{cap}_e(E_2)$ if $E_1 \subset E_2$. Furthermore, elliptic capacity is preserved under Möbius transformations of the form (56), since the pseudoelliptic metric is itself invariant under such mappings.

For example, the elliptic capacity of the disk $E = \{z: |z| \leq r < 1\}$ is found to be $\text{cap}_e(E) = r$. More generally, if E is any closed connected elliptically schlicht set and the ring domain Ω “between” E and E^* is mapped conformally onto an annulus $r < |w| < 1/r$, then Kühnau [95] showed $\text{cap}_e(E) = r$. In particular, $\text{cap}_e(E) < 1$ for every elliptically schlicht continuum E .

A closed elliptically schlicht set E is said to be *elliptically separated* if there is a diametrically symmetric Jordan curve on the sphere that separates E from E^* . We can then speak of the domain Ω between E and E^* , meaning the largest diametrically symmetric domain in the complement of $E \cup E^*$ that contains this separating curve. In this context, the set $E \cap \partial\Omega$ will be called the outer boundary of E . If E has finitely many components and is separated from E^* by some Jordan curve, it is always possible to choose a separating curve that is diametrically symmetric. This is a consequence of the theorem [97, Satz 4.1] that a finitely-connected diametrically-symmetric domain always admits a diametrically symmetric conformal mapping onto the extended complex plane with circular slits centered at the origin. For an explicit example of an elliptically schlicht set that is not elliptically separated, consider the two circles $E_1 = \{z: |z| = 1/4\}$ and $E_2 = \{z: |z| = 2\}$. Then $E = E_1 \cup E_2$ is elliptically schlicht but is not elliptically separated, since the circle $|z| = 1/2$ lies between E_1 and E_2 .

If a closed set E is elliptically separated, then $\text{cap}_e(E) < 1$. Indeed, such a set E is contained in some closed connected set $\tilde{E}$, for which $\text{cap}_e(\tilde{E}) < 1$, as mentioned above.

The elliptic capacity can be related with extremal length of a family of curves as follows. Let E be a closed elliptically separated set whose outer boundary C consists of a finite number of analytic Jordan curves. Let Ω be the domain between E and E^* . Then [25]

$$\text{cap}_e(E) = \exp\{-\pi\lambda_\Omega(C, C^*)\}, \quad (57)$$

where $\lambda_\Omega(C, C^*)$ denotes the extremal distance between C and C^* with respect to Ω , see (16). In particular, $\text{cap}_e(E) = \text{cap}_e(C)$.

As a consequence of (57) one can show that elliptic capacity remains invariant under diametrically-symmetric conformal mapping. More precisely, if f is a diametrically-symmetric conformal mapping of Ω onto a region $f(\Omega)$, then $\text{cap}_e(f(C)) = \text{cap}_e(C)$ by the conformal invariance of extremal length and the fact that diametrically symmetric mappings will preserve the family of curves connecting antipodal boundary components. This vastly generalizes the rotation invariance noted earlier.

It is interesting to compare the elliptic and hyperbolic capacities of sets $E \subset \mathbb{D}$, which are always elliptically schlicht. Let E be a closed-connected subset of the unit disk $\mathbb{D}$ with outer boundary C . Then [25] it holds

$$\text{cap}_e(E) \leq \text{cap}_h(E) \quad (58)$$

with equality if and only if $C = -C$. Note that equality in (58) also holds for arbitrary closed sets $E \subset \mathbb{D}$ with $E = -E$, given that the domain between E and the unit circle is a finitely-connected Jordan domain with analytic boundary; see [25]. This is a consequence of the extremal length formulation of the capacities.

Finally, we remark that the elliptic energy of a positive charge distribution μ on an elliptic schlicht compact set E can be represented as logarithmic energy of the signed measure ν as follows

$$\iint_E \log \frac{1}{[z, \zeta]_e} d\mu(z) d\mu(\zeta) = \frac{1}{2} \iint_{E \cup E^*} \log \frac{1}{|z - \zeta|} d\nu(z) d\nu(\zeta), \quad (59)$$

where $\nu = \mu$ on E and $\nu(e) := -\mu(e^*)$ ($e \in E^*$). This identity remains valid for the hyperbolic case replacing E^* by the hyperbolic reflection of a closed set $E \subset \mathbb{D}$, and $[z, \zeta]_e$ by $[z, \zeta]_h$. Therefore, as in the Euclidean case the elliptic and hyperbolic transfinite diameter arise from a discrete version of a least-energy description of the logarithmic capacity of a condenser. Furthermore, if $E \cup E^*$ is compact, from (57) and (11) it follows that the kernels $\log 1/[z, \zeta]_e$ and $\log 1/[z, \zeta]_h$ satisfy an energy principle.

6.4. Green capacity

In the following let $G \subset \mathbb{C} \cup \{\infty\}$ be a domain such that either G is bounded or $\infty \in G$ and $\text{cap}(\partial G) > 0$. The Green function $g_G(z, \zeta)$ of G with pole at $\zeta \in G$ is defined as the (unique) function on G satisfying the following properties:

- (i) $g_G(z, \zeta) > 0$ and harmonic in $G \setminus \{\zeta\}$ and bounded as z stays away from ζ ,

(ii) $g_G(z, \zeta) + \log |z - \zeta|$ is bounded in a neighborhood of ζ ,

(iii) $\lim_{z \rightarrow \zeta} g_G(z, \zeta) = 0$ for q.e. $\zeta \in \partial G$.

It is well known that the Green function is symmetric $g_G(z, \zeta) = g_G(\zeta, z)$, $z \neq \zeta$.

Let E be a compact subset of G . Define

$$V_E^G := \inf \iint g_G(z, \zeta) d\mu(z) d\mu(\zeta), \quad (60)$$

where the infimum is taken over all positive unit Borel measures μ on E .

The *Green capacity* of E with respect to G is defined as $\text{cap}_G(E) = e^{-V_E^G}$ if $V_E^G < \infty$ and $\text{cap}_G(E) = 0$ if $V_E^G = \infty$. The last is true if and only if the logarithmic capacity $\text{cap}(E) = 0$. Since $g_G(z, \zeta) > 0$ for $z \neq \zeta$, it follows $V_E^G > 0$ and $\text{cap}_G(E) < 1$. The quantity $1/V_E^G$ is often referred to as the *condenser capacity* $C(E, F)$ of the condenser (E, F) with the plates E and $F := \mathbb{C} \setminus G$.

If $\text{cap}_G(E) > 0$ then there exists a unique minimizing measure μ_E^G on E , called the *Green equilibrium measure*, whose *Green conductor potential*

$$u_{\mu_E^G}(z) = \int g_G(z, \zeta) d\mu(\zeta)$$

has the property $u_{\mu_E^G}(z) \leq V_E^G$ on G and $u_{\mu_E^G}(z) = V_E^G$ q.e. on E .

As in the case of the logarithmic kernel, the Green kernel $g_G(z, \zeta)$ satisfies an energy principle, see (11).

For example, if G is the unit disk $\mathbb{D}$ then

$$g_G(z, \zeta) = \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right|,$$

and, therefore the Green capacity is equivalent to the hyperbolic capacity.

The treatment of Green potentials and Green energy is classical and in this general form it is based primarily on the work of Frostman [37]; see also Helms [70, Chapter 11].

A discrete version of the energy minimum problem (60) leads again to extremal points systems like Fekete points. For discrepancy estimates for Green equilibrium measure and quantitative distribution assertions for such extremal points systems we refer to [43].

The Green equilibrium measure μ_E^G is related to the following minimum logarithmic energy problem for signed measures. Let $E \subset G$ be a compact set such that $\text{cap}(E) > 0$. Set

$$V_{E,F} := \inf \iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta). \quad (61)$$

Here the infimum is taken over all *signed measures* $\mu = \mu_E - \mu_F$, where μ_E and μ_F are positive unit Borel measures on E and $F = \mathbb{C} \setminus G$, respectively. Then [160, p. 393] there exists a unique such signed measure $\mu^* = \mu_E^* - \mu_F^*$ for which $V_{E,F}$ is attained.

Furthermore, the logarithmic potential $u_{\mu^*}(z)$ is constant q.e. on E and constant q.e. on F . It turns out that

$$\mu_E^G = \mu_E^* \quad \text{and} \quad V_E^G = V_{E,F}.$$

Another expression for the condenser capacity $C(E, F) = 1/V_{E,F}$ is

$$C(E, F) = \inf \frac{1}{2\pi} \iint_{\Omega} |\nabla u|^2 dx dy, \quad (62)$$

where the infimum is taken for all continuously differentiable functions in $\Omega := \mathbb{C} \setminus (E \cup F)$ that have boundary values 1 at E and 0 at F . Apparently it was Bagby [5] who first noticed the connection of the quantities in (60)–(62). One advantage of this connection is that logarithms of Euclidean distances appear in the integrand of (61), making it possible to relate the Green capacity of E with respect to G to the logarithmic capacity of E and F and geometric evident domain quantities, see [5].

The discrete version of (61) leads to the Green transfinite diameter of the compact set $E \subset G$ with respect to G . Bagby [5] defined this quantity as

$$d_G(E) = \lim_{n \rightarrow \infty} \max \left\{ \prod_{1 \leq j < k \leq n} \frac{|z_j - z_k| |\zeta_j - \zeta_k|}{|z_j - \zeta_k| |\zeta_j - z_k|} \right\}^{1/\binom{n}{2}}, \quad (63)$$

where the maximum is taken over all $z_1, \dots, z_n \in E$ and $\zeta_1, \dots, \zeta_n \in F = \mathbb{C} \setminus G$. Several estimates for the speed of convergence in (63) can be found in [91]. The points z_{nk}, ζ_{nk} ($k = 1, 2, \dots, n$) for which the maximum in (63) is attained are called *Bagby points*. These points z_{nk} and ζ_{nk} ($k = 1, 2, \dots, n; n \in \mathbb{N}$) are asymptotically distributed according to μ_E^* and μ_F^* , respectively. They can be used for good points of interpolation and good poles for rational interpolants to analytic functions, see [6].

Let $r(z)$ denote any rational function of the form

$$r(z) = \frac{z^n + \dots}{z^n + \dots}.$$

Following Chebyshev's concept, Gonchar [47] defined the Green–Chebyshev constant of the compact set E with respect to G as $\tau_G(E) = \lim_{n \rightarrow \infty} \tau_{G,n}(E)$, where

$$\tau_{G,n}(E) = \inf_{\deg(r) \leq n} \left\{ \frac{\max\{|r(z)| : z \in E\}}{\min\{|r(z)| : z \in F\}} \right\}^{1/n}. \quad (64)$$

Apparently it was Zolotarjov [188] who first investigated the minimum problem (64) for the special case $E = [-1, 1]$ and $F = \{x \in \mathbb{R} : |x| \geq 1/k\}$ where $0 < k < 1$, and determined the exact value of $\tau_{G,n}(E)$ for all n expressed in terms of the complete elliptic integral of first type of modulus k .

In [5,47] it was shown that, for every compact set $E \subset G$,

$$\text{cap}_G(E) = d_G(E) = \tau_G(E).$$

Let now Ω be a doubly-connected domain such that each (compact) component of its complement contains at least two points. It is well known that Ω can be conformally mapped onto an annulus $r_1 < |w| < r_2$. The ration r_2/r_1 is a conformal invariant and it is called the modulus of ring domain Ω . We get the following numerical procedure for finding a conformal mapping of Ω onto an annulus. This procedure is numerically stable and can be used regardless of smoothness assumptions on the boundary of the sets involved, but for the same reason, we cannot expect it to converge very rapidly even if the boundary is smooth: Starting from any point $z_0 \in E$ and $\zeta_0 \in F$, we successively define the points $z_n \in E$ and $\zeta_n \in F$ as points where the expression

$$\left\{ \prod_{j=0}^{n-1} \frac{|z - z_j| |\zeta - \zeta_j|}{|z - \zeta_j| |\zeta - z_j|} \right\}^{1/n}$$

takes its maximum A_n for $z \in \partial E$ and $\zeta \in \partial F$. Then with that branch of the n th root that is positive of positive values, the expression

$$\left(\prod_{j=0}^{n-1} \frac{z - z_j}{z - \zeta_j} \right)^{1/n} \quad (65)$$

converges to a conformal mapping φ of Ω onto an annulus $r_1 < |w| < r_2$ as $n \rightarrow \infty$, and it holds

$$\lim_{n \rightarrow \infty} A_n = \frac{r_1}{r_2}.$$

In [91] Kloeke proved estimates for the speed of the convergence of (65), which show that for $z \in \Omega$ satisfying $r_1 + 1/\sqrt{n} < \varphi(z) < r_2 - 1/\sqrt{n}$ the “rate” is not worse than $O((\log n)/\sqrt{n})$.

For further generalizations of the energy problem for signed measures $\mu = \mu^+ - \mu^-$, in particular its extension to the presence of an external field we refer to [160, Chapter VIII]. We note that all these generalizations are based on the assumption that the positive measures $\mu^\pm$ are supported on compact sets having positive distance. Later we shall discuss generalizations without this assumption, see Section 6.6.

6.5. Robin capacity

Suppose that Ω is an unbounded Jordan domain whose boundary consists of a finite number of analytic Jordan curves. Let the boundary be partitioned into a pair of disjoint sets A and B , each consisting of a finite number of subarcs of boundary curves. With respect to such a partition, the *Robin function* $R(z, \zeta)$ of Ω with pole at $\zeta \in \Omega$ is defined by the following properties:

- (i) $R(z, \zeta)$ is harmonic in Ω except at ζ , where $R(z, \zeta) + \log |z - \zeta|$ is harmonic. For $\zeta = \infty$ the definition is modified to require that $R(z, \infty) - \log |z|$ be harmonic at infinity.

- (ii) $R(z, \zeta)$ is continuous together with its first partial derivatives up to the boundary of Ω .
- (iii) The boundary function satisfies $R(z, \zeta) = 0$ for $z \in A$ and its inner normal derivative satisfies $\frac{\partial}{\partial n} R(z, \zeta) = 0$ for $z \in B$.

The Robin function may be viewed as a generalization of the Green function, to which it reduces when the set B is empty. It can be used to solve the mixed boundary-value problem for harmonic functions. This is the problem of finding the function $u(z)$ harmonic in Ω and of class $C^1(\overline{\Omega})$, with prescribed values on A and prescribed normal derivative on B . Note that $R(z, \infty)$ is invariant under precomposition with an arbitrary conformal mapping of Ω of the form

$$f(z) = z + a_0 + a_1 z^{-1} + \dots$$

near infinity. Such mappings will be called *admissible*. The conformal invariance allows a definition of the Robin function even for domains with rough boundary, where no normal direction is defined. Successive applications of the Riemann mapping theorem show that an arbitrary finitely-connected domain with nondegenerated boundary components can be mapped conformally onto a Jordan domain with analytic boundary.

The *Robin capacity* of A with respect to Ω is now defined as $\delta(A) = e^{-\rho(A)}$, where

$$\rho(A) = \lim_{z \rightarrow \infty} \{R(z, \infty) - \log |z|\}$$

is the *Robin constant* of A with respect to Ω .

For example, let $\Omega = \{z \in \mathbb{C}: |z| > 1\}$ and A be an arc on $|z| = 1$ which subtends an angle α at $z = 0$. Then it can be shown by conformal mapping that the transfinite diameter $d(A) = \sin(\alpha/4)$, while $\delta(A) = \sin^2(\alpha/4)$. Thus $\delta(B) = \cos^2(\alpha/4)$, so that $\delta(A) + \delta(B) = 1$. On the other hand $d(A \cup B) = 1$. On the basis of this example, a general principle emerges. Let Ω be the exterior of an arbitrary Jordan curve C partitioned into a pair of disjoint arcs A and B . Then $\delta(A) + \delta(B) = d(C)$. This is a simple consequence of the conformal invariance of all quantities under admissible mappings and the above-mentioned example. The relation breaks down when the subset A and B are not single arcs of C , but it is still true that $\delta(A) + \delta(B) \leq 2d(C)$. The constant 2 is best possible; see [24] for details.

Although the logarithmic capacity of the full boundary of Ω is invariant under admissible conformal mappings, the logarithmic capacity $d(A)$ of any fixed closed subset $A \subset \partial\Omega$ may well be distorted. Under the assumption that A is the union of finitely many arcs on $\partial\Omega$, Duren [28] proved

$$\delta(A) = \min d(f(A)),$$

where the minimum is taken over all admissible conformal mappings f . This explains and generalizes the inequality (28). In particular, $\delta(A) \leq d(A)$.

Note, that in the case A coincides with any full boundary component of Ω the Robin capacity of A with respect to Ω is related with the unique admissible univalent mapping f of Ω onto the exterior of some circle $|w| = R$ slit along segments on radii emanating

from the origin which correspond to the boundary components on $B = \partial\Omega \setminus A$. Since $R(z, \infty) = \log(f(z)/R)$ on Ω , from definition it follows $\delta(A) = R$.

Robin capacity can be expressed in terms of reduced extremal distance in a similar way as logarithmic capacity. Again surround $\partial\Omega$ by a sufficiently large circle C_r , and let $\Omega_r = \Omega \cap \{z \in \mathbb{C}: |z| < r\}$. Then the limit $m(A) := \lim_{r \rightarrow \infty} \{2\pi\lambda_{\Omega_r}(A, C_r) - \log r\}$ exists, where $\lambda_{\Omega_r}(A, C_r)$ denotes the extremal distance from A to C_r , see (16), and the Robin capacity of A with respect to Ω is given by

$$\delta(A) = e^{-m(A)}.$$

On the basis of the extremal-length definition of Robin capacity, it is easy to establish some comparison theorems which are not so accessible from the potential-theoretic description. For instance, if $A \subset \tilde{A} \subset \partial\Omega$, then $\delta(A) \leq \delta(\tilde{A})$. It is also of interest to ask how the Robin capacity of a boundary set is affected by change of domain. Here it is convenient to adopt the more specific notation $\delta_\Omega(A)$ for the Robin capacity of a boundary set A with respect to the domain Ω . Let Ω and $\tilde{\Omega}$ be smooth bounded Jordan domains with $\Omega \subset \tilde{\Omega}$. Then [24]

(i) If $A \subset (\partial\Omega) \cap (\partial\tilde{\Omega})$, then $\delta_\Omega(A) \leq \delta_{\tilde{\Omega}}(A)$.

(ii) If $B \subset (\partial\Omega) \cap (\partial\tilde{\Omega})$ and $A = (\partial\Omega) \setminus B$, $\tilde{A} = (\partial\tilde{\Omega}) \setminus B$, then $\delta_{\tilde{\Omega}}(\tilde{A}) \leq \delta_\Omega(A)$.

We now turn to the following least-energy description of Robin capacity. Let Ω be a finitely-connected domain containing the point at infinity, bounded by analytic Jordan curves. Let $A = C_1 \cup \dots \cup C_k$ and $B = C_{k+1} \cup \dots \cup C_n$, where $1 \leq k \leq n$. Let $N(z, \zeta)$ be the *Neumann function* of the larger domain G bounded by B . This is the function harmonic in G except at ζ and infinity, with the properties:

- (i) $N(z, \zeta) + \log|z - \zeta|$ is harmonic near ζ ,
- (ii) $N(z, \zeta) + \log|z|$ is harmonic near infinity,
- (iii) $\lim_{z \rightarrow \infty} (N(z, \zeta) + \log|z|) = 0$,
- (iv) $\frac{\partial}{\partial n} N(z, \zeta) = 0$ for $z \in B$.

(See for instance Henrici [71, p. 270]. The terminology is not standard.) For example, it can be verified that Neumann's function for the domain outside the unit disk has the form

$$N(z, \zeta) = \log \frac{|z\zeta|}{|z - \zeta||1 - \bar{\zeta}z|}.$$

The Neumann function can be viewed as the potential at z of a point-charge at ζ in the presence of perfectly insulating "islands" with boundary B . An elementary calculation shows that the Neumann function is symmetric: $N(z, \zeta) = N(\zeta, z)$.

Let E be a compact set with boundary A , and define

$$V_N := \inf \iint N(z, \zeta) d\mu(z) d\mu(\zeta),$$

where the infimum is taken over all positive unit Borel measures μ on E . Then [27]

$$\delta(A) = e^{-V_N}.$$

As we have seen Robin capacity arises as the minimum of logarithmic capacity of the image of a specified subset A of the boundary under normalized conformal mappings of the domain. Thurman [176–178] has investigated the associated maximum problem in some detail, giving a formula for the maximum capacity in terms of standard conformal invariants such as harmonic measure and the Riemann matrix of periods of their harmonic conjugates. A new phenomenon arises for the maximum problem: if the subset A contains even a single point of a given boundary component, it may as well contain the entire boundary component. Thurman [177] has given an extremal-length description of maximum capacity in terms of a notion of “bridged extremal distance”.

Betsakos [10] has studied the effect on Robin capacity of a certain type of symmetrization known as polarization. Here the difficulty is to devise methods of symmetrization which will respect the given partition of the boundary.

6.6. Capacity and conformal maps of multiply-connected domains

In the following we restrict us to the case of the conformal parallel slit mapping. Relating to other canonical conformal mappings as radial, circular and parabola slit mappings we refer to [102,76]. Let Ω be a finitely-connected domain containing the point at infinity, bounded by analytic Jordan curves $C_1, \dots, C_m$. Set $C = \bigcup_{r=1}^m C_r$. Then, for each real θ ($0 \leq \theta < \pi$), there exists a unique function $g_\theta(z)$ analytic and univalent in Ω with expansion

$$z + A_{1,\theta} z^{-1} + \dots$$

near infinity which maps Ω onto a domain bounded by rectilinear slits which has argument θ . Let $\Sigma(\Omega)$ denote the class of all functions $f(z)$ analytic and univalent in Ω having expansion

$$z + A_1 z^{-1} + \dots$$

near infinity. It is well known [61,155] that the parallel slit mapping $g_\theta(z)$ is the unique solution of the extremal problem

$$\max_{f \in \Sigma(\Omega)} \operatorname{Re}(e^{2i\theta} A_1) = \operatorname{Re}(e^{2i\theta} A_{1,\theta}).$$

It is easily seen that for all θ ($0 \leq \theta < \pi$) it holds

$$g_\theta(z) = e^{i\theta} (\cos \theta g_0(z) - i \sin \theta g_{\pi/2}(z)), \quad z \in \Omega, \quad (66)$$

and, comparing the coefficients of z^{-1} of the expansions of both sides of (64), it follows

$$A_{1,\theta} = e^{i\theta} (\cos \theta A_{1,0} - i \sin \theta A_{1,\pi/2}).$$

Varying θ from 0 to π the coefficient $A_{1,\theta}$ describes the boundary of a closed disk $K(\Omega)$ of diameter $S(\Omega) := A_{1,0} - A_{1,\pi/2}$. This quantity $S(\Omega)$ is called the *conformal span* of the domain Ω . Grötzsch [61] proved that

$$K(\Omega) = \{A_1: f \in \Sigma(\Omega)\}.$$

Next we shall characterize the parallel slit mapping $g_\theta(z)$ by the Gauss–Thomson principle of minimal energy of signed charge distributions under the presence of an external dipolar field. In contrast to previous problems to minimize the energy of signed measures $\mu = \mu^+ - \mu^-$ we now drop the assumption that the supports of $\mu^\pm$ have positive distance. Let E_r denote the compact set with boundary C_r ($r = 1, 2, \dots, m$), and set $E = \bigcup_{r=1}^m E_r$. Further we denote by $\mathcal{M}_0(E)$ the set of all signed Borel measures $\mu = \mu^+ - \mu^-$ on E such that the logarithmic energy of $\mu^\pm$ is finite and $\mu(E_r) = 0$ for $r = 1, 2, \dots, m$. Kühnau [102] showed that for given θ ($0 \leq \theta < \pi$) and for all $\mu \in \mathcal{M}_0(E)$,

$$\begin{aligned} & -\operatorname{Re}(e^{-2i\theta} A_{1,\theta}) \\ & \leq \iint \log \frac{1}{|z - \zeta|} d\mu(z) d\mu(\zeta) - 2 \int \operatorname{Re}(ie^{-i\theta} z) d\mu(z). \end{aligned} \quad (67)$$

Here equality holds if and only if $\mu = \mu_\theta$ is supported on the boundary C and has the form

$$d\mu_\theta(z) = \frac{1}{2\pi} \frac{\partial}{\partial n} \operatorname{Re}(ie^{-i\theta} g_\theta(z)) |dz|, \quad (68)$$

where n is the outer normal at C with respect to Ω and $|dz|$ denotes the arclength measure on C . Furthermore, the connection between the parallel slit mapping $g_\theta(z)$ and the extremal measure μ_θ in (68) is given by

$$ie^{-i\theta} g_\theta(z) = ie^{-i\theta} z - \int_C \log \frac{1}{z - \zeta} d\mu_\theta(\zeta), \quad z \in \Omega. \quad (69)$$

The function $ie^{-i\theta} g_\theta(z)$ can be viewed as the complex-valued potential at z induced by the electrostatic field of a dipole at the point of infinity in the presence of grounded conductor plates $C_1, \dots, C_m$ embedded in a homogeneous medium. Integrating both sides of (69) with respect to $d\mu_\theta(z)$ then, since $\operatorname{Re}(ie^{-i\theta} g_\theta(z))$ is constant on C_r ($r = 1, 2, \dots, m$), we get together with (67)

$$\operatorname{Re}(e^{-2i\theta} A_{1,\theta}) = \int_C \operatorname{Re}(ie^{-i\theta} z) d\mu_\theta(z).$$

This relation admits the physical interpretation that the quantity $\operatorname{Re}(e^{-2i\theta} A_{1,\theta})$ is the electrostatic moment of the extremal measure μ_θ from (68) in the direction which has argument $\theta + \pi/2$. Substituting $\mu \in \mathcal{M}_0(E)$ in (67) by $t\mu \in \mathcal{M}_0(E)$, the choice of a minimizing t leads to

$$\operatorname{Re}(e^{-2i\theta} A_{1,\theta}) \geq \frac{1}{I(\mu)} \left(\int \operatorname{Re}(ie^{-i\theta} z) d\mu(z) \right)^2 \quad (70)$$

for all $\mu \in \mathcal{M}_0(E)$, $\mu \neq 0$, where $I(\mu)$ denotes the logarithmic energy (10) of μ . Equality in (70) holds if and only if μ/μ_θ is a constant $\neq 0$.

Define the θ -width of a continuum to be the width of the smallest parallel strip containing it in the direction which has argument θ . Under all boundary curves $C_1, \dots, C_m$ of largest θ -width l_θ let C_r that curve of smallest diameter D_θ . By a suitable choice of $\mu \in \mathcal{M}_0(C_r)$, from (70) we get the estimate [76]

$$\operatorname{Re}(e^{-2i\theta} A_{1,\theta}) \geq \frac{l_\theta^2}{12 + 8 \log(2D_\theta/l_\theta)} =: b_\theta. \quad (71)$$

Other lower estimates of this type were derived in [61] and more explicitly in [38, p. 237].

Adding the inequalities in (70) for $\theta = 0$ and $\theta = \pi/2$ we get the sharp estimate for the conformal span

$$S(\Omega) \geq \frac{1}{I(\mu_1)} \left(\int \operatorname{Im}(z) d\mu_1(z) \right)^2 + \frac{1}{I(\mu_2)} \left(\int \operatorname{Re}(z) d\mu_2(z) \right)^2$$

for $\mu_1, \mu_2 \in \mathcal{M}_0(E)$ and $\mu_1, \mu_2 \neq 0$. In particular, from (71) it follows $S(\Omega) \geq b_0 + b_{\pi/2}$.

Another extremal characterization of the conformal span was derived by Schiffer [162] by means of variational techniques:

$$S(\Omega) = \max_{f \in \Sigma(\Omega)} \frac{2}{\pi} \operatorname{mes}(\mathbb{C} \setminus f(\Omega)),$$

where $\operatorname{mes}(E)$ denotes the inner measure of the set E . In particular, we have $S(\Omega) \geq \frac{2}{\pi} \operatorname{mes}(\mathbb{C} \setminus \Omega)$ where equality holds for the case C is a circle.

Next we give a discretized version of (67). To simplify matters we restrict to the case that Ω is a simply-connected domain. Suppose its boundary is a closed analytic Jordan curve C . For given θ ($0 \leq \theta < \pi$), we define the weight function

$$w_\theta(z) := \exp\{\operatorname{Re}(ie^{-i\theta} z)\}, \quad z \in C.$$

Further let ε_n be a sequence of positive numbers tending to zero such that

$$\varepsilon_n^{1/n} \rightarrow 1.$$

Choose a system of $2n+1$ points $z_1, \dots, z_n, \zeta_1, \dots, \zeta_n \in C$, $\lambda > 0$ and form the expression

$$\left\{ \prod_{1 \leq j < k \leq n} \left(\frac{|z_j - z_k| |\zeta_j - \zeta_k|}{(|z_j - \zeta_k| + \varepsilon_n)(|\zeta_j - z_k| + \varepsilon_n)} \right)^{\lambda^2} \left(\frac{w_\theta(z_j) w_\theta(z_k)}{w_\theta(\zeta_j) w_\theta(\zeta_k)} \right)^\lambda \right\}^{1/\binom{n}{2}}. \quad (72)$$

Let $d_{\theta n}(C)$ be the maximum of (72) which is taken over all point systems $z_1, \dots, z_n$ and $\zeta_1, \dots, \zeta_n$ lying on arbitrary distinct arcs on C , and all λ satisfying

$$\frac{D}{2\pi} \leq \lambda \leq \frac{D}{\pi},$$

where D denotes the diameter of C . Let $z_{n1}, \dots, z_{nn}, \zeta_{n1}, \dots, \zeta_{nn}, \lambda_n$ be such points for which the maximum in (72) is attained. Further we may assume that the numbering of the points $z_{n1}, \dots, z_{nn}$ on C corresponds to the orientation of C in mathematical positive sense. In [77] it was shown that for given θ ($0 \leq \theta < \pi$)

$$\lim_{n \rightarrow \infty} d_{\theta n}(C) = \exp \operatorname{Re}(e^{-2\pi} a_{1,\theta})$$

and

$$\lim_{n \rightarrow \infty} \lambda_n = \frac{2}{\pi} d(C),$$

where $d(C)$ is the usual transfinite diameter of C . Moreover, the sequences z_{n1} and z_{nn} converge as $n \rightarrow \infty$ to those two points on C , respectively which are mapped by $g_\theta(z)$ onto the end-points of the slit $g_\theta(C)$.

6.7. Capacity and quasiconformal maps

Potentials in inhomogeneous isotropic medium. The starting point of classical potential theory in one complex variable is Laplace's equation and its solutions. As we have seen in the previous considerations canonical conformal mappings as the Riemann mapping function, circular, radial and parallel slit mapping can be viewed as complex-valued potentials of plane electrostatic fields in homogeneous medium, i.e., the dielectric constant p is equal to 1. Now we shall outline how potentials of plane electrostatic fields in *inhomogeneous isotropic medium*, i.e., the dielectric constant $p(z)$ depends on $z \in \mathbb{C}$, are related with the theory of quasiconformal mappings. As in the conformal case the Gauss principle of minimal energy can be used to give extremal characterizations for certain canonical quasiconformal mappings. On the other hand, such quasiconformal mappings are solutions of certain function-theoretic extremal problems which are closely related with the system

$$v_y = p u_x, \quad -v_x = p u_y \quad (73)$$

or, in complex form,

$$w_{\bar{z}} = -\frac{p-1}{p+1} \bar{w}_z, \quad (74)$$

where $z = x + iy$ and $w = u + iv$. Solutions of (73) are called *p-analytic*. For a good introduction into the theory of *p-analytic* functions we refer to [8] and [144].

Let Ω be a finitely-connected domain containing the point at infinity, bounded by analytic Jordan curves $C_1, \dots, C_m$. Set $C = \bigcup_{r=1}^m C_r$. Further let p be a real-valued function on the extended complex plane which satisfies

$$0 < m \leq p(z) \leq M < \infty, \quad z \in \mathbb{C},$$

and is smooth outside of C , where p is constant on a neighborhood to the right and left of each boundary component of Ω , respectively. Set

$$v(z) = \frac{p-1}{p+1}, \quad z \in \mathbb{C}.$$

Next we define a logarithmic basic solution of (74). Let $r(z, \zeta)$ denote the (uniquely determined) quasiconformal mapping of the extended plane onto itself satisfying $r(\infty, \zeta) = \infty$, $r(\zeta, \zeta) = 0$ and for which $w(\cdot) = \log r(\cdot, \zeta)$ is a solution of (74) in $\mathbb{C} \setminus \{\zeta\}$, where

$$\log r(z, \zeta) - (1 - v(\infty))^{-1} [\log z - v(\infty) \overline{\log z}] \rightarrow 0, \quad z \rightarrow \infty,$$

locally uniformly on $\mathbb{C}$ with respect to ζ , and

$$\log r(z, \zeta) - (1 - v(\zeta))^{-1} [\log(z - \zeta) - v(\zeta) \overline{\log(z - \zeta)}] \rightarrow c(\zeta), \quad z \rightarrow \zeta,$$

for all points $\zeta \notin \mathbb{C}$. For the proof of the existence and further properties of $r(z, \zeta)$, see [98, 102]. In particular, the *pseudo-metric*

$$[z, \zeta] := |r(z, \zeta)|$$

is symmetric $[z, \zeta] = [\zeta, z]$ and $[z, \zeta] \geq 0$ where equality holds if and only if $z = \zeta$. Moreover, $r(z, \zeta)$ and $[z, \zeta]$ are continuous functions on $\mathbb{C} \times \mathbb{C}$. Concerning the potential theory with respect to the equation

$$\operatorname{div}(p \nabla u) = 0$$

we note that its basic solution $\log \frac{1}{[z, \zeta]}$ satisfies an energy principle (11), see [76].

An explicit representation of the function $[z, \zeta]$ is only known in special cases, see [102]. For example, let C be the unit circle $|z| = 1$ and set

$$v(z) = \begin{cases} q, & |z| > 1, \\ 0, & |z| < 1 \end{cases}$$

for a given constant $q \in (-1, 1)$. Then, for $|z| < 1$, $|\zeta| < 1$, it holds

$$[z, \zeta] = \frac{|z - \zeta|}{|1 - \bar{\zeta}z|^q}. \quad (75)$$

In the limit case $q \rightarrow 1$ we get the pseudohyperbolic metric. Therefore, varying q from 0 to 1 the expression in (75) describes a one-parametric family of pseudo-metrics connecting the Euclidean metric with the pseudohyperbolic metric. For more general investigations concerning the hyperbolic metric as limit case of a metric defined by quasiconformal mappings, see [104].

The p -transfinite diameter $d(E, p)$, the p -Chebyshev constant $\tau(E, p)$ and the p -capacity $\text{cap}(E, p)$ of a compact set E can be defined as in the Euclidean case ($p \equiv 1$) replacing the term $(z - \zeta)$ in (1), (3) and (10) by $r(z, \zeta)$. All three quantities coincide again, see [74, 102].

In the following, we assume in addition $p(z) \geq 1$ ($z \in \mathbb{C}$) and $p(z) = 1$ in a neighborhood of infinity.

Alternate description of p -transfinite diameter can be given in terms of *weighted extremal length* due to Ohtsuka [140]; the connection with the system (73) goes back to [23]. The definition of the weighted extremal length of a family of curves is given by a modification of (14) and (16) multiplying the integrands by the weight $p(z)$. Then, an analogous formula (15) for the p -transfinite diameter remains valid. In particular, it shows that $d(E, p)$ is independent of the values of $p(z)$ on E although the corresponding pseudo-metric $[z, \zeta]$ (for fixed z and ζ) is not. Moreover, from the extremal length description of the p -transfinite diameter it follows the monotonicity property $d(E, p) \leq d(E, p)$ if $p(z) \leq p(z)$, $z \in \mathbb{C}$. For example, if C is the unit circle $|z| = 1$ and $p(z) = Q > 1$ inside of C , $p(z) = 1$ outside of C , and $1 \leq p(z) \leq p(z)$ on $\mathbb{C}$ then for every compact set E inside of C it holds the following estimate

$$d(E, 1) \leq d(E, p) \leq 2^{q/Q} (d(E, 1))^{1/Q},$$

which is asymptotically sharp for $q := (Q - 1)/(Q + 1) \rightarrow 0$. Here the right inequality follows from a simple estimate of the explicitly known pseudo-metric $[z, \zeta]$, see [102] corresponding with the definition of $d(E, p)$.

Let G be the outer domain of a given continuum E . Further let $\mathcal{A}_p(G)$ be the class of all univalent quasiconformal mappings $w(z)$ of G which satisfy

$$|w_z| \leq \frac{p(z) - 1}{p(z) + 1} |w_{\bar{z}}|, \quad z \in G,$$

and have expansion

$$z + a_0 + a_1 z^{-1} + \dots$$

near infinity where $p(z) = 1$ brings conformality with it. Let $w_p(z) \in \mathcal{A}_p(G)$ denote the uniquely-determined mapping from G onto $|w| > R$ for which $\log w_p(z)$ is a solution of (74). The quantity $R = R(G, p)$ is called p -quasiconformal mapping radius of the domain G with respect the point at infinity. In [102, 74] it was shown

$$d(E, p) = R(G, p).$$

By means of variational methods for quasiconformal mappings one can prove that $w_p(z)$ is a solution of the following maximum problem. Thus, for every continuum E we have the representation

$$d(E, p) = \max\{d(\mathbb{C} \setminus w(G)) : w \in \mathcal{A}_p(G)\}.$$

Kühnau [93] investigated the extremal problem

$$\max d(E, p), \quad (76)$$

where the maximum is taken over all continua E containing $n \geq 2$ given distinct points $z_1, \dots, z_n$. As in the conformal case ($p \equiv 1$) there exists a unique extremal continuum which consists of finitely many Jordan arcs, where the densities of the extremal charge distribution on both sides of the arcs coincide. A complete solution of this problem was given in the case $n = 2$. It turns out that in this case the extremal continuum consists of a Jordan arc connecting the points z_1 and z_2 which is smooth in those points z wherever $p(z)$ is smooth. Estimates of the geometrical shape of the extremal continuum were given in [75] by means of distortion theorems for quasiconformal mappings.

For given θ ($0 \leq \theta < \pi$), there exists a unique univalent quasiconformal mapping $j_\theta(z)$ from Ω onto a domain bounded by rectilinear slits in the direction having argument θ for which $ie^{-i\theta} j_\theta(z)$ is a solution of (74) in Ω , where $j_\theta(z)$ has expansion

$$j_\theta(z) = z + a_{1,\theta} z^{-1} + \dots$$

near infinity. Finally, let $j_\theta(z)$ define exactly the same way as $j_\theta(z)$, where Ω is replaced by the complex plane $\mathbb{C}$, and $j_\theta(z)$ has the expansion

$$j_\theta(z) = z + a_{1,\theta} z^{-1} + \dots$$

near infinity. In [96] it was proved that the quasiconformal mappings $j_\theta(z)$ and $j_\theta(z)$ are the unique solutions of the extremal problem

$$\max \operatorname{Re}(e^{2i\theta} a_1),$$

where the maximum is taken over $\mathcal{A}_p(\Omega)$ and $\mathcal{A}_p(\mathbb{C})$, respectively.

Analogously to the conformal case (67) we can now formulate the following extremal characterization for the quasiconformal parallel slit mapping j_θ , see [102]:

$$\begin{aligned} & \operatorname{Re}[e^{-2i\theta} (a_{1,\theta} - a_{1,\theta})] \\ & \leq \iint \log \frac{1}{[z, \zeta]} d\mu(z) d\mu(\zeta) - 2 \int \operatorname{Re}(ie^{-i\theta} j_\theta(z)) d\mu(z) \end{aligned} \quad (77)$$

for all μ of the class $\mathcal{M}_0(E)$ defined before (67). Equality holds if and only if $\mu = \mu_\theta$ is supported on the boundary C of Ω and has the form

$$d\mu_\theta(z) = \frac{p(z)}{2\pi} \frac{\partial}{\partial n} \operatorname{Re}(ie^{-i\theta} j_\theta(z)) |dz|,$$

where n is the outer normal at C with respect to Ω and $|dz|$ denotes the arclength measure on C . For a discretized version of (77), see [77].

The function $j_\theta(z)$ can be viewed as a complex-valued potential at z induced by the electrostatic field of a dipole at the point of infinity in the presence of grounded conductor plates $C_1, \dots, C_m$ embedded in an inhomogeneous isotropic medium, i.e., the dielectric constant is given by $p(z)$, $z \in \Omega$. Analogously, the function $j_\theta(z)$ has the same electrostatic interpretation as $j_\theta(z)$ without the presence of conductor plates. In this situation “imaginary” charges arise particularly on those arcs along the dielectric constant $p(z)$ (as function z) has jumps. These charges can also be characterized by a principle of minimal weighted energy. For simplicity we formulate this extremal principle for a special case. Let $p(z) = 1$ on Ω and $p(z)$ is equal to a constant $Q > 1$ on $E = \mathbb{C} \setminus \Omega$. Define $\mathcal{H}_0(C)$ as the set of all real-valued Hölder continuous functions $\mu(z)$ on the boundary C of Ω satisfying

$$\int_C \mu(z) ds_z = 0.$$

For $\mu \in \mathcal{H}_0(C)$ we set

$$\begin{aligned} J_Q(\mu) &:= \pi(Q+1) \int_C \int_C \log \frac{1}{|z-\zeta|} \mu(z) \mu(\zeta) ds_z ds_\zeta \\ &\quad + \pi(Q-1) \int_C \int_C K(z, \zeta) \mu(z) \mu(\zeta) ds_z ds_\zeta \end{aligned}$$

with the convolution kernel

$$K(z, \zeta) = -\frac{1}{\pi} \int_C \log |z-w| \frac{\partial}{\partial n_w} \log |w-\zeta| ds_w, \quad (78)$$

where n denotes the outer normal at $z \in C$ with respect to Ω . Further, for every θ ($0 \leq \theta < \pi$), we introduce the potential weight function

$$u_\theta(z) = \int_C \log |z-\zeta| \frac{\partial}{\partial n} \operatorname{Re}(ie^{-i\theta}\zeta) ds_\zeta, \quad z \in C.$$

Then [99], with the above notations

$$\begin{aligned} &(Q-1) \left(\operatorname{mes}(E) + 2 \int_C u_\theta(z) \mu(z) ds_z \right) + J_Q(\mu) \\ &\geq 2\pi \operatorname{Re}(e^{-2i\theta} a_{1,\theta}) \\ &\geq (1-1/Q) \left(\operatorname{mes}(E) + 2 \int_C u_{\theta+\pi/2}(z) \mu(z) ds_z \right) - J_{1/Q}(\mu) \end{aligned} \quad (79)$$

for all $\mu \in \mathcal{H}_0(C)$, where equality on the left- and right-hand side of (78) holds if and only if

$$\mu(z) = \frac{Q-1}{2\pi} \frac{\partial}{\partial n} \operatorname{Re}(ie^{-i\theta} j_\theta(z)) \quad \text{and} \quad \mu(z) = \frac{1/Q-1}{2\pi} \frac{\partial}{\partial n} \operatorname{Im}(ie^{-i\theta} j_\theta(z))$$

on C , respectively. Here n denotes the outer normal at $z \in C$ with respect to Ω . Note that the quantity $J_\tau(\mu) \geq 0$ for all $\mu \in \mathcal{H}_0(C)$ and $\tau > 0$ since otherwise the left- and right-hand side of (79) would not be bounded from below and from above, respectively. In particular, this implies

$$\left| \int_C \int_C K(z, \zeta) \mu(z) \mu(\zeta) ds_z ds_\zeta \right| \leq \int_C \int_C \log \frac{1}{|z - \zeta|} \mu(z) \mu(\zeta) ds_z ds_\zeta$$

for all $\mu \in \mathcal{H}_0(C)$. We remark that the convolution kernel $K(z, \zeta)$ in (78) was studied by Carleman in greater detail, see [38, pp. 27–28]. Choosing $\mu \equiv 0$ in (79), for all real θ we get the estimate

$$(1 - 1/Q) \operatorname{mes}(E) \leq 2\pi \operatorname{Re}(e^{-2i\theta} a_{1,\theta}) \leq (Q - 1) \operatorname{mes}(E)$$

which is asymptotically sharp for $Q \rightarrow 1$.

Finally, we refer to different extremal characterizations of energy type for the above-mentioned and other canonical quasiconformal mappings in [100], in particular, [103] involving Fredholm eigenvalues of the curve system C , and further generalizations in [99] relating to inequalities such as of Grunsky and Goluzin type.

Potentials in inhomogeneous nonisotropic medium. The previous considerations can be extended to the case of plane electrostatic fields in inhomogeneous nonisotropic medium, i.e. the dielectric constant is a given tensor $(p_{ik}(z))$, $i, k = 1, 2$. Let G be a doubly-connected domain with nondegenerating boundary components. It can be interpreted as a conductor with its boundary components as conductor plates. Let $p_{11}(z)$, $p_{12}(z) = p_{21}(z)$ and $p_{22}(z)$ be real-valued and piecewise smooth functions on G which satisfy for given positive constants m and M

$$m \leq p_{11} \leq M, \quad p_{22} \leq M, \quad p \equiv \sqrt{p_{11}p_{22} - p_{12}^2} \geq m$$

on G . Instead of (73), now the following more general system arises

$$v_y = p_{11}u_x + p_{12}u_y, \quad -v_x = p_{12}u_x + p_{22}u_y. \quad (80)$$

The domain G can be mapped univalently onto an annulus $r < |w| < R$ such that the logarithm $w = u + iv$ of the mapping function satisfies (80) on G . The ratio $R/r > 1$ is uniquely determined. Kühnau [94] defined *capacity* of the condenser G as $2\pi/\log(R/r)$.

This quantity can be expressed in terms of a *generalized extremal length* of a family of curves defined as follows. Let Γ be a family of locally rectifiable curves $\gamma \subset G$ connecting

both boundary components of G . An *admissible* metric of Γ is a Borel measurable function $\rho(z) \geq 0$ which satisfies $\int_{\gamma} dS \geq 1$ for all $\gamma \in \Gamma$, where

$$dS^2 = \frac{\rho^2}{p} (p_{22} dx^2 - 2p_{12} dx dy + p_{11} dy^2).$$

Then [94]

$$\frac{2\pi}{\log(R/r)} = \inf \iint_G p \rho^2 dx dy,$$

where the infimum is taken over all admissible metrics ρ of Γ .

It turns out that the determination of the capacity and the corresponding equilibrium potential of the condenser G in a nonisotropic medium can be reduced to the determination of the capacity and the corresponding equilibrium potential of a condenser G' in a certain isotropic medium, where G' is the image of G by a suitable quasiconformal mapping determined by the nonisotropic medium. This fact admits to generalize and to solve the extremal problem (76) in the case of nonisotropic medium. See [94,101] for more details.

6.8. Capacity in $\mathbb{C}^N$

The previous notions of capacity rely on potential theory in one complex variable. For the multidimensional generalizations pluripotential theory is used. This theory has been developed over the last thirty years and in particular gives the “correct” version of capacity of sets in $\mathbb{C}^N$. In the following we review some basic facts from pluripotential theory, see [90].

Let $\mathbb{C}^N$ denote the complex N -dimensional space, $N > 1$. We use $z = (z_1, \dots, z_N)$ with $z_k \in \mathbb{C}$ as coordinates for $\mathbb{C}^N$. We may identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ (Euclidean $2N$ -dimensional space). Under this identification of $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ all the usual concepts from real analysis in Euclidean space (e.g., Lebesgue $2N$ -dimensional measure) apply to $\mathbb{C}^N$. The Euclidean norm of a point $z \in \mathbb{C}^N$ is given by $|z| := (|z_1|^2 + \dots + |z_N|^2)^{1/2}$. The open ball of a center $\zeta \in \mathbb{C}^N$ and radius $r > 0$ is

$$B(\zeta, r) := \{z \in \mathbb{C}^N : |z - \zeta| < r\}.$$

An N -multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N -tuple of nonnegative integers. The monomial $(z_1^{\alpha_1}) \cdots (z_N^{\alpha_N})$ is denoted by z^α . It is a monomial of degree $|\alpha| = \alpha_1 + \dots + \alpha_N$. A polynomial $q(z) = \sum_{|\alpha| \leq n} c_\alpha z^\alpha$ is of degree n if at least one of the coefficients $c_\alpha \in \mathbb{C}$ with $|\alpha| = n$ is nonzero. Let G be an open subset of $\mathbb{C}^N$.

A function $u : G \rightarrow [-\infty, \infty)$ is called upper semicontinuous on G if for every $\zeta \in G$, $\limsup_{z \rightarrow \zeta} u(z) \leq u(\zeta)$. Further a function $u : G \rightarrow [-\infty, \infty)$ is said *plurisubharmonic* if it is upper semicontinuous on G , $u \not\equiv -\infty$ on any component of G , and, for every $a \in G$, $b \in \mathbb{C}^N$, the function of the single complex variable $\lambda \rightarrow u(a + \lambda b)$ is subharmonic

or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C}: a + \lambda b \in G\}$. For example, let f be an analytic function on G with $f \not\equiv 0$. Then $\log |f(z)|$ is plurisubharmonic on G .

The starting point for potential theory in one complex variable is Laplace's equation and its solutions. In pluripotential theory, the corresponding role is played by the homogeneous complex *Monge–Ampère equation*

$$\det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) = 0, \quad (81)$$

where $i, j = 1, \dots, N$, and $\det$ denotes determinant. This is essentially because “free upper envelope” of plurisubharmonic functions must satisfy (81). For $N = 1$, (81) reduces to Laplace's equation. In contrast, however to the one variable case, for $N > 1$ the solutions of (81) are not necessarily real analytic.

Let $C^2(G)$ denote the class of all real-valued and twice-continuously-differentiable functions on G . Further let dV denote the standard $2N$ -dimensional volume form on $\mathbb{R}^{2N}$. The operator

$$\mu^N : u \rightarrow 4^N N! \det \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right) dV \quad (82)$$

acting on $C^2(G)$ has an extension in the sense of distributions to locally bounded plurisubharmonic functions on G . For every locally bounded plurisubharmonic function u on G , $\mu^N(u)$ is a locally finite positive Borel measure on G . The proof of this fact is not a standard application of the theory of distributions as (81) shows that one must consider a product of distributions. Note that for $N > 1$ the operator (81) is nonlinear.

The next definition provides the generalization to several variables of sets of capacity zero. A set $F \subset \mathbb{C}^N$ is said *pluripolar* if, for all $a \in F$, there is a neighborhood B of a and a plurisubharmonic function u on B such that $F \cap B \subset \{z \in B: u(z) = -\infty\}$. For example, if $F \subset \{z \in \mathbb{C}^N: f(z) = 0\}$ and f is analytic on $\mathbb{C}^N$, $f \not\equiv 0$, then F is pluripolar since $\log |f|$ is plurisubharmonic on $\mathbb{C}^N$. As in the one variable case, we say that a property holds *quasi*everywhere (q.e.) on a set S if it holds on $S \setminus F$, where F is pluripolar. An important property of capacity of sets in one variable which holds in several variables is that a countable union of pluripolar sets is again pluripolar.

We will now give a generalization to several variables of the Green function with pole at infinity. Let $E \subset \mathbb{C}^N$ be compact. The *pluricomplex Green function* of E is defined for $z \in \mathbb{C}^N$ by

$$v_E(z) := \sup u(z),$$

where the supremum is taken over all plurisubharmonic functions u on $\mathbb{C}^N$ for which $u(z) \leq 0$ on E and $u(z) - \log(1 + |z|)$ is bounded from above on $\mathbb{C}^N$. Note that v_E is not, in general, upper semicontinuous. The function v_E^* denotes its upper semicontinuous regularization defined by

$$v_E^*(z) := \limsup_{\zeta \rightarrow z} v_E(\zeta), \quad z \in \mathbb{C}^N.$$

Then E is pluripolar if and only if $v_E^*(z) \equiv \infty$. If E is not pluripolar, the function v_E^* has the following properties:

- (i) $v_E^*(z)$ is nonnegative and satisfies the equation (81) on $\mathbb{C}^N \setminus E$,
- (ii) $v_E^*(z) - \log(1 + |z|)$ is bounded on $\mathbb{C}^N$,
- (iii) $v_E^*(z) = 0$ q.e. on E .

The Monge–Ampère measure $\mu^N(v_E^*)$ has total mass $(2\pi)^N$ and its support is contained in E . This measure is referred to as the equilibrium measure of E .

Next we will give two examples of compact sets, pluricomplex Green functions and equilibrium measures without details of calculations. Further examples and a sample of calculations involved can be found in [90].

EXAMPLE 1. Let $E_1 = \{z \in \mathbb{C}^N: |z| \leq 1\}$ be the unit Euclidean ball in $\mathbb{C}^N$. Then $v_{E_1}(z) = v_{E_1}^*(z) = \log^+ |z|$ and $\mu^N(v_{E_1})$ is (up to normalization) the surface area on the sphere $\{z \in \mathbb{C}^N: |z| = 1\}$, where $\log^+ |z| := \max\{\log |z|, 0\}$.

EXAMPLE 2. Let $E_2 = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N: |z_j| \leq 1, j = 1, \dots, N\}$ be the unit polydisk in $\mathbb{C}^N$. Then $v_{E_2}(z) = v_{E_2}^*(z) = \max\{\log^+ |z_j|: j = 1, \dots, N\}$ and $\mu^N(v_{E_2})$ is (up to normalization) the measure $d\theta_1 \cdots d\theta_N$ on $\{(z_1, \dots, z_N) \in E_2: |z_j| = 1, j = 1, \dots, N\}$, where $\theta_1, \dots, \theta_N$ are the angular parts of polar coordinates for $z_1, \dots, z_N$.

It is well known that the pluricomplex Green function has the following representation

$$v_E(z) = \sup \left\{ \frac{\log |q(z)|}{\deg(q)} \right\}, \quad z \in \mathbb{C}^N,$$

where the supremum is taken over all polynomials $q(z)$ of degree $\deg(q) \geq 1$ satisfying $|q(z)| \leq 1$ on E , see [172]. This formula is a crucial step in using pluripotential theory to obtain results on approximation by polynomials in several variables, see [13–15]. For weighted approximation we refer to [17, 172].

By means of pluricomplex Green function $v_E(z)$ the notion of capacity of a compact set $E \subset \mathbb{C}^N$ can be generalized to the case of several variables. Put

$$\psi(z) := \max\{\log |z_j|: j = 1, \dots, N\}, \quad z = (z_1, \dots, z_N) \in \mathbb{C}^N.$$

Then the quantity

$$\rho^{(N)}(E) := \limsup_{|z| \rightarrow \infty} (v_E(z) - \psi(z))$$

is the analogue of the Robin constant of E for $N = 1$. Zaharjuta [186] defined the *multivariate capacity* of E as

$$\text{cap}^{(N)}(E) := \exp(-\rho^{(N)}(E)).$$

It is easily shown that either $v_E(z) \equiv \infty$ or $v_E(z) - \log(1 + |z|)$ is bounded on $\mathbb{C}^N$. Therefore, $\text{cap}^{(N)}(E) = 0$ if and only if $v_E(z) \equiv \infty$.

Let $q(z) = \sum_{|\alpha| \leq n} c_\alpha z^\alpha$ be a polynomial of degree n , $a := \sum_{|\alpha|=n} |c_\alpha| > 0$, and let E be a compact set contained in $\{z \in \mathbb{C}^N : |q(z)| \leq r\}$. Then [186] the following estimate holds

$$\text{cap}^{(N)}(E) \leq (r/a)^{1/n}.$$

If $E = E_1 \times \cdots \times E_r \subset \mathbb{C}^N$, E_j compact in $\mathbb{C}^{N_j}$, $N = \sum_{j=1}^r N_j$, then [186]

$$\text{cap}^{(N)}(E) = \inf \{ \text{cap}^{(N_j)}(E_j) : j = 1, \dots, r \}.$$

Let $\mathring{\mathcal{S}} := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 + \cdots + x_N = 1, x_1, \dots, x_N > 0\}$, and its closure we denote by $\mathcal{S}$. Define

$$h_E(x) := \sup \left\{ \sum_{j=1}^N x_j \log |z_j| : (z_1, \dots, z_N) \in E \right\}, \quad x \in \mathring{\mathcal{S}}. \quad (83)$$

If the compact set E has the property

$$(P): \quad (z_1, \dots, z_N) \in E \implies (c_1 z_1, \dots, c_N z_N) \in E \\ \text{for all } c_k \in \mathbb{C}, |c_k| \leq 1, k = 1, \dots, N,$$

then [186]

$$\text{cap}^{(N)}(E) = \inf \{ \exp h_E(x) : x \in \mathcal{S} \}. \quad (84)$$

For other notions of capacity in $\mathbb{C}^N$ and its application in complex analysis we refer to [7, 18].

Now we give a discrete version of multivariate capacity. First we consider the monomials to be ordered lexicographically. That is $z^\alpha > z^\beta$ if $|\alpha| > |\beta|$ or if $|\alpha| = |\beta|$ and $\alpha_i = \beta_i$ for $i = 1, \dots, j$, but $\alpha_{j+1} > \beta_{j+1}$. We use the notation $e_k(z)$ for the k th monomial under this ordering. For $e_k(z) = z^\alpha$ we write $\alpha = \alpha(k)$. For example, in $\mathbb{C}^2$, the first six monomial under this ordering are

$$e_1 = 1, \quad e_2 = z_1, \quad e_3 = z_2, \quad e_4 = z_1^2, \quad e_5 = z_1 z_2 \quad \text{and} \quad e_6 = z_2^2.$$

Let E be compact. Let $n (\geq 2)$ be a positive integer, and $\zeta_1, \dots, \zeta_n$ points in $\mathbb{C}^N$. The Vandermonde determinant of order n is defined by

$$V(\zeta_1, \dots, \zeta_n) := \det(e_k(\zeta_l))_{k,l=1}^n.$$

It may be considered as a polynomial in $n \cdot N$ variables (the coordinates of $\zeta_1, \dots, \zeta_n$). Put $V_n := \sup \{ |V(\zeta_1, \dots, \zeta_n)| : \zeta_1, \dots, \zeta_n \in E \}$. Leja [121] defined the *multivariate transfinite diameter* of E as

$$d^{(N)}(E) := \limsup_{s \rightarrow \infty} V_{m_s}^{1/l_s}, \quad (85)$$

where $m_s = \binom{N+s}{N}$ is the number of multiindices α with length $|\alpha| \leq s$ and $l_s = \sum_{k=1}^s k(m_k - m_{k-1})$. Zaharjuta [186] proved that the limit in (85) exists. Besides, he introduced a multidimensional analogue of the classical Chebyshev constant. Set

$$M_k := \inf \left\{ \max_{z \in E} |q(z)| : q(z) = z^{\alpha(k)} + \sum_{j < k} c_j z^{\alpha(j)} \right\}.$$

The Chebyshev constant of E with respect to the direction $x \in \mathcal{S}$ is the constant

$$\tau(E, x) := \limsup_{k \rightarrow \infty} M_k^{1/|\alpha(k)|}, \quad \frac{\alpha(k)}{|\alpha(k)|} \rightarrow x.$$

Here the usual limit exists for all $x \in \mathring{\mathcal{S}}$. Then the *multivariate Chebyshev constant* of E is defined as

$$\tau^{(N)}(E) := \exp \left\{ \frac{1}{\text{mes } \mathcal{S}} \int_{\mathcal{S}} \log \tau(E, x) dm(x) \right\}.$$

In [186] it was shown that for every compact set $E \subset \mathbb{C}^N$,

$$\text{cap}^{(N)}(E) \leq d^{(N)}(E) = \tau^{(N)}(E). \quad (86)$$

All three quantities in (86) have monotonicity property, in particular $\text{cap}^{(N)}(E) \leq \text{cap}^{(N)}(F)$ if $E \subset F$. This follows immediately from their definitions.

Suppose the compact set E has the property (P). Then

$$\tau(E, x) = \exp h_E(x), \quad x \in \mathcal{S},$$

where $h_E(x)$ is defined in (83), and it holds

$$\tau^{(N)}(E) := \exp \left\{ \frac{1}{\text{mes } \mathcal{S}} \int_{\mathcal{S}} h_E(x) dm(x) \right\}. \quad (87)$$

Obviously, the unit Euclidean ball E_1 and the unit polydisk E_2 (see Examples 1 and 2) have the property (P). One easily calculates

$$h_{E_1}(x) = \frac{1}{2} \sum_{j=1}^N x_j \log x_j, \quad h_{E_2}(x) = 0, \quad x \in \mathring{\mathcal{S}}.$$

Hence, by (84), (86) and (87) we get

$$\text{cap}^{(N)}(E_1) = \frac{1}{\sqrt{N}} < d^{(N)}(E_1) = \tau^{(N)}(E_1) = \exp \left(-\frac{1}{2} \sum_{k=2}^N \frac{1}{k} \right);$$

see also [73], and

$$\text{cap}^{(N)}(E_2) = d^{(N)}(E_2) = \tau^{(N)}(E_2) = 1.$$

Finally, we mention the paper [16] in which several new results on multivariate transfinite diameter and its connection with pluripotential theory were proven. For example, if $E \subset \mathbb{C}^M$ and $F \subset \mathbb{C}^N$ are compact, then

$$d^{(M+N)}(E \times F) = (d^{(M)}(E)^M \cdot d^{(N)}(F)^N)^{1/(M+N)}.$$

This is an essential improvement of an old result due to Schiffer and Siciak [164]. Further, if E and F are regular compact subsets of $\mathbb{C}^N$ (i.e., their pluricomplex Green functions $v_E(z)$ and $v_F(z)$ with pole at infinity are continuous on $\mathbb{C}$) then

$$\inf_{|z|=1} \frac{e^{-\rho_E(z)}}{e^{-\rho_F(z)}} \leq \frac{d^{(N)}(E)}{d^{(N)}(F)} \leq \sup_{|z|=1} \frac{e^{-\rho_E(z)}}{e^{-\rho_F(z)}}, \quad (88)$$

where $\rho_E(z) := \limsup_{t \rightarrow \infty} [v_E(tz) - \log |tz|]$, $z \in \mathbb{C} \setminus \{0\}$. Let $E \subset \mathbb{C}^N$ be a regular compact set, $R > 1$ and $E_R := \{z \in \mathbb{C}^N : v_E(z) \leq \log R\}$. Then, from (88) it follows

$$d^{(N)}(E_R) = R d^{(N)}(E).$$

This formula gives an answer to an old question due to Zaharjuta [186].

References

- [1] L.V. Ahlfors, *Bounded analytic functions*, Duke Math. J. **14** (1947), 1–11.
- [2] L.V. Ahlfors, *Open Riemann surfaces and extremal problems on compact subregions*, Comment. Math. Helv. **24** (1950), 100–134.
- [3] L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [4] L.V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1950), 101–129.
- [5] T.H. Bagby, *The modulus of a plane condenser*, J. Math. Mech. **17** (1967), 315–329.
- [6] T.H. Bagby, *On interpolation by rational functions*, Duke Math. J. **36** (1969), 95–104.
- [7] E. Bedford and B.A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier (Grenoble) **38** (1988), 133–171.
- [8] L. Bers, *Theory of pseudo-analytic functions*, New York Univ., New York (1953) (mimeographed).
- [9] A. Besicovitch, *On sufficient conditions for a function to be analytic and on behaviour of analytic functions in the neighborhood of non-isolated singular points*, Proc. London Math. Soc. **2** (1931), 1–9.
- [10] D. Betsakos, *Polarization, conformal invariants, and Brownian motion*, Ann. Acad. Sci. Fenn. **23** (1998), 59–82.
- [11] A. Beurling, *Ensembles exceptionnels*, Acta Math. **72** (1940), 1–13.
- [12] L. Bieberbach, *Über einige Extremalprobleme im Gebiete der konformen Abbildung*, Math. Ann. **77** (1916), 153–172.
- [13] T. Bloom, *On the convergence of multivariable Lagrange interpolants*, Constr. Approx. **5** (1989), 415–435.

- [14] T. Bloom, *On families of polynomials which approximate the pluricomplex Green function*, Indiana Univ. Math. J. **50** (2001), 1545–1566.
- [15] T. Bloom, L. Bos, C. Christensen and N. Levenberg, *Polynomial interpolation of holomorphic functions in $\mathbb{C}$ and $\mathbb{C}^N$* , Rocky Mountain J. Math. **22** (1992), 441–470.
- [16] T. Bloom and J.-P. Calvi, *On the multivariate transfinite diameter*, Ann. Polon. Math. **72** (1999), 285–305.
- [17] T. Bloom and N. Levenberg, *Weighted pluripotential theory in $\mathbb{C}^N$* , Amer. J. Math. **125** (2003), 57–103.
- [18] U. Cegrell, *Capacities in Complex Analysis*, Aspects of Mathematics, Vol. 14, Vieweg, Braunschweig, Wiesbaden (1988).
- [19] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1955), 131–295.
- [20] A. Denjoy, *Sur les fonctions analytiques uniformes qui restent continues sur ensemble parfait discontinu de singularités*, C. R. Acad. Sci. Paris **148** (1909), 1154–1156.
- [21] P. Dragnev and E.B. Saff, *Constrained energy problems with applications to orthogonal polynomials of a discrete variable*, J. Anal. Math. **72** (1997), 223–259.
- [22] V.N. Dubinin, *Symmetrization in the geometric theory of functions of a complex variable*, Russian Math. Surveys **49** (1994), 1–79.
- [23] R.J. Duffin, *The extremal length of a network*, J. Math. Anal. Appl. **5** (1962), 200–215.
- [24] P. Duren, *Robin capacity*, Computational Methods and Function Theory, 1997, Nicosia, Ser. Approx. Decompos., Vol. 11, World Scientific, River Edge, NJ (1999), 177–190.
- [25] P. Duren and R. Kühnau, *Elliptic capacity and its distortion under conformal mapping*, J. Anal. Math. **89** (2003), 317–335.
- [26] P. Duren and J. Pfaltzgraff, *Hyperbolic capacity and its distortion under conformal mapping*, J. Anal. Math. **78** (1999), 205–218.
- [27] P. Duren, J. Pfaltzgraff and R.E. Thurman, *Physical interpretation and further properties of Robin capacity*, St. Petersburg Math. J. **9** (1997), 607–614.
- [28] P. Duren and M. Schiffer, *Robin functions and distortion of capacity under conformal mapping*, Complex Var. Theory Appl. **21** (1993), 189–196.
- [29] M. Eiermann, R.S. Varga and W. Niethammer, *Iteration methods for nonsymmetric systems of equations and approximation methods in the complex plane*, Jahresber. Deutsch. Math.-Verein. **89** (1987), 1–32.
- [30] P. Erdős and J. Gillis, *Note on the transfinite diameter*, J. London Math. Soc. **12** (1937), 185–192.
- [31] D.K. Faddejew and V.N. Faddejewa, *Computational Methods of Linear Algebra*, Fizmatgiz, Moscow, (1960) (1st edn); (1963) (2nd edn); English transl.: Freeman and Co., San Francisco–London (1963).
- [32] L. Fejér, *Interpolation und konforme Abbildung*, Göttinger Nachr. **46** (1918), 319–331.
- [33] M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), 228–249.
- [34] M. Fekete, *Über den transfiniten Durchmesser ebener Punktmengen, II*, Math. Z. **32** (1930), 215–221.
- [35] S. Fisher, *On Schwarz's lemma and inner functions*, Trans. Amer. Math. Soc. **138** (1969), 229–240.
- [36] O. Frostman, *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions* (Thesis), Meddel. Lunds Univ. Mat. Sem. **3** (1935), 1–118.
- [37] O. Frostman, *La méthode de variation de Gauss et les fonctions sousharmoniques*, Acta Sci. Math. **8** (1936–1937), 149–159.
- [38] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1964).
- [39] D. Gaier, *Lectures on Complex Approximation*, Birkhäuser, Boston (1987).
- [40] P. Garabedian, *Schwarz's lemma and the Szegő kernel function*, Trans. Amer. Math. Soc. **67** (1949), 1–35.
- [41] J. Garnett, *Analytic Capacity and Measure*, Springer-Verlag, Berlin–Heidelberg–New York (1972).
- [42] C.F. Gauss, *Allgemeine Lehrsätze*, Werke 5, Dieterich, Göttingen (1877).
- [43] M. Götz, *Diskrepanzabschätzungen für Maße und quantitative Verteilungsaussagen für Energieextremale Punktsysteme (Discrepancy Estimates for Measures and Quantitative Distribution Assertions for Energy Extremal Point Systems)*, Shaker, Aachen (1998).
- [44] M. Götz, *Potential and discrepancy estimates for weighted extremal points*, Constr. Approx. **16** (2000), 541–557.
- [45] M. von Golitschek, G.G. Lorentz and Y. Makovoz, *Asymptotics of weighted polynomials*, Progress in Approx. Theory, A.A. Gonchar and E.B. Saff, eds, Springer-Verlag, New York (1992), 431–451.

- [46] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Nauka, Moscow (1952); German transl.: VEB Deutsch. Verlag Wissensch., Berlin (1957); 2nd edn, Nauka, Moscow (1966); English transl.: Transl. Math. Monographs, Vol. 29, Amer. Math. Soc., Providence, RI (1969).
- [47] A.A. Gonchar, *On the speed of rational approximation of some analytic functions*, Math. USSR-Sb. **34** (1978), 131–145.
- [48] A.A. Gonchar and E.A. Rakhmanov, *Equilibrium measure and the distribution of zeros of extremal polynomials*, Mat. Sb. **125** (1984), 117–127 (in Russian).
- [49] A.A. Gonchar and E.A. Rakhmanov, *On the equilibrium problem for vector potentials*, Uspekhi Mat. Nauk **40** (1985), 155–156; English transl.: Russian Math. Surveys **40** (1985), 183–184.
- [50] J. Górski, *Méthode des points extrémaux de résolution du problème de Dirichlet dans l'espace*, Ann. Polon. Math. **1** (1950), 418–429.
- [51] J. Górski, *Remarque sur le diamètre transfini des ensembles plans*, Ann. Soc. Polon. Math. **23** (1950), 90–94.
- [52] J. Górski, *Sur certaines fonctions harmoniques jouissant des propriétés extrémales par rapport à un ensemble*, Ann. Soc. Math. Polon. **23** (1950), 259–271.
- [53] J. Górski, *Sur un problème de F. Leja*, Ann. Soc. Polon. Math. **25** (1952), 273–278.
- [54] J. Górski, *Sur certaines propriétés de points extrémaux liés à un domaine plan*, Ann. Polon. Math. **3** (1957), 32–36.
- [55] J. Górski, *Sur la représentation conforme d'un domaine multiplement connexe*, Ann. Polon. Math. **3** (1957), 218–224.
- [56] J. Górski, *Distributions restreintes des points extrémaux liés aux ensembles dans l'espace*, Ann. Polon. Math. **4** (1957–1958), 325–339.
- [57] J. Górski, *Les suites de points extrémaux liées aux ensembles dans l'espace à 3 dimensions*, Ann. Polon. Math. **4** (1957–1958), 14–20.
- [58] J. Górski, *Solution of some boundary-value problems by the method of F. Leja*, Ann. Polon. Math. **8** (1960), 249–257.
- [59] J. Górski, *Application of the extremal points method to some variational problems in the theory of schlicht functions*, Ann. Polon. Math. **17** (1965), 141–145.
- [60] H. Grötzsch, *Über ein Variationsproblem der konformen Abbildung*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **82** (1930), 251–263.
- [61] H. Grötzsch, *Über das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche*, Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **84** (1932), 15–36.
- [62] H. Grunsky, *Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten, I*, Jahresber. Deutsch. Math.-Verein. **50** (1940), 230–255.
- [63] H. Grunsky, *Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten, II*, Jahresber. Deutsch. Math.-Verein. **52** (1942), 118–132.
- [64] H. Grunsky, *Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten, III. Ein Einzigkeitssatz*, Math. Z. **51** (1949), 586–615.
- [65] H. Grunsky, *Über beschränkte Funktionen*, Jahresber. Deutsch. Math.-Verein. **55** (1951), 4–9.
- [66] K. Haliste, *On an extremal configuration for capacity*, Ark. Mat. **27** (1989), 97–104.
- [67] S.Ya. Havinson, *Analytic capacity of sets related to the nontriviality of various classes of analytic functions, and on Schwarz's lemma in arbitrary domains*, Mat. Sb. (N.S.) **54** (96) (1961), 3–50.
- [68] W.K. Hayman, *Some applications of the transfinite diameter to the theory of functions*, J. Anal. Math. **1** (1951), 155–179.
- [69] W.K. Hayman, *Transfinite Diameter and Its Application*, Notes by K.R. Unni, Second printing MatScience Report No. 45, Inst. of Math. Sci., Madras (1966).
- [70] L.L. Helms, *Introduction to Potential Theory*, Wiley, New York (1969).
- [71] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, Wiley, New York (1986).
- [72] E. Hille, *Analytic Function Theory II*, Ginn and Co., Boston (1962).
- [73] M. Jędrzejowski, *The homogeneous transfinite diameter of a compact subset on $\mathbb{C}^N$* , Ann. Polon. Math. **55** (1991), 191–205.
- [74] S. Kirsch, *Ein verallgemeinerter transfiniter Durchmesser im Zusammenhang mit einer quasikonformen Normalabbildung*, Banach Center Publ. **11** (1983), 121–129.

- [75] S. Kirsch, *Lageabschätzung für einen Kondensator minimaler Kapazität*, Z. Anal. Anwendungen **3** (1984), 119–131.
- [76] S. Kirsch, *Extremalprinzipien zur Charakterisierung von quasikonformen Normalabbildungen und ihre Anwendung zur Abschätzung von Gebietsfunktionalen*, Z. Anal. Anwendungen **3** (1984), 555–568.
- [77] S. Kirsch, *Ein transfiniter Durchmesser bei quasikonformen Normalabbildungen mehrfach zusammenhängender Gebiete*, Z. Anal. Anwendungen **5** (1986), 445–456.
- [78] S. Kirsch, *On the boundary distortion under conformal mapping*, Complex Var. **42** (2000), 269–288.
- [79] M. Klein, *Estimates for the transfinite diameter with applications to conformal mapping*, Pacific J. Math. **22** (1967), 267–279.
- [80] W. Kleiner, *Démonstration du théoreme de Osgood–Carathéodory par la méthode des points extrémaux*, Ann. Polon. Math. **2** (1955), 67–72.
- [81] W. Kleiner, *Démonstration du théoreme de Carathéodory par la méthode des points extrémaux*, Ann. Polon. Math. **11** (1961–1962), 217–224.
- [82] W. Kleiner, *Sur l’approximation du diamètre transfini*, Ann. Polon. Math. **12** (1962), 171–173.
- [83] W. Kleiner, *Sur la condensation de masses*, Ann. Polon. Math. **15** (1964), 85–90.
- [84] W. Kleiner, *Sur la détermination numérique des points extrémaux de Fekete–Leja*, Ann. Polon. Math. **15** (1964), 91–96.
- [85] W. Kleiner, *Sur l’approximation de la représentation conforme par la méthode des points extrémaux de F. Leja*, Ann. Polon. Math. **14** (1964), 131–140.
- [86] W. Kleiner, *Sur les approximations de F. Leja dans le problème plan de Dirichlet*, Ann. Polon. Math. **15** (1964), 203–209.
- [87] W. Kleiner, *Une condition de Dini–Lipschitz dans la théorie du potentiel*, Ann. Polon. Math. **14** (1964), 117–130.
- [88] W. Kleiner, *Une variante de la méthode de F. Leja pour l’approximation de la représentation conforme*, Ann. Polon. Math. **15** (1964), 211–216.
- [89] W. Kleiner, *A variant of Leja’s approximations in Dirichlet’s plane problem*, Ann. Polon. Math. **16** (1965), 201–211.
- [90] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford (1991).
- [91] H. Kloeke, *On the capacity of a plane condenser and conformal mapping*, J. Reine Angew. Math. **358** (1985), 179–201.
- [92] Y. Komatu, *Über eine Verschärfung des Löwnerschen Hilfssatzes*, Proc. Imperial Acad. Japan **18** (1942), 354–359.
- [93] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien*, J. Reine Angew. Math. **231** (1968), 101–113.
- [94] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien, II*, J. Reine Angew. Math. **238** (1969), 61–66.
- [95] R. Kühnau, *Transfinites Durchmesser, Kapazität und Tschebyschewsche Konstante in der Euklidischen, hyperbolischen und elliptischen Geometrie*, J. Reine Angew. Math. **234** (1969), 216–220.
- [96] R. Kühnau, *Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung*, Math. Nachr. **40** (1969), 1–11.
- [97] R. Kühnau, *Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag der Wissenschaften, Berlin (1974).
- [98] R. Kühnau, *Identitäten bei quasikonformen Normalabbildungen und eine hiermit zusammenhängende Kernfunktion*, Math. Nachr. **73** (1976), 73–106.
- [99] R. Kühnau, *Verzerrungsaussagen bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung und ein Extremalprinzip der Elektrostatik in inhomogenen Medien*, Comment. Math. Helv. **53** (1978), 408–428.
- [100] R. Kühnau, *Einige Verzerrungsaussagen bei quasikonformen Abbildungen endlich vielfach zusammenhängender Gebiete*, L’Enseign. Math. **24** (1978), 189–201.
- [101] R. Kühnau, *Zur Moduländerung eines Vierecks bei quasikonformer Abbildung*, Math. Nachr. **93** (1979), 249–258.
- [102] R. Kühnau, *Gauss–Thomsonsches Prinzip minimaler Energie, verallgemeinerte transfinite Durchmesser und quasikonforme Abbildungen*, Proc. Romanian–Finnish Sem. on Complex Analysis, Bucharest, 1976, Lecture Notes in Math., Vol. 743 (1979), 140–164.

- [103] R. Kühnau, *Funktionalabschätzungen bei quasikonformen Abbildungen mit Fredholmschen Eigenwerten*, Comment. Math. Helv. **56** (1981), 297–306.
- [104] R. Kühnau, *Die hyperbolische Metrik als Grenzfall einer durch quasikonforme Abbildungen definierten Metrik*, Complex Analysis, Banach Center Publ., Vol. 11, Warsaw (1983), 211–216.
- [105] N.S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss., Springer-Verlag, Berlin (1972).
- [106] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin (1965).
- [107] F. Leja, *Sur les suites de polynômes bornés presque partout sur la frontière d'un domaine*, Math. Ann. **108** (1933), 517–524.
- [108] F. Leja, *Sur les suites de polynômes, les ensembles fermés et la fonction de Green*, Ann. Soc. Math. Polon. **12** (1934), 57–71.
- [109] F. Leja, *Sur une famille de fonctions harmoniques dans le plan liées à une fonction donnée sur la frontière d'un domaine*, Bull. Internat. Acad. Polon. Sci. Lett. A **108** (1936), 79–92.
- [110] F. Leja, *Sur les polynômes de Tchebycheff et la fonction de Green*, Ann. Soc. Polon. Math. **19** (1946), 1–6.
- [111] F. Leja, *Une condition de régularité et d'irrégularité des frontières dans le problème de Dirichlet*, Ann. Soc. Math. Polon. **20** (1947), 223–228.
- [112] F. Leja, *Sur les coefficients des fonctions analytiques univalentes dans le cercle et les points extrémaux des ensembles*, Ann. Soc. Polon. Math. **23** (1950), 69–78.
- [113] F. Leja, *Une méthode d'approximation des fonctions réelles d'une variable complexe par des fonctions harmoniques*, Rend. Accad. Naz. Lincei **8** (1950), 292–302.
- [114] F. Leja, *Une méthode élémentaire de résolution du problème de Dirichlet dans le plan*, Ann. Soc. Math. Polon. **23** (1950), 230–245.
- [115] F. Leja, *Sur une famille de fonctions analytiques extrémales*, Ann. Soc. Math. Polon. **25** (1952), 1–16.
- [116] F. Leja, *Polynômes extrémaux et la représentation conforme des domaines doublement connexes*, Ann. Polon. Math. **1** (1955), 13–28.
- [117] F. Leja, *Distributions libres et restreintes de points extrémaux dans les ensembles plans*, Ann. Polon. Math. **3** (1957), 147–156.
- [118] F. Leja, *Propriétés des points extrémaux des ensembles plans et leur application à la représentation conforme*, Ann. Polon. Math. **3** (1957), 319–342.
- [119] F. Leja, *Teoria Funkcji Analitycznych*, Państwowe Wydawnictwo Naukowe, Warszawa (1957) (in Polish).
- [120] F. Leja, *Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme*, Ann. Polon. Math. **4** (1957–1958), 8–13.
- [121] F. Leja, *Problems à résoudre posés à la conférence*, Colloq. Math. **7** (1959), 153.
- [122] F. Leja, *Sur les moyennes arithmétiques, géométriques et harmoniques des distances mutuelles des points d'un ensemble*, Ann. Polon. Math. **9** (1961), 211–218.
- [123] A. Levenberg and L. Reichel, *A generalized ADI iterative method*, Numer. Math. **66** (1993), 215–233.
- [124] R. Maskus, *Über die geometrische Gestalt eines Kontinuums minimalen elliptischen transfiniten Durchmessers*, Rev. Roumaine Math. Pures Appl. **25** (1980), 876–890.
- [125] M.S. Melnikov, *A bound for the Cauchy integral along an analytic curve*, Mat. Sb. **71** (1966), 503–515.
- [126] M.S. Melnikov, *Analytic capacity and the Cauchy integral*, Soviet Math. Dokl. **8** (1967), 20–23.
- [127] M.S. Melnikov, *Analytic capacity: A discrete approach and the curvature of measure*, Mat. Sb. **186** (1995), 57–76; Transl.: Sb. Math. **186** (1995), 827–846.
- [128] K. Menke, *Extremalpunkte und konforme Abbildung*, Math. Ann. **195** (1972), 292–308.
- [129] K. Menke, *Bestimmungen von Näherungen für die konforme Abbildung mit Hilfe von stationären Punktsystemen*, Numer. Math. **22** (1974), 111–117.
- [130] K. Menke, *Über die Verteilung von gewissen Punktsystemen mit Extremaleigenschaften*, J. Reine Angew. Math. **283/284** (1976), 421–435.
- [131] K. Menke, *On the distribution of Tsuji points*, Math. Z. **190** (1985), 439–446.
- [132] K. Menke, *Tsuji points and conformal mapping*, Ann. Polon. Math. **46** (1985), 183–187.
- [133] H.N. Mhaskar and E.B. Saff, *Extremal problems for polynomials with exponential weights*, Trans. Amer. Math. Soc. **285** (1984), 204–234.
- [134] H.N. Mhaskar and E.B. Saff, *Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)*, Constr. Approx. **1** (1985), 71–91.

- [135] H.N. Mhaskar and E.B. Saff, *Weighted analogues of capacity, transfinite diameter and Chebyshev constant*, Constr. Approx. **8** (1992), 105–124.
- [136] P.J. Myrberg, *Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche*, Acta Math. **61** (1933), 39–79.
- [137] Z. Nehari, *Bounded analytic functions*, Bull. Amer. Math. Soc. **57** (1951), 354–366.
- [138] R. Nevanlinna, *Analytic Functions*, Springer-Verlag, Berlin (1970).
- [139] N. Ninomiya, *Méthode de variation du minimum dans la théorie du potentiel*, Sem. Théory du Potentiel **5** (1958–1959).
- [140] M. Ohtsuka, *Sur un théorème étoilé de Gross*, Nagoya Math. J. **9** (1955), 191–207.
- [141] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A I Math. **25** (1961), 135–352.
- [142] P. Painlevé, *Sur les lignes singulières des fonctions analytiques*, Ann. Fac. Sci. Univ. Toulouse **2** (1888).
- [143] U. Pirl, *Über die geometrische Gestalt eines Extremalkontinuums aus der Theorie der konformen Abbildung*, Math. Nachr. **39** (1969), 297–312.
- [144] G.M. Položii, *Generalization of the Theory of Analytic Functions of a Complex Variable. p -Analytic and (p, q) -Analytic Functions and some of Their Applications*, Kiev. Univ., Kiev (1965); 2nd edn: Naukova Dumka, Kiev (1973) (in Russian).
- [145] G. Pólya, *Beiträge zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete, I, II*, S.-Ber. Preuss. Akad. Wiss. Berlin (1928), 228–232, 280–282.
- [146] G. Pólya, *Beiträge zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete, III*, S.-Ber. Preuss. Akad. Wiss. Berlin (1929), 55–62.
- [147] Ch. Pommerenke, *Über die analytische Kapazität*, Arch. Math. **11** (1960), 270–277.
- [148] Ch. Pommerenke, *On the hyperbolic capacity and conformal mapping*, Proc. Amer. Math. Soc. **13** (1963), 941–947.
- [149] Ch. Pommerenke, *Konforme Abbildungen und Fekete-Punkte*, Math. Z. **89** (1965), 422–438.
- [150] Ch. Pommerenke, *Polynome und konforme Abbildung*, Monatsh. Math. **69** (1965), 58–61.
- [151] Ch. Pommerenke, *Über die Verteilung der Fekete-Punkte*, Math. Ann. **168** (1967), 111–127.
- [152] Ch. Pommerenke, *On the logarithmic capacity and conformal mapping*, Duke Math. J. **35** (1968), 321–326.
- [153] Ch. Pommerenke, *Über die Verteilung der Fekete-Punkte, II*, Math. Ann. **179** (1969), 212–218.
- [154] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
- [155] R. de Possel, *Zum Parallelschlitztheorem unendlich-vielfach zusammenhängender Gebiete*, Nachr. Ges. Wiss. Göttingen (1931), 192–202.
- [156] E.A. Rakhmanov, *On asymptotic properties of polynomials orthogonal on the real axis*, Mat. Sb. **119** (1982), 163–203; English transl.: Math. USSR-Sb. **47** (1984), 155–193.
- [157] E.A. Rakhmanov, *Equilibrium measure and zero distribution of extremal polynomials of a discrete variable*, Mat. Sb. **187** (1996), 109–124 (in Russian).
- [158] H. Renggli, *An inequality for logarithmic capacities*, Pacif. J. Math. **11** (1961), 313–314.
- [159] H. Royden, *The boundary values of analytic and harmonic functions*, Math. Z. **78** (1962), 1–24.
- [160] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren Math. Wiss., Bd. 316, Springer-Verlag, Berlin–Heidelberg (1997).
- [161] M. Schiffer, *On the subadditivity of the transfinite diameter*, Proc. Cambridge Phil. Soc. **37** (1941), 373–383.
- [162] M. Schiffer, *The span of multiply connected domains*, Duke Math. J. **10** (1943), 109–216.
- [163] M. Schiffer, *Hadarnard's formula and variation of domain functions*, Amer. J. Math. **68** (1946), 417–448.
- [164] M. Schiffer and J. Siciak, *Transfinite diameter and analytic continuation of functions of two complex variables*, Stud. Math. Analysis and Related Topics, Stanford Univ. Press (1962), 341–358.
- [165] I. Schur, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **1** (1918), 377–402.
- [166] Y.C. Shen, *On interpolation and approximation by rational function with preassigned poles*, J. Chinese Math. Soc. **1** (1936), 154–173.
- [167] J. Siciak, *Sur la distribution des points extrémaux dans les ensembles plans*, Ann. Polon. Math. **4** (1957–1958), 214–219.

- [168] J. Siciak, *On some extremal functions and their applications in the theory of functions of several complex variables*, Trans. Amer. Math. Soc. **105** (1962), 322–357.
- [169] J. Siciak, *Some applications of the method of extremal points*, Colloq. Math. **11** (1964), 209–249.
- [170] J. Siciak, *Asymptotic behaviour of harmonic polynomials bounded on a compact set*, Ann. Polon. Math. **20** (1968), 267–278.
- [171] J. Siciak, *Degree of convergence of some sequences in the conformal mapping theory*, Colloq. Math. **16** (1976), 49–59.
- [172] J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^N$* , Ann. Polon. Math. **39** (1981), 175–211.
- [173] J. Siciak, *Extremal Plurisubharmonic Functions and Capacities in $\mathbb{C}^N$* , Sophia Kōkyuroku in Mathematics, Vol. 14 (1982).
- [174] J. Siciak, *A remark on Tchebysheff polynomials in $\mathbb{C}^N$* , Preprint, Jagellonian University (1996).
- [175] G. Szegő, *Bemerkungen zu einer Arbeit von Herrn Fekete: “Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten”*, Math. Z. **21** (1924), 203–208.
- [176] R.E. Thurman, *Upper bound for distortion of capacity under conformal mapping*, Trans. Amer. Math. Soc. **346** (1994), 605–616.
- [177] R.E. Thurman, *Bridged extremal distance and maximal capacity*, Pacific. J. Math. **176** (1996), 507–528.
- [178] R.E. Thurman, *Maximal capacity, Robin capacity, and minimum energy*, Indiana Univ. Math. J. **46** (1997), 621–636.
- [179] V. Totik, *Weighted Approximation with Varying Weights*, Lectures Notes in Math., Vol. 1300, Springer-Verlag, Berlin–Heidelberg–New York (1994).
- [180] M. Tsuji, *Some metrical theorems on Fuchsian groups*, Japan. J. Math. **19** (1947), 483–516.
- [181] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959).
- [182] A.G. Vitushkin, *Conditions on a set which are necessary and sufficient in order that any continuous function, analytic at its interior points, admit uniform approximation by rational approximation*, Dokl. Akad. Nauk SSSR **171**, 1255–1258; Transl.: Soviet Math. Dokl. **7** (1966), 1622–1625.
- [183] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th edn, Amer. Math. Soc. Colloq. Publ., Vol. 20, Amer. Math. Soc., Providence, RI (1965).
- [184] J.L. Walsh and H.G. Russell, *On the convergence and overconvergence of sequences of polynomials of best simultaneous approximation to several functions analytic in distinct regions*, Trans. Amer. Math. Soc. **36** (1934), 13–28.
- [185] J. Weisel, *Lösung singulärer Variationsprobleme durch die Verfahren von Ritz und Galerkin mit finiten Elementenanwendungen in der konformen Abbildung*, Mitt. Math. Sem. Gießen **138** (1979), 156 S.
- [186] V.P. Zaharjuta, *Transfinite diameter, Chebyshev constants, and capacity for compacta in $\mathbb{C}^N$* , Math. USSR-Sb. **25** (1975), 350–364 (in Russian).
- [187] L. Zalcman, *Analytic Capacity and Rational Approximation*, Springer-Verlag, Berlin–Heidelberg–New York (1968).
- [188] E.I. Zolotarjov, *Collected Works II*, USSR Acad. Sci. (1932) (in Russian).

CHAPTER 7

Some Special Classes of Conformal Mappings

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1. Preliminary results

In this chapter, we investigate some families of univalent functions for which the image domain has a special geometric property. Among the families considered are convex and starlike mappings, close-to-convex mappings, spirallike mappings and typically real mappings. We will consider the connection between the geometry of the image domains and analytic properties of the mapping function. In this connection, some results of Robertson [24] are useful. The statements given in Theorems 1.1 and 1.2 are slightly different from Theorems A and B in [24] but they are clearly equivalent to those results. We denote the unit disk and the unit interval by $\Delta = \{z: |z| < 1\}$ and $I = [0, 1]$, respectively. In addition, if f and g are analytic in Δ , we use the symbol $g \prec f$ to mean that $g(z) = f(\omega(z))$, where $\omega(0) = 0$ and $|\omega| < 1$ on Δ .

THEOREM 1.1. *Suppose $\omega: \Delta \times I \rightarrow \mathbb{C}$ and that $\omega(z, t)$ is analytic as a function of z for each fixed $t \in I$ and continuous as a function of t . Further, assume $\omega(0, t) = 0$ for all $t \in I$ and $\omega(z, 0) = z$ for all $z \in \Delta$. Finally, assume*

$$\lim_{t \rightarrow 0+} \frac{z - \omega(z, t)}{zt} = \Omega(z)$$

for all $z \in \Delta$ and that Ω is analytic in Δ . Then $\operatorname{Re}(\Omega(z)) \geq 0$.

PROOF. Since $\omega(\cdot, t)$ satisfies the hypotheses of Schwarz lemma for each $t \in I$, we know that $|\omega(z, t)| \leq |z|$. Therefore

$$\begin{aligned} \operatorname{Re}(\Omega(z)) &= \lim_{t \rightarrow 0+} \left(\frac{1}{t} \left(1 - \operatorname{Re} \frac{\omega(z, t)}{z} \right) \right) \\ &\geq \lim_{t \rightarrow 0+} \left(\frac{1}{t} \left(1 - \frac{|\omega(z, t)|}{|z|} \right) \right) \\ &\geq 0. \end{aligned}$$

□

THEOREM 1.2. *Let $f: \Delta \rightarrow \mathbb{C}$ be analytic and univalent with $f(0) = 0$. Assume $F: \Delta \times I \rightarrow \mathbb{C}$ is analytic as a function of z for each $t \in I$, and continuous as a function of t . Further, assume $F(z, 0) = f(z)$, $F(0, t) = 0$ and suppose $F(z, t) \prec f(z)$ for each $t \in I$. If*

$$\lim_{t \rightarrow 0+} \frac{f(z) - F(z, t)}{t} = G(z)$$

exists and is analytic in Δ , then $\operatorname{Re}(\frac{G(z)}{zf'(z)}) \geq 0$.

PROOF. From the hypotheses, there is a function $v: \Delta \times I \rightarrow \Delta$ that is analytic as a function of the first variable, with $v(0, t) = 0$, $v(z, 0) = z$, and such that $F(z, t) = f(v(z, t))$.

Expanding $f(v(z, t))$ about z , we have $f(v(z, t)) = f(z) + f'(z)(v(z, t) - z) + R(z, t)$, where $\frac{R(z, t)}{v(z, t) - z} \rightarrow 0$ as $t \rightarrow 0+$. Therefore,

$$\frac{f(z) - F(z, t)}{t} = \frac{z - v(z, t)}{t} \left(f'(z) - \frac{R(z, t)}{z - v(z, t)} \right).$$

It is now clear that $v(z, t)$ satisfies the hypotheses of Theorem 1.1 with $\Omega(z) = \frac{G(z)}{zf'(z)}$ and the proof is complete. $\square$

2. Starlike and convex domains

A domain D is starlike with respect to a point $z_0 \in D$ if for every point $z \in D$ and all t , $0 < t < 1$, we have $tz_0 + (1 - t)z \in D$. We consider here univalent functions, f , on the disk Δ that satisfy the conditions $f(0) = 0$ and $f(\Delta)$ is starlike with respect to the origin.

DEFINITION 2.1. A function f , that is analytic and univalent on the unit disk with $f(0) = 0$ is starlike if $(1 - t)f < f$ for all t , $0 \leq t \leq 1$. The family of all analytic functions that are starlike with the additional normalization, $f'(0) = 1$, will be denoted by S^* .

Using Theorem 1.2 we easily prove a necessary analytic condition for a function f , to be starlike.

THEOREM 2.2. *Suppose f is starlike. Then*

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0. \quad (2.1)$$

PROOF. Set $F(z, t) = (1 - t)f(z)$ and apply Theorem 1.2. Since G of Theorem 1.2 is f , we conclude $\operatorname{Re} \left(\frac{f(z)}{zf'(z)} \right) \geq 0$. However, using the fact that f is univalent, we know that $f'(z)$ is never 0. Therefore $\lim_{z \rightarrow 0} \left(\frac{f(z)}{zf'(z)} \right) = 1$ and hence application of the minimum principle for harmonic functions implies $\operatorname{Re} \left(\frac{f(z)}{zf'(z)} \right) > 0$. The theorem now easily follows. $\square$

Actually, the converse of Theorem 2.2 is also true. The precise statement is as follows.

THEOREM 2.3. *Suppose f is analytic on Δ , with $f(z) = a_1z + a_2z^2 + \dots$, $a_1 \neq 0$, and assume (2.1) holds. Then f is starlike.*

PROOF. Since the only singularity of $\frac{f'(z)}{f(z)}$ is a simple pole at $z = 0$, for $r < 1$, the residue theorem yields the result

$$\int_{|z|=r} \frac{f'(z)}{f(z)} dz = 2\pi i.$$

Therefore, if $0 < \theta_2 - \theta_1 < 2\pi$ (using $z = re^{i\theta}$),

$$0 < \int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{zf'(z)}{f(z)} d\theta < \int_{\theta_1}^{\theta_1+2\pi} \operatorname{Re} \frac{zf'(z)}{f(z)} d\theta = \operatorname{Re} \int_{|z|=r} \frac{-if'(z)}{f(z)} dz = 2\pi. \quad (2.2)$$

We allow θ_2 to be variable and note that the first integral in (2.2) is $\Phi(\theta_2) = \arg(f(re^{i\theta_2})) - \arg(f(re^{i\theta_1}))$. It is clear that Φ is an increasing function and that the image of the circle $\{|z| = r\}$ is a simple closed curve that encloses a domain that is starlike with respect to the origin. The theorem now follows by applying the argument principle. $\square$

REMARK 2.4. Note that we have shown above that if f is starlike, and $0 < r < 1$, then the image of the disk $\{z: |z| < r\}$ under the mapping f is also starlike with respect to the origin.

We include here a useful formula known as Herglotz formula for analytic functions, P , that have positive real part in the unit disk. This, of course, includes the case $P(z) = \frac{zf'(z)}{f(z)}$, where f is starlike.

THEOREM 2.5. Suppose P is analytic in the unit disk with $P(0) = 1$ and $\operatorname{Re}(P(z)) > 0$ for all $z \in \Delta$. Then there exists a nondecreasing function $F: [-\pi, \pi] \rightarrow \mathbb{R}$ such that $F(\pi) - F(-\pi) = 2\pi$ and

$$P(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} dF(t). \quad (2.3)$$

REMARK 2.6. Alternatively, we may express the formula (2.3) in terms of measures by replacing $\frac{1}{2\pi} dF(t)$ by a probability measure $d\mu(t)$ on the unit circle.

We refer the reader to three different proofs of this formula. The first is [34], a standard advanced calculus approach arriving at the Stieltjes integral (2.3). The second is [25] which uses extreme point theory to arrive at an integral expressed in terms of probability measures. The third is [11] which has some features in common with each of the first two.

The analytic condition for convexity is also readily derived from Theorem 1.2. We use the fact that if f is a univalent mapping of the unit disk onto a convex domain then

$$\frac{f(ze^{it}) + f(ze^{-it})}{2} < f(z) \quad (2.4)$$

for $z \in \Delta$ and $t \in I$. The theorem is as follows.

THEOREM 2.7. Suppose f is an analytic and univalent function that maps the unit disk onto a convex domain. Then

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0 \quad (2.5)$$

for all $z \in \Delta$. Conversely, if f is analytic on the unit disk, $f'(0) \neq 0$ and (2.5) holds then f is a univalent mapping of Δ onto a convex domain.

PROOF. Assume f is analytic and univalent with $f(\Delta)$ convex. Let $\tau = \sqrt{t} \in [0, 1]$ and define $F(z, t) = \frac{f(ze^{i\tau}) + f(ze^{-i\tau})}{2}$. Expanding $F(\cdot, t)$ in a series about z yields $F(z, t) = f(z) + zf'(z)(\cos(\tau) - 1) + \frac{1}{2}z^2f''(z)(\cos(2\tau) - 2\cos(\tau) + 1) + o(t)$. It now follows that $F(z, t)$ satisfies the hypotheses of Theorem 1.2 with $G(z) = \frac{1}{2}(zf'(z) + z^2f''(z))$. This proves (2.5).

Now assume (2.5) holds and $f'(0) \neq 0$. Without loss of generality, we may assume f is normalized so that $f(0) = 0$ and $f'(0) = 1$. Now (2.5) yields the result that $zf'(z)$ is starlike. Thus, if $r = |z|$ is fixed, $0 < r < 1$, then the normal $(zf'(z))$ to the curve $f(z)$, $0 < \arg(z) < 2\pi$, turns monotonically with $\arg(z)$. We want to show that f is univalent. To that end, it is sufficient to show that $\{f(re^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ is a simple closed curve for each r , $0 < r < 1$. Choose $z_j = re^{i\theta_j}$, $j = 1, 2$, $0 < \theta_2 - \theta_1 < 2\pi$. We know

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z) dz = \int_{\theta_1}^{\theta_2} izf'(z) d\theta, \quad z = re^{i\theta}.$$

Choose a branch of $\arg(w)$ in a neighborhood of $z_1f'(z_1)$, and let $\phi = \arg(z_1f'(z_1))$. Now choose $z_3 = re^{i\theta_3}$ so that $\theta_1 < \theta_3$ and so that $\arg(z_3f'(z_3)) = \phi + \pi$. Then $\operatorname{Re}(e^{-i\phi}zf'(z)) > 0$ when $\theta_1 < \theta < \theta_3$ and $\operatorname{Re}(e^{-i\phi}zf'(z)) < 0$ when $\theta_3 < \theta < \theta_1 + 2\pi$, where $z = re^{i\theta}$. Now, if $\theta_1 < \theta_2 \leq \theta_3$, we have

$$\operatorname{Re}(-ie^{-i\phi}(f(z_2) - f(z_1))) = \int_{\theta_1}^{\theta_2} \operatorname{Re}(e^{-i\phi}zf'(z)) d\theta > 0$$

while

$$\operatorname{Re}(-ie^{-i\phi}(f(z_2) - f(z_1))) = \int_{\theta_2}^{\theta_1+2\pi} \operatorname{Re}(e^{-i\phi}zf'(z)) d\theta < 0$$

when $\theta_3 < \theta_2 < \theta_1 + 2\pi$. Thus, we have shown that the image of the circle $|z| = r$ is a simple closed curve and it follows that f is univalent. To show that $f(\Delta)$ is convex, it is sufficient to show that $f(|z| \leq r)$ is convex for each r , $0 < r < 1$. We use the fact that if ϕ is the angle between the positive real axis and the tangent line to the curve $\gamma(\theta) = \{f(re^{i\theta})\}$, $0 \leq \theta \leq 2\pi$ for fixed r , then

$$\frac{\partial \phi}{\partial \theta} = \operatorname{Re}\left(\frac{f''(z)}{f'(z)} + 1\right) > 0, \quad z = re^{i\theta}. \quad (2.6)$$

Assume there is an $r < 1$ such that $\Omega_r = f(|z| < r)$ is not convex. Then there is a line that is tangent to γ at more than one point. This contradicts (2.6). $\square$

DEFINITION 2.8. A function, f , that is analytic in the unit disk Δ is convex if it is univalent, $f(0) = 0$ and $f(\Delta)$ is convex. The family of all analytic functions that are convex with the additional normalization $f'(0) = 1$ will be denoted by K .

Comparing the conditions (2.5) and (2.1) for convexity and starlikeness, the following result is clear.

COROLLARY 2.9. *Suppose f is analytic on the unit disk. Then $f \in K$ if and only if $zf'(z) \in S^*$.*

REMARK 2.10. It is clear from the above proof that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $|z| < 1$, is convex if and only if the image of every disk $|z| < r$, $0 < r < 1$ is convex.

The boundary behavior of conformal mappings of the disk, is considered in most graduate level texts in Complex Analysis. We mention the following theorem of Carathéodory [10, Theorem 4, p. 44]. See also [7] and [22, Theorem 2.1, p. 20].

THEOREM 2.11. *A univalent conformal mapping of a domain B bounded by a closed Jordan curve onto the disk $|\zeta| < 1$ maps $\overline{B}$ bicontinuously onto the closed disk $|\zeta| \leq 1$.*

For convex mappings, we obtain the following theorem essentially due to Study [29, pp. 106–107] concerning the boundary behavior. The precise statement given here is in [31, Theorem 1, p. 796].

THEOREM 2.12. *Let $f \in K$. Either*

- (a) *f extends to the closed disk, $|z| \leq 1$, and is continuous and one-to-one,*
or
- (b) *f extends to the closed disk $|z| \leq 1$ and is continuous and one-to-one with the exception of one point, say z_0 , $|z_0| = 1$. For the exceptional point, $\lim_{z \rightarrow z_0} f(z) = \infty$,*
or
- (c) *$f(\Delta)$ is an infinite strip. In this case, f assumes the value infinity at two points on the boundary of the disk.*

REMARK 2.13. We note for future reference that in (c) above, the normalized functions that map the unit disk to an infinite strip are all rotations of the functions

$$f(z) = \frac{1}{2i \sin(\alpha)} \log \left(\frac{1 - ze^{-i\alpha}}{1 - ze^{i\alpha}} \right), \quad 0 < \alpha < \pi. \quad (2.7)$$

Marx and Stroh  cker [18] and [28] obtained some interesting and useful geometric properties of convex functions. The theorem which follows, see [27,30] and [32], is interesting and in addition leads to simple proofs of the Marx–Stroh  cker results.

THEOREM 2.14. *Assume f is convex. Then $\operatorname{Re}(F(z, \zeta)) > 0$ for all $z, \zeta \in \Delta$, where*

$$F(z, \zeta) = \begin{cases} \left(\frac{2zf'(z)}{f(z)-f(\zeta)} - \frac{z+\zeta}{z-\zeta} \right) & \text{if } z \neq \zeta, \\ \frac{zf''(z)}{f'(z)} + 1 & \text{if } z = \zeta. \end{cases} \quad (2.8)$$

PROOF. Since f is univalent, the function $F(z, \zeta)$ is analytic in both z and ζ when $|z| < 1$, $|\zeta| < 1$ and $z \neq \zeta$. However, it is easy to check that $\lim_{z \rightarrow \zeta} F(z, \zeta) = F(z, z)$ so that $F(z, \zeta)$ is analytic in both z and ζ when $z, \zeta \in \Delta$. For fixed ζ , $f(\Delta)$ is starlike with respect to $f(\zeta)$. Therefore when $|\zeta| < |z| < 1$, we have

$$\frac{\partial(\arg(f(z) - f(\zeta)))}{\partial \arg(z)} = \operatorname{Re} \frac{zf'(z)}{f(z) - f(\zeta)} > 0.$$

By continuity, it follows that $\operatorname{Re} \frac{zf'(z)}{f(z) - f(\zeta)} \geq 0$ when $|z| = |\zeta|$, $z \neq \zeta$. Since $\operatorname{Re}(\frac{z+\zeta}{z-\zeta}) = 0$ when $|z| = |\zeta|$, $z \neq \zeta$, the inequality $\operatorname{Re}(F(z, \zeta)) \geq 0$ follows on the distinguished boundary $|z| = |\zeta| = r$ where we have used continuity at $z = \zeta$. Now apply the minimum principle for harmonic functions (of ζ) to conclude that (2.8) holds when $|\zeta| \leq |z|$. Similarly, the minimum principle for harmonic functions (of z) implies that (2.8) holds when $|z| \leq |\zeta|$. This concludes the proof. $\square$

COROLLARY 2.15 (Marx [18] and Stroh  cker [28]). *If f is convex, then*

- (a) $\operatorname{Re}(\frac{zf'(z)}{f(z)}) > \frac{1}{2}$
- and
- (b) $\operatorname{Re}(\frac{f(z)}{z}) > \frac{1}{2}$.
- (c) *Suppose, in addition that f is not of the form $\frac{z}{1-\gamma z}$, $|\gamma| = 1$. Then there exists $\alpha > 1/2$ such that $\operatorname{Re}(zf'(z)f(z)) > \alpha$.*

PROOF. To prove (a), set $\zeta = 0$ in the definition of $F(z, \zeta)$.

To prove (b), fix ζ and consider $F(z, \zeta)$ as a power series in z . We will use the well-known fact if $P(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $\operatorname{Re}(P(z)) > 0$ for $z \in \Delta$ then $|a_n| \leq 2$. Since $F(0, \zeta) = 1$ and $\frac{1}{z}(F(z, \zeta) - 1)|_{z=0} = 2(\frac{1}{\zeta} - \frac{1}{f(\zeta)})$, we conclude $|\frac{1}{f(\zeta)} - \frac{1}{\zeta}| \leq 1$. Thus, $|\frac{\zeta}{f(\zeta)} - 1| \leq |\zeta| < 1$ when $|\zeta| < 1$. The inequality (b) now follows from the fact that under the mapping $w \rightarrow \frac{1}{w}$ the disk $|w - 1| < 1$ maps to the half-plane $\operatorname{Re}(w) > 1/2$.

Now assume there is a convergent sequence $\{z_n\} \subset \Delta$ such that $\lim_{n \rightarrow \infty} (\frac{z_n f'(z_n)}{f(z_n)}) = c$, where $\operatorname{Re}(c) = 1/2$. Using (a), we see that $\lim_{n \rightarrow \infty} z_n = w$ where $|w| = 1$. Define $F(z, \zeta)$ as in (2.8). If $f(z_n) \rightarrow \infty$ then $F(z_n, \zeta) \rightarrow 2c - \frac{w+\zeta}{w-\zeta}$. When $\zeta = 0$, this last quantity is purely imaginary. In view of the fact that $\operatorname{Re}(F(z, \zeta)) > 0$ when $|z| < 1$ and $|\zeta| < 1$, it follows that $\frac{w+\zeta}{w-\zeta}$ must be independent of ζ . Since this is not true, using Theorem 2.12, we conclude $\{f(z_n)\}$ has a finite limit $f(w)$ and $z_n f'(z_n) \rightarrow cf(w)$. Since the limit value $F(w, 0)$ is purely imaginary, we conclude that $F(w, \zeta) = 2c - 1$ for all ζ in the disk. Thus we have

$$\frac{2cf(w)}{f(w) - f(\zeta)} - \frac{w + \zeta}{w - \zeta} = 2c - 1.$$

Solving this equation for $f(\zeta)$ yields

$$f(\zeta) = \frac{\zeta f(w)}{cw + (1 - c)\zeta}.$$

Dividing by ζ and letting $\zeta \rightarrow 0$, we see that $f(w) = cw$ and $f(\zeta) = \frac{\zeta}{1-\gamma\zeta}$, where $\gamma = \frac{c-1}{cw}$. Since $|c-1| = |c|$, the proof is now complete. $\square$

3. Coefficient inequalities and growth rates

In this section we find the bounds on the coefficients and the rate of growth for normalized convex and starlike functions.

THEOREM 3.1. *If $f(z) = \sum_{n=1}^{\infty} a_n z^n \in K$, then $|a_n| \leq 1$ for $2 \leq n$, with equality if and only if $f(z) = \frac{z}{1-\gamma z}$ for some γ , $|\gamma| = 1$.*

PROOF. Since $f \in K$, the function g given by

$$g(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(ze^{2i\pi k/n}) \quad (3.1)$$

is subordinate to f .

Thus we have

$$\begin{aligned} g(z) &= a_n z^n + a_{2n} z^{2n} + \cdots \\ &= f(\omega(z)) \\ &= \omega(z) + a_2 \omega(z)^2 + \cdots, \end{aligned} \quad (3.2)$$

where $|\omega(z)| \leq |z|^n$ and in fact, $\omega(z) = b_n z^n + \cdots$ with $|b_n| \leq 1$. Equating coefficients, we see that $a_n = b_n$ and the first part of the theorem is proved.

Now assume $|a_n| = 1$ for some n , $2 \leq n$. Then by an obvious generalization of Schwarz lemma, it follows that $\omega(z) = \alpha z^n$ for some α , $|\alpha| = 1$. From (2.7), we know that for functions that map to a strip domain, the n th coefficient has absolute value $|\frac{\sin(n\alpha)}{n \sin(\alpha)}|$ for some α , $0 < \alpha < \pi$ which is strictly less than one. Thus f satisfies (a) or (b) of Theorem 2.12. After possibly rotating f , we may assume that f is continuous on the arc $\{e^{i\theta}: 0 < \theta < 2\pi\}$. Now choose θ so that $0 < \theta < 2\pi/n$, and consider (3.1) and (3.2) with $z = e^{i\theta}$. Since $\alpha z^n = \omega(z)$ is a boundary point, it follows that the arc $\{e^{i\phi}: \theta < \phi < \theta + 2(n-1)\pi/n\}$ maps to a line, L . By continuity, the entire unit circle except for the point 1 maps into L . It now follows that $\Omega = f(\Delta)$ is a half-plane, for if $\zeta \in \Omega$ then by convexity, the line through ζ parallel to L is entirely in Ω . This completes the proof. $\square$

Using the fact that $f \in K$ if and only if $zf' \in S^*$, the following corollary is clear.

COROLLARY 3.2. *If $f \in S^*$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_n| \leq n$ with equality if and only if $f(z) = \frac{z}{(1-\gamma z)^2}$ for some γ , $|\gamma| = 1$.*

THEOREM 3.3. (a) If $f \in K$, then $\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|}$ with equality at $z = -1$ and 1 , respectively, when $f(z) = \frac{z}{1-z}$.

(b) If $f \in S^*$, then $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$ with equality at $z = -1$ and 1 , respectively, when $f(z) = \frac{z}{(1-z)^2}$.

PROOF. The upper bound in (a) follows from

$$|f(z)| \leq \sum_{n=1}^{\infty} |a_n| |z|^n \leq \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1-|z|}.$$

The lower bound in (a) follows from the fact that $\operatorname{Re}\left(\frac{f(z)}{z}\right) > 1/2$ (Corollary 2.15). This implies that $\frac{f(z)}{z} < \frac{1}{1-z}$ so that $f(z) = \frac{z}{1-\omega(z)}$ where $|\omega(z)| \leq |z|$ and the lower bound in (a) now easily follows.

The bounds in (b) are the bounds for the whole class S of normalized univalent functions. $\square$

4. Radii of starlikeness and convexity for S

Suppose $f \in S$. That is, f is univalent in the unit disk Δ and is normalized by assuming $f(0) = 0$, $f'(0) = 1$. Thus, any $f \in S$ has an expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

valid for $|z| < 1$.

For an arbitrary $f \in S$ we have $\frac{zf'(z)}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k$. Thus, $\frac{zf'(z)}{f(z)}$ has a removable singularity at $z = 0$ and $\left[\frac{zf'(z)}{f(z)}\right]_{z=0} = 1$. It follows that for each $f \in S$, there is a number $\rho > 0$ such that $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ when $|z| < \rho$. Under these conditions, f maps the disk $\{|z| < \rho\}$ onto a domain that is starlike with respect to the origin.

Similar comments can be made with respect to the quantity $\frac{zf''(z)}{f'(z)} + 1$ to conclude that for every $f \in S$ there is a $\rho > 0$ such that f maps the disk $\{|z| < \rho\}$ onto a convex domain.

DEFINITION 4.1. Suppose $\mathcal{F}$ is a subfamily of the class S . The radius of starlikeness of the family $\mathcal{F}$ is $\sup\{\rho \leq 1: \text{for every } f \in \mathcal{F}, f(|z| < \rho) \text{ is starlike with respect to the origin}\}$. Similarly, the radius of convexity of $\mathcal{F}$ is $\sup\{\rho \leq 1: \text{for every } f \in \mathcal{F}, f(|z| < \rho) \text{ is convex}\}$.

Of course, it is not immediately obvious that the radius of starlikeness or convexity of the family S is positive. However, both radii are positive and we have the following theorems.

THEOREM 4.2. The radius of convexity of the family S is $2 - \sqrt{3}$.

PROOF. We use the fact that whenever $f \in S$ and $|a| < 1$ the function

$$g(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{f'(a)(1-|a|^2)}$$

is also in the family S . Further, we know that the second coefficient, a_2 , for functions in S is bounded by 2. Applying this bound to g yields

$$\left| \frac{1}{2} \frac{f''(a)}{f'(a)} (1 - |a|^2) - \bar{a} \right| \leq 2.$$

Multiplying by $\frac{2a}{1-|a|^2}$ yields

$$\left| \frac{af''(a)}{f'(a)} - \frac{2|a|^2}{1-|a|^2} \right| \leq \frac{4|a|}{1-|a|^2}$$

or

$$\left| \frac{af''(a)}{f'(a)} + 1 - \left(\frac{1+|a|^2}{1-|a|^2} \right) \right| \leq \frac{4|a|}{1-|a|^2}$$

and finally,

$$\frac{1-4|a|+|a|^2}{1-|a|^2} \leq \operatorname{Re} \left(\frac{af''(a)}{f'(a)} + 1 \right). \quad (4.1)$$

From (4.1), one can readily see that $\operatorname{Re}(\frac{af''(a)}{f'(a)} + 1) > 0$ when $|a| < 2 - \sqrt{3}$. This implies that the radius of convexity of S is at least $2 - \sqrt{3}$. Now take $f(z) = \frac{z}{(1-z)^2}$. This is the Koebe function and it satisfies $\frac{zf''(z)}{f'(z)} + 1 = \frac{1+4z+z^2}{(1-z^2)}$. The last quantity is zero when $z = -(2 - \sqrt{3})$ and this shows that the radius of convexity is no larger than $2 - \sqrt{3}$. $\square$

It is somewhat more challenging to find the radius of starlikeness for the family S .

We begin with an extremal problem concerning $|\arg(\frac{f(z)}{z})|$

THEOREM 4.3. *If $f \in S$, then $|\arg(\frac{f(z)}{z})| \leq \log(\frac{1+|z|}{1-|z|})$ and this result is sharp.*

PROOF. Two methods of proving this result are by use of the Löwner equation [17] or [10, p. 112] and by use of the theory concerning conformal mapping of multiply-connected domains [19, Chapter VII, Section 2, Exercise 11]. We will use the first approach. From the Löwner theory we obtain the following. Given a continuous function $k(t)$, $0 \leq t$, $|k(t)| = 1$ for all t , the problem

$$\frac{\partial f(z, t)}{\partial t} = -f(z, t) \frac{1+k(t)f(z, t)}{1-k(t)f(z, t)}, \quad f(z, 0) = z, \quad (4.2)$$

has a solution that is analytic and univalent in the variable z , $|z| < 1$. Further, $f(0, t) = 0$, and $\frac{\partial f(0, t)}{\partial z} = e^{-t}$. Finally, the limit

$$F(z) = \lim_{t \rightarrow \infty} e^t f(z, t)$$

(the uniform limit on compact sets) exists and is in the family S . In addition, the set of all F obtained in this way is dense in the family S . We use these results to complete the proof. Following Golusin, divide (4.2) by $f(z, t)$, $z \neq 0$, to get the local result

$$\frac{\partial(\log(f(z, t)))}{\partial t} = \frac{1 + k(t)f(z, t)}{1 - k(t)f(z, t)}. \quad (4.3)$$

Taking real and imaginary parts of both sides of (4.3) yields

$$\frac{\partial|f(z, t)|}{\partial t} = -|f(z, t)| \frac{1 - |f(z, t)|^2}{|1 + k(t)f(z, t)|^2} \quad (4.4)$$

and

$$\frac{\partial \arg(f(z, t))}{\partial t} = -\frac{2 \operatorname{Im}(k(t)f(z, t))}{|1 + k(t)f(z, t)|^2}. \quad (4.5)$$

Thus, $|f(z, t)|$ decreases from $|z|$ to 0 as t increases from 0 to ∞ . Since the factor e^t will not affect the argument,

$$\begin{aligned} \left| \int_0^\infty \frac{\arg(e^t f(z, t))}{\partial t} dt \right| &\leq \int_0^\infty \frac{2|f(z, t)|}{|1 - k(t)f(z, t)|^2} dt \\ &= \int_0^\infty -\frac{1}{1 - |f(z, t)|^2} \frac{\partial|f(z, t)|}{\partial t} dt. \end{aligned}$$

Using the fact that $f(z, 0) = z$ and $f(z, t) \rightarrow 0$ as $t \rightarrow \infty$, the integration completes the proof of the inequality in Theorem 4.3. To see that it is sharp, use (4.3) and (4.4) assuming $k(t)f(z, t) = -|f(z, t)|$ (for the fixed z) in order to find $k(t)$. Using this assumption, and the fact that $f(z, 0) = z$ we integrate (4.4) to see that

$$\frac{|f(z, t)|}{1 - |f(z, t)|^2} = e^{-t} \frac{|z|}{1 - |z|^2}. \quad (4.6)$$

Since the function $\frac{x}{1-x^2}$ is an increasing function, any solution of (4.6) decreases as t increases. That is, $|f(z, t)|$ is determined as a function of t . Thus, combining (4.4) and (4.5), $\arg(\frac{f(z, t)}{z})$ is determined as a function of t with $\arg(\frac{f(z, 0)}{z}) = 0$. Now define $k(t)$ by $|k(t)| = 1$ and $\arg(k(t)) = -\arg(f(z, t)) - \frac{\pi}{2}$. For this choice of $k(t)$ equality holds in each step of the proof of the inequality in Theorem 4.3 and the theorem is proved. $\square$

THEOREM 4.4. *The sharp inequality*

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| \leq \log \left(\frac{1+|z|}{1-|z|} \right)$$

holds for $f \in S$ and the radius of starlikeness for the family S is

$$r = \tanh \left(\frac{\pi}{4} \right).$$

PROOF. For arbitrary a , $|a| < 1$, the equation

$$g(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{f'(a)(1-|a|^2)}$$

yields a function $g \in S$ whenever $f \in S$ and conversely determines a function $f \in S$ whenever g is a given function of S . If we choose $z = -a$, then from Theorem 4.3 the sharp inequality

$$\begin{aligned} \log \left(\frac{1+|a|}{1-|a|} \right) &\geq \left| \arg \left(\frac{g(-a)}{-a} \right) \right| \\ &= \left| \arg \left(\frac{f(a)}{af'(a)(1-|a|^2)} \right) \right| \\ &= \left| \arg \left(\frac{af'(a)}{f(a)} \right) \right| \end{aligned} \quad (4.7)$$

follows. This proves the first part of Theorem 4.4.

For the second part, observe that the left-hand side of (4.7) is $\frac{\pi}{2}$ when $|a| = \tanh(\frac{\pi}{4})$. □

5. Further properties of starlike functions

The family of functions that are starlike of order $\alpha \leq 1$ (see the definition below) can be characterized in terms of positive semidefinite forms.

DEFINITION 5.1. Suppose f is analytic in the unit disk with $f(0) = 0$ and $f'(0) = 1$. Then for a given $\alpha \leq 1$, f is starlike of order α if $\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \alpha$. The family of functions that are starlike of order α will be denoted by St_α .

REMARK 5.2. (a) It is clear that the only function that is starlike of order 1 is $f(z) = z$. If $\alpha < 1$ then we may use strict inequality in the definition.

(b) A function that is starlike of order $\alpha < 0$, may not be univalent. However the definition still applies.

In [26, pp. 220–221] Schur proves the following results. Given a function

$$f(z) = \frac{a_0 + a_1 z + \cdots}{b_0 + b_1 z + \cdots}$$

that is analytic in the unit disk, for a sequence $Z = \{z_n\}$ with $\limsup |z_n|^{1/n} < 1$, define $\mathcal{H}(Z) = \mathcal{H}(z_0, z_1, \dots)$ by

$$\begin{aligned} \mathcal{H}(z_0, z_1, \dots) &= \sum_{k=0}^{\infty} \left(\left| \sum_{n=0}^{\infty} b_n z_k + b_{n+1} z_{k+1} + \cdots \right|^2 - \left| \sum_{n=0}^{\infty} a_n z_k + a_{n+1} z_{k+1} + \cdots \right|^2 \right), \\ \mathcal{H}_v(z_0, z_1, \dots, z_v) &= \mathcal{H}(z_0, z_1, \dots, z_v, 0, 0, \dots) \\ &= \sum_{k=0}^v (|b_0 z_k + b_1 z_{k+1} + \cdots + b_{v-k} z_v|^2 - |a_0 z_k + a_1 z_{k+1} + \cdots + a_{v-k} z_v|^2). \end{aligned}$$

THEOREM 5.3. *The Hermitian form $\mathcal{H}$ is positive semidefinite if and only if $|f(z)| \leq 1$ for all $z \in \Delta$. In addition, if $|f(z)| < 1$ for all $z \in \Delta$, $\mathcal{H}_k$ is positive definite when $k < v$ but $\mathcal{H}_v$ is not positive definite, i.e., $\mathcal{H}_v(z_0, z_1, \dots, z_v) = 0$ for some nontrivial sequence $\{z_k\}_{k=0}^v$ then $\mathcal{H}_k$ is not positive definite for $k = v + 1, v + 2, \dots$, and f has the form*

$$f(z) = \gamma \prod_{n=1}^v \frac{z + c_n}{1 + \bar{c}_n z}, \quad (5.1)$$

where $|\gamma| = 1$.

We apply Theorem 5.3 to the family St_α as follows. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in the unit disk and that $\alpha < 1$. The following are equivalent:

$$\begin{aligned} \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) &> \alpha, \\ \left| \frac{zf'(z)}{f(z)} - (2\alpha - 1) \right| &> \left| \frac{zf'(z)}{f(z)} - 1 \right|, \\ |zf'(z) - (2\alpha - 1)f(z)| &> |zf'(z) - f(z)|, \\ 1 &> \left| \frac{f'(z) - f(z)/z}{f'(z) - (2\alpha - 1)f(z)/z} \right|, \\ 1 &\geq \left| \frac{\frac{1}{z}(f'(z) - f(z)/z)}{f'(z) - (2\alpha - 1)f(z)/z} \right|. \end{aligned} \quad (5.2)$$

Now let

$$\omega(z) = \frac{\frac{1}{z}(f'(z) - f(z)/z)}{f'(z) - (2\alpha - 1)f(z)/z}$$

and apply Theorem 5.3 to ω . The result is the following theorem.

THEOREM 5.4. (a) $f \in St_\alpha$ if and only if

$$\mathcal{H}(Z) = \sum_{k=0}^{\infty} \left(\left| \sum_{n=1}^{\infty} (n+1-2\alpha) a_n z_{n+k} \right|^2 - \left| \sum_{n=1}^{\infty} n a_{n+1} z_{n+k} \right|^2 \right)$$

is positive semidefinite where $Z = \{z_n\}_{n=1}^{\infty}$, and $\limsup |z_n|^{1/n} < 1$.

(b) If $\mathcal{H}$ is positive semidefinite, but $\mathcal{H}_k$ is not positive definite for some k , then f has the form

$$f(z) = \frac{z}{\prod_{n=1}^k (1 - e^{i\phi_n} z)^{s_n}},$$

where $s_n \geq 0$ for each n and $\sum s_n = 2 - 2\alpha$.

PROOF. (a) above is clear. To prove (b), assume $\mathcal{H}$ is positive semidefinite and that $\mathcal{H}_k$ is not positive definite. By Theorem 5.3, ω is a product of holomorphic automorphisms of the disk. Solving for $\frac{zf'(z)}{f(z)}$ we find that

$$\frac{zf'(z)}{f(z)} = \alpha + (1 - \alpha) \frac{1 + z\omega}{1 - z\omega}.$$

The quantity $P(z) = \frac{1+z\omega}{1-z\omega}$ is rational of degree $\leq k$, and has positive real part. Note that ω has degree less than or equal to $k-1$ rather than k because of the change in indexing Z . Since P is purely imaginary on the unit circle, P has the form

$$P(z) = \sum_{n=1}^k t_n \frac{1 + e^{i\phi_n} z}{1 - e^{i\phi_n} z},$$

where each $t_n > 0$ and $\sum_{n=1}^k t_n = 1$. Part (b) now easily follows. $\square$

For a different approach to this result, see [33, Theorem 1]

THEOREM 5.5. Let $n \geq 2$ be a fixed integer, and assume $\alpha < 1$. Further, assume $a_1, a_2, \dots, a_n$ are given with $a_1 = 1$. Then there exist $a_{n+1}, a_{n+2}, \dots$ such that the function $f(z) = \sum_{k=1}^{\infty} a_k z^k \in St_\alpha$ if and only if the quadratic form $\mathcal{H}_{n-1}$ of Theorem 5.4 (this form depends only on the first n coefficients) is positive semidefinite.

PROOF. This result follows from the corresponding result in [26] for bounded functions. Also see [33, Theorem 2]. $\square$

Coefficient bounds follow readily for the families St_α .

THEOREM 5.6. *If $\alpha < 1$ is fixed and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in St_\alpha$, then*

$$|a_n| \leq \frac{\Gamma(n+1-2\alpha)}{\Gamma(2-2\alpha)(n-1)!}, \quad n = 2, 3, \dots,$$

with equality if and only if

$$f(z) = \frac{z}{(1-\gamma z)^{2-2\alpha}}, \quad |\gamma| = 1. \quad (5.3)$$

PROOF. For a fixed positive integer k choose $z_k = 1$ and $z_j = 0$, if $j \neq k$. From Theorem 5.4 we conclude

$$\sum_{j=1}^k |(j+1-2\alpha)a_j|^2 \geq \sum_{j=1}^k |ja_{j+1}|^2.$$

This is equivalent to

$$4(1-\alpha) \sum_{j=1}^k (j-\alpha)|a_j|^2 \geq k^2 |a_{k+1}|^2.$$

The inequality now follows by induction. In order for equality to hold, it must be true that $|a_2| = 2 - 2\alpha$. However this means $\mathcal{H}_1 = 0$ and an application of part (b) of Theorem 5.4 completes the proof. $\square$

The functions (5.3) are the extreme points in the family St_α [5].

Some extremal problems in the family St_α with the side condition $a_2 = 0$, have solutions that are odd functions. See [33, p. 442].

THEOREM 5.7. *If $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ (observe that $a_2 = 0$) is starlike of order α , then*

$$\frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)(k-1)!} \geq |a_{2k-1}| \quad (5.4)$$

and

$$\frac{2k-2}{2k-1} \frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)(k-1)!} \geq |a_{2k}|. \quad (5.5)$$

The inequality (5.4) is sharp with equality for the odd function $f(z) = \frac{z}{(1-z^2)^{1-\alpha}}$ while (5.5) is sharp only when $k = 2$ with equality in this case for $f(z) = \frac{z}{(1-z^3)^{2(1-\alpha)/3}}$.

PROOF. First consider $\mathcal{H}_2(0, \lambda) = ((2 - 2\alpha)^2 - 4|a_3|^2)|\lambda|^2$, where we have used the fact that $a_2 = 0$. (5.4) now follows for $k = 2$.

Now consider $\mathcal{H}_3(\lambda, 0, \mu) = |(2 - 2\alpha)\lambda + (3 - 2\alpha)a_3\mu|^2 + (2 - 2\alpha)^2|\mu|^2 - 4|a_3|^2|\mu|^2 - 9|a_4|^2|\mu|^2 \geq 0$. Choose λ to make the first term = 0. The result is $(2 - 2\alpha)^2 - 4|a_3|^2 \geq 9|a_4|^2$. Inequality (5.5) now follows for $k = 2$. In general consider $\mathcal{H}_n(z_1, 0, 0, \dots, z_n)$ where again, z_1 is chosen to make the first term = 0. The result is $(2 - 2\alpha)^2 + \sum_{k=3}^{n-1} [(k + 1 - 2\alpha)^2 - (k - 1)^2]|a_k|^2 - (n - 1)^2|a_n|^2 - n^2|a_{n+1}|^2 \geq 0$. This simplifies to $4(1 - \alpha) \sum_{k=1}^{n-1} (k - \alpha)|a_k|^2 - (n - 1)^2|a_n|^2 \geq n^2|a_{n+1}|^2$. The proof is completed by induction on the statements

$$4(1 - \alpha) \sum_{k=1}^{2j} (k - \alpha)|a_k|^2 \leq \frac{(2j)^2 \Gamma^2(j + 1 - \alpha)}{\Gamma^2(1 - \alpha)((j - 1)!)^2}$$

and

$$4(1 - \alpha) \sum_{k=1}^{2j-1} (k - \alpha)|a_k|^2 \leq \frac{(2j)^2 \Gamma^2(j + 1 - \alpha)}{\Gamma^2(1 - \alpha)(j!)^2}$$

using the fact that $a_1 = 1$ and $a_2 = 0$. The statements concerning equality are straightforward to check. $\square$

6. Close to convex functions

The family of close to convex functions was introduced by Kaplan [13] in 1952 although some of these ideas had been used prior to that time. See, for example, Alexander [1] and Biernacki [4].

We begin with a theorem of Noshiro and Warschawski; see [12,20] and [35].

THEOREM 6.1. *Suppose f is analytic in a convex domain D and that $\operatorname{Re}(f'(z)) > 0$ for all $z \in D$. Then f is univalent on D .*

PROOF. Let $z_1, z_2 \in D$, $z_1 \neq z_2$. Then

$$f(z_2) - f(z_1) = \int_{\Gamma} f'(z) dz,$$

where Γ is the line segment, $z(t) = tz_2 + (1 - t)z_1$ and

$$\operatorname{Re}\left(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right) = \int_0^1 \operatorname{Re}(f'(z(t))) dt > 0.$$

Thus $f(z_2) \neq f(z_1)$ and this proves univalence in D . $\square$

Using this theorem we prove the following result of Kaplan [13].

THEOREM 6.2. *Suppose f is analytic on the unit disk, Δ , and assume φ is a conformal mapping of Δ onto a convex domain. Further, assume $\operatorname{Re}(\frac{f'(z)}{\varphi'(z)}) > 0$ for all $z \in \Delta$. Then f is univalent on Δ .*

PROOF. Let $D = \varphi(\Delta)$. Define $g: D \rightarrow \mathbb{C}$ by $g(z) = f \circ \varphi^{-1}(z)$. Then $\operatorname{Re}(g'(z)) = \operatorname{Re}(\frac{f'}{\varphi}(w)) > 0$ where $w = \varphi^{-1}(z)$. Using Theorem 6.1, the theorem now follows. $\square$

DEFINITION 6.3. The class $\mathcal{C}$ of close to convex functions is the family of analytic functions, f , on Δ such that $f(0) = 0$, $f'(0) = 1$ and

$$\operatorname{Re}\left(\frac{f'(z)}{\varphi'(z)}\right) > 0 \quad \text{for all } z \in \Delta, \quad (6.1)$$

for some convex mapping φ .

REMARK 6.4. (a) The condition (6.1) is equivalent to the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for all } z \in \Delta,$$

where g is a starlike mapping.

(b) It is not assumed that $\varphi'(0)$ (or $g'(0)$) is real.

EXAMPLE 6.5. Define

$$f(z) = \frac{z(1 - \gamma z)}{(1 - z)^2}$$

where $\gamma = \frac{1 - e^{2i\alpha}}{2}$, $|\alpha| < \frac{\pi}{2}$. Set $g(z) = \frac{e^{i\alpha}z}{(1 - z)^2}$. Then g is starlike and $\frac{zf'(z)}{g(z)} = \frac{e^{-i\alpha}(1 + e^{2i\alpha}z)}{1 - z}$ is a function with positive real part in the disk. Therefore, f is close to convex. It is not difficult to check that $f(\Delta)$ is the complement of the slit that goes through the point $-\frac{1}{2} = f(\pm ie^{-i\alpha})$ with the tip of the slit at $f(-e^{-2i\alpha}) = -\frac{1}{4} + i\frac{\tan(\alpha)}{4}$. These functions together with their rotations contain the extreme points in the family $\mathcal{C}$ [6], see Theorem 9.6. As shown in [6], the extreme points are the rotations of the functions above for which the boundary point that is nearest the origin is the tip of the slit. This means, $|\alpha| \leq \frac{\pi}{4}$.

Suppose f is close to convex and g is starlike with

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for all } z \in \Delta. \quad (6.2)$$

For fixed r , $0 < r < 1$, let C_r be the circle $\{|z| = r\}$. Then since g is starlike, $\arg(g(z))$ is an increasing function of $\arg(z)$ on C_r . In view of (6.2), this means that on any arc of C_r , say $\{z: |z| = r \text{ and } \arg z_1 = \theta_1 \leq \arg z \leq \theta_2 = \arg z_2\}$, we must have

$$\arg(z_2 f'(z_2)) - \arg(z_1 f'(z_1)) > -\pi.$$

A proof of the converse can be found in [13]. The idea of the proof of the converse is to use the hypotheses to find a starlike function g_r for each r , $0 < r < 1$, such that $\operatorname{Re}(\frac{zf'(z)}{g_r(z/r)}) > 0$ when $|z| \leq r$ and then use the normality of the family of starlike mappings to complete the proof. The function g_r is given by Herglotz formula

$$\frac{zg'_r(z)}{g_r(z)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} dF(t),$$

where F is a nondecreasing function on $[-\pi, \pi]$, with $F(\pi) - F(-\pi) = 2\pi$. By a result of Keogh [14, Equation 7, p. 483], taking $z = \rho e^{i\theta}$, we have $\lim_{\rho \rightarrow 1-} \arg g_r(z) = \frac{1}{2}[F(\theta+) + F(\theta-)] + \arg(g'_r(0))$ with an appropriate branch of $\arg(g_r(z))$ defined locally. The problem then becomes one of choosing F to be nondecreasing so that $|F(\theta) - \arg(zf'(z))| = |F(\theta) - (\arg(f'(re^{i\theta}) + \theta)| < \frac{\pi}{2}$. As Kaplan shows, the choice $F(\theta) = \sup_{\theta' \leq \theta} (\arg(f'(re^{i\theta'}) + \theta'))$ will suffice.

THEOREM 6.6. *Suppose f is analytic in the unit disk with $f(0) = 0$ and $f'(0) = 1$. Then f is close to convex if and only if for fixed r , $0 < r < 1$, we have $\arg(z_2 f'(z_2)) - \arg(z_1 f'(z_1)) > -\pi$ whenever $|z_1| = |z_2| = r$ and $\arg z_1 = \theta_1 \leq \theta_2 = \arg z_2$.*

We may also characterize the close to convex functions in terms of subordination chains. We use the following result of Pommerenke [21].

LEMMA 6.7. *Assume the following:*

- (a) $F(z, t)$ is analytic as a function of z for each $t \in [a, b]$,
- (b) $F(z, a)$ is univalent,
- (c) $\frac{\partial F(z, t)}{\partial z}|_{z=0} > 0$,

and

- (d) $F(z, \cdot) \in C^1[a, b]$.

If $h(z, t)$ defined by

$$\frac{\partial F(z, t)}{\partial t} = h(z, t)z \frac{\partial F(z, t)}{\partial z}$$

satisfies the condition $\operatorname{Re}[h(z, t)] > 0$ for all $z \in \Delta$ and $t \in [a, b]$, then $F(z, t)$ is a univalent subordination chain. That is, $F(z, t)$ is univalent in the unit disk for each fixed t and $F(z, s) \prec F(z, t)$ whenever $a \leq s < t \leq b$.

THEOREM 6.8. *If f is close to convex with respect to a starlike function g , then the function $F(z, t)$ given by $F(z, t) = f(z) + (e^t - 1)g(z)$ is a univalent subordination chain.*

PROOF. The theorem follows from Pommerenke's result above since

$$\frac{\partial F(z, t)}{\partial t} = \frac{e^t g(z)}{zf'(z) + (e^t - 1)zg'(z)} z \frac{\partial F(z, t)}{\partial z}. \quad \square$$

From the above results, the following theorem of Lewandowski [15] and [16] is not surprising.

THEOREM 6.9. *Suppose f is close to convex. Then $f(\Delta)$ has the property that $\mathbb{C} \setminus D$ is the union of rays with the property that no two of them intersect except possibly at an end point. Conversely, suppose D is a simply connected domain with $0 \in D$ and such that $\mathbb{C} \setminus D$ is the union of rays with the property that no two of them intersect except possibly at an end point. Then there is a close to convex function f with the property that $cf(\Delta) = D$ for some complex constant c .*

Robertson's theorem (Theorem 1.2) can also be used to characterize close to convex mappings.

THEOREM 6.10. *Let $f(z) = z + a_2 z^2 + \dots$ and assume g is starlike. Then f is close to convex with respect to g if and only if, for each r , $0 < r < 1$, we have $F(z, t) = f(z) - tg(z) < f(z)$ for $t \in [0, t_0]$ and $|z| < r$ (where t_0 depends on r).*

PROOF. The direction $\Rightarrow$ follows from Theorem 1.2. The converse follows from the fact that the function

$$v(z, t) = f^{-1}(f(z) - tg(z)) = z \left(1 - t \frac{g(z)}{zf'(z)} \right) + o(t)$$

satisfies Schwarz lemma for each t in an interval $t \in [0, t_0]$. We have $|v(z, t)|^2 = |z|^2(1 - 2t \operatorname{Re}(\frac{g(z)}{zf'(z)})) + o(t)$. Using the continuity of $v(z, t)$ on the circle $|z| = r$ and applying the maximum principle to $\frac{z}{v(z, t)}$, the result now follows. $\square$

7. Spirallike functions

DEFINITION 7.1. Suppose f is analytic on Δ with $f(0) = 0$ and $f'(0) = 1$. If there exists γ , with $|\gamma| = 1$ such that

$$\operatorname{Re} \left(\frac{\gamma z f'(z)}{f(z)} \right) > 0, \quad (7.1)$$

then f is spirallike with respect to γ . Clearly, $\gamma = e^{i\alpha}$ for some α such that $|\alpha| < \frac{\pi}{2}$.

THEOREM 7.2. *Spirallike functions are univalent.*

PROOF. We use an argument similar to that in Theorem 2.3. In (2.2), replace $\frac{zf'(z)}{f(z)}$ by $\frac{\gamma zf'(z)}{f(z)}$ under the assumption that (7.1) holds. The conclusion is that

$$\begin{aligned} 0 &< \operatorname{Im}(\gamma \log(f(z_2)) - \log(f(z_1))) \\ &= \cos(\alpha)(\arg(f(z_2)) - \arg(f(z_1))) - \sin(\alpha)(\log|f(z_2)| - \log|f(z_1)|) \\ &< 2\pi \cos(\alpha). \end{aligned}$$

If $f(z_2) = f(z_1)$ then $|f(z_2)| = |f(z_1)|$ and $\arg(f(z_2)) = \arg(f(z_1)) + 2k\pi$ for some integer k . From the inequality above, $0 < k < 1$, which is a contradiction and the proof is complete. $\square$

EXAMPLE 7.3. For constant $\zeta = \bar{\gamma}^2 = e^{-2i\alpha}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ the function $f_\zeta(z) = \frac{z}{(1-z)^{1+\zeta}}$ is extremal in the following sense.

- (a) The function $\frac{\gamma zf'_\zeta(z)}{f_\zeta(z)}$ is a Möbius transformation that maps the unit disk onto the right half-plane. Hence, f_ζ is spirallike with respect to γ .
- (b) The derivative $f'_\zeta(-\gamma) = 0$.
- (c) The function f_ζ maps the unit disk onto the complement of the part of the spiral $\arg(w) = \tan(\alpha) \log|w|$ that starts at $f_\zeta(-\gamma)$.

The following theorem shows the relationship between starlike and spirallike mappings.

THEOREM 7.4. Suppose f is spirallike and satisfies (7.1). Then there is a starlike mapping g with $g'(0) = 1$ such that

$$f(z) = z \left(\frac{g(z)}{z} \right)^{(1+\bar{\gamma}^2)/2}. \quad (7.2)$$

Conversely, if g is starlike with $g'(0) = 1$, and f is given by (7.2), then f is spirallike with respect to γ .

The proof of this result is straightforward.

The family of spirallike functions yields counterexamples to some conjectures that one might make concerning the class S . For instance, one might be led to conjecture that if $f \in S$ then the function g given by $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta$ is also in S . The fact that, if f is starlike, then g is convex lends support to that conjecture. However, we have the following example.

EXAMPLE 7.5. Let $f(z) = \frac{z}{(1-z)^{1-i}}$. Then as shown above, f is spirallike with respect to $e^{i\pi/4}$. Also,

$$g(z) = \int_0^z \frac{1}{(1-\zeta)^{1-i}} d\zeta = i((1-z)^i - 1).$$

If we now choose r , $0 < r < 1$, so that $\log(1+r) = \log(1-r) + 2\pi$, i.e., $r = \tanh(\pi)$, then $g(r) = g(-r)$.

Incidentally, Example 7.5 also shows that the attractive conjecture of Mandelbrot and Schiffer: If $f(z) = \sum_1^\infty a_n z^n \in S$ and $g(z) = \sum_1^\infty b_n z^n \in S$, then $\sum_1^\infty \frac{a_n b_n}{n} z^n \in S$ is false. Just take f as in Example 7.5 and $g(z) = \frac{z}{1-z}$. The Mandelbrot–Schiffer conjecture was made as a possible method of proof of the Bieberbach conjecture (now the de Branges theorem). Of course, there are other known counterexamples [9].

Spirallike functions can also be characterized by subordination.

THEOREM 7.6. *Suppose f is analytic and locally univalent in the unit disk and normalized so that $f(0) = 0$ and $f'(0) = 1$, and assume γ is a complex number with $|\gamma| = 1$ and $\operatorname{Re}(\gamma) > 0$. Then f is spirallike with respect to γ if and only if $e^{-t\tilde{\gamma}} f \prec f$ for $0 < t$.*

PROOF. Assume $e^{-t\tilde{\gamma}} f \prec f$ and apply Theorem 1.2 with $F(z, t) = e^{-t\tilde{\gamma}} f(z)$, where $\tau = \frac{t}{1-t}$. This proves that $\operatorname{Re}(\frac{\tilde{\gamma} f(z)}{z f'(z)}) > 0$ so that f is spirallike with respect to γ .

Now assume f is spirallike with respect to γ and consider the initial value problem

$$\frac{\partial v(z, t)}{\partial t} = -w(v(z, t)), \quad v(z, 0) = z, \quad |z| < 1, 0 \leq t,$$

where $w(z) = \frac{\tilde{\gamma} f(z)}{z f'(z)}$. Thus, $\operatorname{Re}(\frac{w(z)}{z}) > 0$. When $\varepsilon > 0$ and $t \geq 0$, we have

$$\begin{aligned} |v(z, t + \varepsilon)|^2 &= |v(z, t) - \varepsilon w(v(z, t)) + o(\varepsilon)|^2 \\ &= |v(z, t)|^2 \left(1 - 2\varepsilon \operatorname{Re} \frac{w(v(z, t))}{v(z, t)} + o(\varepsilon) \right) \end{aligned}$$

so that $|v(z, t)|$ decreases with t . Since $v(z, t) = f^{-1}(e^{-t\tilde{\gamma}} f(z))$ is the solution of the initial value problem, the subordination $e^{-t\tilde{\gamma}} f \prec f$ now follows. $\square$

Similar considerations show that spirallike functions can be imbedded in a subordination chain.

THEOREM 7.7. *If f is spirallike with respect to γ , then the family $F(z, t) = e^{t\tilde{\gamma}} f(z)$ is a subordination chain.*

PROOF. Apply Lemma 6.7 with $h(z, t) = \frac{\tilde{\gamma} f(z)}{z f'(z)}$. $\square$

8. Typically real functions

The family of typically real functions is a family that contains the functions in the class S that have real coefficients. The definition is as follows.

DEFINITION 8.1. A function, f , is said to be typically real if it is analytic in the unit disk, Δ , with the normalization $f(0) = 0$ and $f'(0) = 1$ and $f(z)$ is real if and only if z is real. The family of typically real functions will be denoted by $\mathbb{T}$.

REMARK 8.2. Assume $f \in \mathbb{T}$.

(a) It is easy to see that $f^{(n)}(x)$ is real for all $x \in (-1, 1)$ and all nonnegative integers, n . Thus the coefficients in the Maclaurin expansion of f must be real.

(b) For $x \in (-1, 1)$, the values of $f(x + \varepsilon(e^{i\theta} - 1))$, $0 < \theta < \pi$, for small ε cannot intersect the real axis. We conclude:

(i) $f'(x) \neq 0$, and

(ii) since $f'(0) = 1$, we must have $f'(x) > 0$. By conformality, the upper (lower) half of the disk maps into the upper (lower) half-plane. Therefore $\text{Im}(z) \text{Im}(f(z)) \geq 0$.

(c) $\text{Re}((\frac{1-z^2}{z})f(z)) > 0$.

PROOF. Using (b)(ii), it follows that for $0 < r < 1$ and $|z| = r$,

$$\text{Re}\left(\left(\frac{r}{z} - \frac{z}{r}\right)f(z)\right) > 0. \quad (8.1)$$

By the minimum principle, (8.1) holds for $|z| < r$. Since $\mathbb{T}$ is a normal family, we may let $r \rightarrow 1^-$ to arrive at (c). $\square$

(d) Robertson [23] proved the following integral representation. There exists a nondecreasing function, T , on $[0, \pi]$ with $T(\pi) - T(0) = 1$ such that

$$f(z) = \int_0^\pi \frac{z}{1 - 2z \cos(t) + z^2} dT(t). \quad (8.2)$$

PROOF. Using (c) and (2.3) we have

$$\frac{1 - z^2}{z} f(z) = P(z) = \int_{-\pi}^\pi \frac{1 + ze^{-it}}{1 - ze^{-it}} dF(t).$$

Using the fact that f has real coefficients, so that $\overline{f(\bar{z})} = f(z)$, we may write

$$\begin{aligned} P(z) &= \frac{1}{2} \int_{-\pi}^\pi \frac{1 + ze^{-it}}{1 - ze^{-it}} + \frac{1 + ze^{it}}{1 - ze^{it}} dF(t) \\ &= \int_{-\pi}^\pi \frac{1 - z^2}{1 - 2z \cos(t) + z^2} dF(t) \\ &= \int_{-\pi}^0 \frac{1 - z^2}{1 - 2z \cos(t) + z^2} dF(t) + \int_0^\pi \frac{1 - z^2}{1 - 2z \cos(t) + z^2} dF(t). \end{aligned}$$

In the second integral, replace t by $-t$ to obtain (8.2) with $T(t) = \frac{F(t)-F(-t)}{2}$. $\square$

(e) Since a function f given by (8.2) is clearly typically real, it easily follows that $f \in \mathbb{T} \Leftrightarrow$ (d) holds $\Leftrightarrow$ (c) holds and f has real coefficients.

(f) Let S_R denote the subset of the family of normalized univalent functions $g \in S$ that have real coefficients. Then $S_R \subset \mathbb{T}$. This follows from the fact that $\overline{g(z)} = g(\bar{z})$ so that $g(z)$ real implies $g(z) = g(\bar{z})$ so that $z = \bar{z}$ if $g \in S_R$.

(g) Using (c) above, given $z + \sum_{n=2}^{\infty} a_n z^n \in \mathbb{T}$, we know that the function $P(z) = 1 + a_2 z + \sum_{n=2}^{\infty} (a_{n+1} - a_{n-1}) z^n$ has positive real part. Since the coefficients of P are bounded by 2, an easy induction shows that $|a_n| \leq n$. Also, the function $k(z) = \frac{z}{(1-z)^2} \in S_R$, shows that the bounds are sharp.

In [2, Problem 647] Clunie posed the problem: Under the assumptions $f \in S$ and f' is univalent in the unit disk, what can be said about $\max |a_n|$, $n \geq 2$? In [3] it was shown that if f and f' (appropriately normalized) are typically real, then a_2 satisfies the sharp inequality

$$|a_2| \leq \frac{3}{2} + \frac{1}{\pi}.$$

Further, the extremal function is actually univalent with a univalent derivative and is therefore a candidate for the solution of Clunie's problem for $n = 2$. We give an indication of the reasoning that leads to a candidate for the solution and then outline the proof.

Let $\mathbb{T}'$ denote the family of typically real functions, f , such that the second coefficient, a_2 , is positive and f' is typically real if properly normalized (i.e., $\frac{f'(z)-1}{a_2}$ is typically real). First observe that if $f \in \mathbb{T}'$ and $f(e^{i\theta}) = u(\theta) + iv(\theta)$ is analytic on an arc of the unit circle then $f'(e^{i\theta}) = e^{-i\theta}(-iu'(\theta) + v'(\theta))$. Now $\text{Im}(f(e^{i\theta_0})) = 0 \Rightarrow v(\theta_0) = 0$ and $v'(\theta_0) = 0 \Rightarrow \text{Im}(f'(e^{i\theta_0})) = -\cos(\theta_0)u'(\theta_0)$. Using conformality, the fact that $v(\theta_0) = 0$ and the fact that f' is typically real, we conclude that $u'(\theta_0) > 0$. This means that $\cos(\theta_0) \leq 0$. Thus, $\text{Im}(f(e^{i\theta}))$ cannot be zero on the right half of the unit circle (except at $z = 1$). A likely candidate for an extremal function is therefore one that has the property $\text{Im}(f'(e^{i\theta})) = 0$ when $-\pi/2 < \theta < \pi/2$ and $\text{Im}(f(e^{i\theta})) = 0$ when $\pi/2 < \theta < 3\pi/2$. Thus, the candidate for the solution should have the properties: $\text{Re}(\frac{zf''(z)}{f'(z)}) = 0$ on the right half of the unit circle and $\text{Re}(\frac{zf''(z)}{f'(z)}) = -1$ on the left half of the unit circle. The normalized function ($f(0) = 0$, $f'(0) = 1$) that has these properties is

$$f'(z) = \frac{1+z}{(1-z)^2} \exp\left(\int_0^z \frac{1}{\pi i w} \log\left(\frac{1+iw}{1-iw}\right) dw\right), \quad f(0) = 0. \quad (8.3)$$

We now have the following theorems [3, Theorems 2 and 3].

THEOREM 8.3. *If $f \in \mathbb{T}'$ with $a_2 > 0$, then $a_2 \leq \frac{3}{2} + \frac{1}{\pi}$ with equality if and only if f is given by (8.3).*

THEOREM 8.4. *If $f \in S_R$ and f' is univalent with $a_2 > 0$ then $a_2 \leq \frac{3}{2} + \frac{1}{\pi}$ with equality if and only if f is given by (8.3).*

PROOFS (Outline). Theorem 8.4 will clearly follow from Theorem 8.3 if an extremal function for Theorem 8.3 is univalent with univalent derivative. We show (8.3) is the unique extremal for Theorem 8.3. Univalence of f and f' follows by analyzing the image of the circle and applying the argument principle [3].

We assume f is extremal for Theorem 8.3 and use Robertson's integral representation for f ,

$$\begin{aligned} f(z) &= \int_0^\pi \frac{z}{1 - 2\cos(t)z + z^2} dT, \\ f'(z) &= \int_0^\pi \frac{1 - z^2}{(1 - 2\cos(t)z + z^2)^2} dT. \end{aligned} \quad (8.4)$$

In (8.4), the coefficient a_2 is given by

$$a_2 = \int_0^\pi \cos(t) dT.$$

This quantity will clearly be increased by replacing the increasing function T by

$$\hat{T} = \begin{cases} \frac{T(t) - T(0)}{T(\pi/2) - T(0)}, & 0 \leq t \leq \pi/2, \\ 1, & \pi/2 < t \leq \pi. \end{cases}$$

Note that $T(\pi/2) > T(0)$ because $a_2 > 0$.

Now define

$$g(z) = \int_0^{\pi/2} \frac{z}{1 - 2\cos(t)z + z^2} d\hat{T}$$

so that

$$g'(z) = \int_0^{\pi/2} \frac{1 - z^2}{(1 - 2\cos(t)z + z^2)^2} d\hat{T}. \quad (8.5)$$

The fact that the function g' given by (8.5) is typically real is proved as follows. First, consider the integrand $k(z, t) = \frac{1 - z^2}{(1 - 2z\cos(t) + z^2)^2}$ when $0 < t < \frac{\pi}{2}$ in the region Ω given by $z = re^{i\theta}$, $0 \leq r \leq 1$, $\frac{\pi}{2} \leq \theta \leq \pi$. On the arc $z = e^{i\theta}$, $\frac{\pi}{2} \leq \theta \leq \pi$, $\text{Im}(k(z, t)) = -\frac{\sin(2\theta)}{4(\cos(\theta) - \cos(t))^2} \geq 0$. On the line segment $-1 \leq x \leq 0$, $\text{Im}(k(x, t)) = 0$. On the line segment $z = i\rho$, $0 \leq \rho \leq 1$, it is easy to verify that $\text{Im}(k(z, t)) \geq 0$. By the minimum principle for harmonic functions, it follows that $\text{Im}(k(z, t)) \geq 0$ on Ω , when $0 \leq t < \frac{\pi}{2}$ and hence $\text{Im}(g'(z)) > 0$ on Ω . Since $k(-\bar{z}, \pi - t) = k(\bar{z}, t) = \overline{k(z, t)}$, it follows that $\text{Im}(k(z, t)) \leq 0$ when $\frac{\pi}{2} < t \leq \pi$, and z is in the region Ω' given by $z = re^{i\theta}$,

$0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. Now using the fact that $f \in \mathbb{T}'$ when $z \in \Omega'$, we have

$$\begin{aligned} 0 &\leq \operatorname{Im}(f'(z)) \\ &= \int_0^\pi \operatorname{Im}(k(z, t)) dT = \int_0^{\pi/2} \operatorname{Im}(k(z, t)) dT + \int_{\pi/2}^\pi \operatorname{Im}(k(z, t)) dT \\ &\leq \int_0^{\pi/2} \operatorname{Im}(k(z, t)) dT = (T(\pi/2) - T(0)) \operatorname{Im}(g'(z)). \end{aligned}$$

Since g has real coefficients, it now follows that $g \in \mathbb{T}'$. Also, it is clear that g has the property:

PROPERTY A. The function g is analytic on the arc $\{e^{i\theta} : \pi/2 < \theta < 3\pi/2\}$ and has zero real part there.

This follows from the fact that the integrand is continuous on that part of the boundary. The proof is completed by showing that if g is typically real with Property A, then

$$(1+z) \exp\left(-\int_0^z \frac{1}{\pi i w} \log\left(\frac{1+iw}{1-iw}\right) dw\right) g'(z)$$

is typically real and omits the negative real axis. This function is then subordinate to $(\frac{1+z}{1-z})^2$ so that

$$2a_2 + 1 - \frac{2}{\pi} \leq 4.$$

□

9. Some integral representations and extreme point theory

To find an integral representation for convex mappings we use Corollary 2.15(b) together with Theorem 2.3, see [6]. Corollary 2.15(b) says that if $f \in K$, then $\operatorname{Re}(\frac{f(z)}{z}) > \frac{1}{2}$. It then follows that $P(z) = \frac{2f(z)}{z} - 1$ has positive real part with $P(0) = 1$ so that P can be represented by Herglotz formula. If we then solve for f , we obtain the following theorem.

THEOREM 9.1. *If $f \in K$, then*

$$f(z) = \int_0^{2\pi} \frac{z}{1 - ze^{-it}} d\mu(t), \quad (9.1)$$

where μ is a probability measure on $[0, 2\pi]$.

Actually, it is easy to see that every function, f , that satisfies $\operatorname{Re}(\frac{f(z)}{z}) > 1/2$ can be represented by an integral of the form given in (9.1). For each fixed t , the integrand in (9.1), $k(z, t) = \frac{z}{1 - ze^{-it}}$, is a rotation of the half-plane mapping $\frac{z}{1-z}$ and is therefore convex. This

result (Theorem 9.1) is a special case of some general results given in [6]. We require a few definitions to state the main theorem from [6].

DEFINITION 9.2. If Ω is a subset of a vector space and $x \in \Omega$, then x is an extreme point of Ω if, for $p, q \in \Omega$ and $t \in (0, 1)$, the equality $x = tp + (1 - t)q$ can only hold when $p = q = x$.

The Krein–Milman theorem which says that if Ω is a compact subset of a locally convex linear topological space, then Ω is contained in the convex hull of the set of extreme points of Ω is an important result in this subject. For a proof of this theorem as well as other results concerning extreme points, see [8].

In Theorem 9.1, the topological space is the space of analytic functions on the unit disk Δ with the topology of uniform convergence on compact subsets of Δ . The set Ω is K .

The general result proved by Brickman, MacGregor and Wilken [6, Theorem 1, p. 93] is as follows.

THEOREM 9.3. Let Δ be the open unit disk in the complex plane $\mathbb{C}$, and let X be any compact Hausdorff space. Let $k: \Delta \times X \rightarrow \mathbb{C}$ have the following three properties:

- (1) For each x in X the map $z \rightarrow k(z, x)$ is analytic in Δ .
- (2) For each z in Δ the map $x \rightarrow k(z, x)$ is continuous on X .
- (3) For each r , $0 < r < 1$, there exists $M_r > 0$ such that $|K(z, x)| \leq M_r$ for $|z| \leq r$ and for x in X .

Let $\mathcal{P}$ denote the set of probability measures on the Borel subsets of X . For μ in $\mathcal{P}$, let

$$f_\mu(z) = \int_X k(z, x) d\mu(x) \quad (z \in \Delta).$$

Finally, let $\mathcal{F} = \{f_\mu: \mu \in \mathcal{P}\}$. Then:

- (a) Each function in $\mathcal{F}$ is analytic in Δ .
- (b) The map $\mu \rightarrow f_\mu$ is continuous (with the relative weak-star topology on $\mathcal{P}$, regarded as a subset of $C(X)^*$, and the topology of uniform convergence on compacta on $\mathcal{F}$).
- (c) $\mathcal{F}$ is compact and is the closed convex hull of the set of functions $\{z \rightarrow k(z, x): x \in X\}$.
- (d) The only possible extreme points of $\mathcal{F}$ are the functions $z \rightarrow k(z, x)$, $x \in X$. If $x_0 \in X$ and $k(z, x_0) = \int_X k(z, x) d\mu(x)$, ($z \in \Delta$) holds only for $\mu = \delta_{x_0}$ (unit point mass at x_0), then the function $z \rightarrow k(z, x_0)$ is an extreme point of $\mathcal{F}$. In particular, if the map $\mu \rightarrow f_\mu$ is one-to-one, then each function $z \rightarrow k(z, x)$, $x \in X$, is an extreme point of $\mathcal{F}$.

The following theorem is now straightforward.

THEOREM 9.4. If $\mathcal{F}$ is a family as defined in Theorem 9.3 and J is a complex-valued continuous linear functional on $\mathcal{F}$, then

$$\max_{f \in \mathcal{F}} \operatorname{Re}(J(f)) = \max_{x \in X} \operatorname{Re}(J(k(\cdot, x))).$$

PROOF. Assume the hypotheses. For any $f_\mu \in \mathcal{F}$, we have

$$\begin{aligned} \operatorname{Re}(J(f_\mu)) &= \int_X \operatorname{Re}(J(k(\cdot, x))) d\mu(x) \\ &\leq \int_X \max_{x \in X} (\operatorname{Re}(J(k(\cdot, x)))) d\mu(x) \\ &= \max_{x \in X} (\operatorname{Re}(J(k(\cdot, x)))). \end{aligned}$$

□

Some of the rate of growth results and coefficient bounds proved in Section 3 for convex and starlike functions as well as similar results for typically real functions and close to convex functions can now be proved using Theorem 9.4. The method is to describe the closed convex hull of the family as a family $\mathcal{F}$ as in Theorem 9.3. If the quantity to be determined can be expressed as the problem of maximizing the real part of a continuous linear functional then the problem is reduced to maximizing the real part of the functional over the family $\{k(\cdot, x): x \in X\}$.

THEOREM 9.5. *Let Ω be one of the families $K, S^*, \mathcal{C}$ or $\mathbb{T}$. Then $\operatorname{cl co} \Omega$ (the closed convex hull of Ω) is the family*

$$\left\{ \int_X k(z, x) d\mu(x) : \mu \text{ is a probability measure on } X \right\},$$

where

- (a) $k(z, t) = \frac{z}{1 - ze^{-it}}$, $X = [0, 2\pi]$ when $\Omega = K$,
- (b) $k(z, t) = \frac{z}{(1 - ze^{-it})^2}$, $X = [0, 2\pi]$ when $\Omega = S^*$,
- (c) $k(z, t) = \frac{z}{1 - 2\cos(t)z + z^2}$, $X = [0, \pi]$ when $\Omega = \mathbb{T}$,
- (d) $k(z, (x, y)) = \frac{z - ((x+y)/2)z^2}{(1 - yz)^2}$, $X = \{(x, y): |x| = |y| = 1\}$ when $\Omega = \mathcal{C}$.

PROOF. Part (a) is Theorem 9.1. To prove (b) use (a) and the fact that $f \in S^*$ if and only if $f(z) = zg'(z)$ for some $g \in K$. Part (c) is Remark 8.2(d). Part (d) is in [6, Theorem 6]. □

THEOREM 9.6. (a) *The extreme points for the families K, S^* and $\mathbb{T}$ are the relevant $k(\cdot, t)$ in Theorem 9.5.*

(b) *The extreme points for the family $\mathcal{C}$ are among the mappings*

$$f(z) = \frac{z(1 - ayz)}{(1 - yz)^2},$$

where $a = \frac{x\bar{y}+1}{2}$, $|x| = |y| = 1$.

PROOF. Part (a) follows from Theorem 9.3(d). In part (b), the given functions map the disk onto the complement of a slit. From [6, p. 104] two points on the slit are equidistant from the origin when $|a| > 1/2$ and these mappings are not extreme points. □

REMARK 9.7. The coefficient bounds for each of the families of functions in Theorem 9.5 can be found using the functionals $J_n(f) = a_n$, $n = 2, 3, \dots$, applied to $k(\cdot, t)$, $t \in X$. This follows from the fact that $\max \operatorname{Re}(a_n) = \max |a_n|$ in each of the families. Similarly, the quantity $\max_{|z|=r} |f(z)|$ can be found using the functional $J_r(f) = f(r)$.

Extreme points for the family of functions starlike of order $\alpha < 1$ (i.e., functions f , such that $\operatorname{Re}(\frac{zf'(z)}{f(z)}) > \alpha$ for all $z \in \Delta$), and functions, f , in S^* that are k fold symmetric, in the sense that $f(z\gamma) = \gamma f(z)$, where $\gamma^k = 1$ are studied in [5].

References

- [1] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12–22.
- [2] J.M. Anderson, K.F. Barth and D.A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. **9** (1977), 129–162.
- [3] R.W. Barnard and T.J. Suffridge, *On the simultaneous univalence of f and f'* , Michigan Math. J. **30** (1983), 9–16.
- [4] M. Biernacki, *Sur la représentation conforme des domaines linéairement accessible*, Prace Frat. Fiz. **44** (1937), 293–314.
- [5] L. Brickman, D.J. Hallenbeck, T.H. MacGregor and D.R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings*, Trans. Amer. Math. Soc. **185** (1973), 413–428.
- [6] L. Brickman, T.H. MacGregor and D.R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91–107.
- [7] C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der Abbildung des Innern einer Jordanschen Kurve auf einen Kreis*, Math. Ann. **73** (1913), 305–320.
- [8] N. Dunford and J.T. Schwartz, *Linear Operators. I: General Theory*, Pure Appl. Math., Vol. 7, Interscience, New York (1958).
- [9] P. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [10] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Transl. Math. Monographs, Vol. 26, Amer. Math. Soc., Providence, RI (1969).
- [11] D.J. Hallenbeck and T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman, London (1984).
- [12] F. Herzog and G. Piranian, *On the univalence of functions whose derivative has a positive real part*, Proc. Amer. Math. Soc. **2** (1951), 625–633.
- [13] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185.
- [14] F.R. Keogh, *Some theorems on conformal mapping of bounded star-shaped domains*, Proc. London Math. Soc. **9** (1959), 481–491.
- [15] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes, I*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **12** (1958), 131–146.
- [16] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes, II*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **14** (1960), 19–46.
- [17] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I*, Math. Ann. **89** (1923), 103–121.
- [18] A. Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann. **107** (1932–1933), 40–67.
- [19] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York (1952).
- [20] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hakkaido Imper. Univ. Sapporo (I) **2** (1934–1935), 129–155.
- [21] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.

- [22] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Grundlehren Math. Wiss., Bd. 299, Springer-Verlag, Berlin (1992).
- [23] M.S. Robertson, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc. **41** (1935), 565–572.
- [24] M.S. Robertson, *Applications of the subordination principle to univalent functions*, Pacific J. Math. **11** (1961), 315–324.
- [25] G. Schober, *Univalent Functions-Selected Topics*, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin (1975).
- [26] I. Schur, *Über Potenzreihen die im Innern des Einheitskreises beschränkt sind*, J. Reine Angew. Math. **147** (1917), 204–232.
- [27] T. Sheil-Small, *On convex univalent functions*, J. London Math. Soc. **1** (2) (1969), 483–492.
- [28] E. Strohäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. **37** (1933), 356–380.
- [29] E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Heft 2; herausgegeben unter Mitwirkung von W. Blaschke: *Konforme Abbildung einfach-zusammenhängender Bereiche*, Teubner, Leipzig und Berlin (1913).
- [30] T.J. Suffridge, *Univalent functions*, Ph.D. Thesis, University of Kansas (1965).
- [31] T.J. Suffridge, *Convolutions of convex functions*, J. Math. Mech. **15** (1966), 795–804.
- [32] T.J. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775–777.
- [33] T.J. Suffridge, *A new criterion for starlike functions*, Indiana Math. J. **28** (1979), 429–443.
- [34] H.S. Wall, *Analytic Theory of Continued Fractions*, Chelsea, Bronx, NY (1973).
- [35] S.E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc. **38** (1935), 310–340.

CHAPTER 8

Univalence and Zeros of Complex Polynomials

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Introduction

A complex-valued function f , defined in a domain G in the complex plane $\mathbb{C}$, is called *univalent* (or *schlicht*) in G if it is holomorphic and one-to-one in G . For the general theory of univalent functions we refer the reader to the books of Duren [8], Goodman [10] and Pommerenke [17].

If G is a Jordan domain, bounded by the rectifiable curve $\gamma : [0, 1] \rightarrow \mathbb{C}$ and f is holomorphic in the closure $\overline{G}$ of G , then f is univalent in G if and only if the image curve $f \circ \gamma$ has winding number 1 with respect to each $w \in G$. This is an immediate consequence of the argument principle (cf. [1, Theorem 10, p. 131]).

In the following we restrict our considerations on polynomials and on the case of the open unit disk E . Univalent polynomials can be helpful as test functions for results or conjectures in Geometric Function Theory. Proving univalence is closely related with the problem to locate the zeros of polynomials, and this is a quite general mathematical question.

1. General criteria

It can be easily seen that a complex polynomial p is univalent in E if and only if the curves given by $\gamma_r(t) := p(re^{it})$ ($0 \leq t \leq 2\pi$) are simply closed for all $0 < r < 1$. The latter is fulfilled iff there is no such r and no real numbers $\alpha \in [0, 2\pi[$, $\beta \in]0, \frac{\pi}{2}]$ such that $p(re^{i(\alpha-\beta)}) = p(re^{i(\alpha+\beta)})$. A short calculation leads to the following theorem.

THEOREM 1 (Dieudonné's criterion). *The complex polynomial*

$$p(z) = \sum_{j=0}^n a_j z^j$$

is univalent in the open unit disk E if and only if all the polynomials

$$p_\beta(z) := \sum_{j=1}^n a_j \frac{\sin j\beta}{\sin \beta} z^{j-1}, \quad \beta \in \left]0, \frac{\pi}{2}\right],$$

have no zeros in E .

Note that $\lim_{\beta \rightarrow 0} p_\beta(z) = p'(z)$.

Dieudonné's criterion reduces the question of univalence of a polynomial in E to test the location of the zeros of a family of polynomials. It would be enough to have detailed information on the coefficient body

$$\Omega(n-1) := \left\{ (b_0, \dots, b_{n-1}) \in \mathbb{C}^n \left| q(z) = \sum_{k=0}^{n-1} b_k z^k \neq 0 \text{ for all } z \in E \right. \right\}$$

and then one has to check if

$$\left(a_1, a_2 \frac{\sin 2\beta}{\sin \beta}, \dots, a_n \frac{\sin n\beta}{\sin \beta}\right) \in \Omega(n-1), \quad \beta \in \left]0, \frac{\pi}{2}\right],$$

in order to obtain univalence of the polynomial $p(z) = \sum_{j=0}^n a_j z^j$ in E . But unfortunately we have only very partial knowledge of $\Omega(n-1)$.

There are criteria using the so called Schur–Cohn determinants to decide if a given polynomial does vanish in E or not (we refer to Marden’s book [15, Chapter X]). But this is only interesting from a theoretical point of view because determinants of this size are hard to calculate.

The following result in some cases helpful.

THEOREM 2 (Cohn’s rule). *Let $p(z) = a_0 + a_1 z + \dots + a_n z^n$ be a polynomial with $0 < |a_n| < |a_0|$ and define*

$$p^*(z) := z^n \overline{p(1/\bar{z})} = \overline{a_n} + \overline{a_{n-1}}z + \dots + \overline{a_0}z^n.$$

Then the polynomials $\tilde{p}(z) := \overline{a_0}p(z) - a_n p^*(z)$ and $p(z)$ have the same number of zeros (counting multiplicity) in E .

Another result comparing zeros of two polynomials in a certain disk is the following theorem.

THEOREM 3 (Grace’s apolarity theorem). *Let $\lambda_0, \dots, \lambda_n, a_0, \dots, a_n$ be complex numbers with $\lambda_n \neq 0$ (here $a_n \neq 0$ is not required) and assume that the “apolarity condition” $\lambda_0 a_n + \lambda_1 a_{n-1} + \dots + \lambda_n a_0 = 0$ is fulfilled. Then every closed disk D in $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$ containing all roots of the polynomial*

$$\Lambda(z) = \lambda_0 - \binom{n}{1} \lambda_1 z^1 + \binom{n}{2} \lambda_2 z^2 - \dots + (-1)^n \lambda_n z^n$$

contains at least one zero of the polynomial

$$p(z) = a_0 + a_1 z + \dots + a_n z^n.$$

The apolarity theorem has many surprising and useful consequences (cf. [21]). We mention the following analogue of Rolle’s theorem in real analysis:

THEOREM 4 (Grace–Heawood). *Let p be a complex polynomial with $p(-1) = p(1)$. Then the derivative p' has some zero in the closed left half-plane as well as in the closed right half-plane.*

The following result, due to Alexander (1915) and Takeya (1917), comes out as a consequence of the Grace–Heawood theorem (cf. [21]).

THEOREM 5. *If p is a complex polynomial of degree n and $p'(z) \neq 0$ for all $|z| \leq \frac{1}{\sin \frac{\pi}{n}}$, then p is univalent in E .*

The bound $1/\sin \frac{\pi}{n}$ is best possible.

If $f: E \rightarrow \mathbb{C}$ is an arbitrary univalent function we see from the argument principle that the partial sum $f_n(z) = \sum_{k=0}^n a_k z^k$ of its power series is univalent in each fixed compact disk $|z| < r < 1$ provided that n is sufficiently large. Setting $p_n(z) := f_n(rz)$ we obtain a univalent polynomial in E . This shows that the set of univalent polynomials is dense with respect to the topology of locally uniform convergence in the class of all univalent functions in E . Szegő discovered a surprising detail (see [8, Section 8.2]):

THEOREM 6 (Szegő). *Suppose that the function $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is univalent in E . Then each partial sum of this power series is univalent in the disk $|z| < \frac{1}{4}$.*

The radius $\frac{1}{4}$ is best possible – consider the second partial sum of the Koebe function $k(z) := \sum_{m=1}^{\infty} m z^m$.

For studies of univalence it is obviously enough to consider polynomials of the form $p(z) = z + a_2 z^2 + \dots + a_n z^n$. The class of the so normalized n th degree polynomials we denote by S_n . The related coefficient body C_n is the set of points $(a_2, \dots, a_n) \in \mathbb{C}^{n-1}$ such that $p \in S_n$. This set is completely known only for cubic polynomials (Kössler [13] in 1951, Brannan [5] in 1967, Cowling and Royster [6] in 1968). Further we describe the coefficient body of univalent trinomials $z + a z^k + b z^n$ for arbitrary natural numbers k, n .

For “small” degree n it is possible to get information on the n th coefficient body by help of a computer. The coefficient vector $(a_2, \dots, a_n)$ is an inner point of C_n with respect to the topology in $\mathbb{C}^{n-1}$ if and only if $p(e^{it})$, $t \in [0, 2\pi]$, is a simple curve without cusps. If it belongs to the boundary of C_n then there is some t_0 such that $p'(e^{it_0}) = 0$ (and in this case $p(e^{it})$ has a cusp in t_0) or if there exist $0 \leq t_1 < t_2 < 2\pi$ with $p(e^{it_1}) = p(e^{it_2})$. Cum grano salis this can be decided by a suitable computer program.

2. Extremal univalent polynomials

To check a conjecture or to find an idea it is often helpful to use polynomials belonging to the boundary of C_n . The boundary points of the trinomial coefficient bodies (see below) give such examples. Suffridge [20] introduced the polynomials

$$q_n(z) = z + \sum_{k=2}^n \frac{n+1-k}{n} \frac{\sin k \frac{\pi}{n+1}}{\sin \frac{\pi}{n+1}} z^k.$$

He proved that the k th coefficient ($k = 2, \dots, n$) of q_n maximizes $|a_k|$ in the class of univalent polynomials $z + \sum_{k=2}^{n-1} z^k + \frac{1}{n} z^n$ with real coefficients. Note that the product of the derivative zeros of a polynomial $p(z) = z + a_2 z^2 + \dots + a_n z^n$ is $1/|na_n|$. This shows that $|a_n| \leq \frac{1}{n}$ is necessary for p in order to be in S_n . The derivative of q_n has all its zeros on

the unit circle. There are more nice geometric mapping properties. For a more detailed description we refer to [20].

Brandt [2–4] found interesting methods to determine extremal univalent polynomials. Let $S_n^{\mathbb{R}}$ denote the set of polynomials in S_n having only real coefficients, and let $\mathbf{S}_n$ and $\mathbf{S}_n^{\mathbb{R}}$ be the corresponding subclasses where $|a_n|$ attains its maximum, i.e., where $|a_n| = 1/n$.

In [2] local variations in S_n and $S_n^{\mathbb{R}}$ are investigated. They give some results concerning relations between extremal polynomials in S_n and $\mathbf{S}_n$: For fixed $r \in]0, 1[$ and sufficiently large n no polynomial which maximizes $|f(r)|$ or $|f'(r)|$ in S_n belongs to $\mathbf{S}_n$, but if $r \in [1, \infty[$ and $n > 1$ any polynomial maximizing $|f(r)|$ or $|f'(r)|$ in S_n is in $\mathbf{S}_n$. For $m > 1$ and $n > 2m$ no polynomial which maximizes $|a_m|$ in S_n lies in $\mathbf{S}_n$. In the case $m > 1$ and $n = m + 1$ or $n = 2m - 1$ each polynomial maximizing $|a_m|$ in S_n belongs to $\mathbf{S}_n$.

Brandt's method also leads to explicit formulas like

$$\min_{f \in S_n^{\mathbb{R}}} f(1) = \frac{1}{4 \cos^2(\frac{\pi}{n+2})}, \quad \max_{f \in S_n^{\mathbb{R}}} \max_{|z|=1} |f(z)| = \frac{n+1}{4n \sin^2(\frac{\pi}{2(n+1)})}$$

for $n > 1$, as well as to the following growth estimations: For fixed $r \in (0, 1)$ and fixed $m > 1$, it follows

$$\begin{aligned} \max_{f \in S_n^{\mathbb{R}}} \max_{|z|=r} |f(z)| &= r(1-r)^{-2} - 4\pi^2 r^2 (1-r)^{-4} n^{-2} + O(n^{-3}), \\ \max_{f \in S_n^{\mathbb{R}}} \max_{|z|=r} |f'(z)| &= (1+r)(1-r)^{-3} - 8\pi^2 r(1+r)(1-r)^{-5} n^{-2} + O(n^{-3}), \\ \max_{f \in S_n^{\mathbb{R}}} |a_m| &= m - \frac{2}{3}\pi^2 (m^3 - m)n^{-2} + O(n^{-3}). \end{aligned}$$

In [3] parametric representations for typically real polynomials are given, which allow to obtain some sharp estimates concerning polynomials in $S_n^{\mathbb{R}}$, e.g., the sharp upper bound for $\max_{|z|=1} |f(z)|$, cf. above.

3. Univalent trinomials

In this section we fix natural numbers k, n and assume $1 < k < n$. We give the exact set of the coefficient pairs $(a, b) \in \mathbb{C}^2$ such that the trinomial $f(z) = z + az^k + bz^n$ is univalent in E .

By a suitable rotation $f_\alpha(z) = e^{-i\alpha} f(ze^{i\alpha})$ we see that it is enough to consider the case $b \geq 0$. Let some $b > 0$ for the moment be fixed and define

$$C_{k,n}(b) := \{a \in \mathbb{C} \mid z + az^k + bz^n \text{ is univalent in } E\}$$

as well as

$$C_{k,n} := \{(a, b) \mid 0 \leq b, a \in C_{k,n}(b)\}.$$

Note that $C_{k,n}(b)$ is empty if $b \geq 1/n$.

In order to determine $C_{k,n}(b)$ we study, according to our general remark above, the equation

$$f(e^{i(\alpha+\beta)}) = f(e^{i(\alpha-\beta)}) \quad (*)$$

which has a unique solution $a = a_1 + ia_2$. It is

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= F_b(\alpha, \beta) \\ &:= \frac{\sin \beta}{\sin k\beta} \begin{pmatrix} -\cos(k-1)\alpha - b \frac{\sin n\beta}{\sin \beta} \cos(n-k)\alpha \\ -\sin(k-1)\alpha - b \frac{\sin n\beta}{\sin \beta} \sin(n-k)\alpha \end{pmatrix} \end{aligned}$$

and it is sufficient to consider $0 \leq \alpha < 2\pi$, $0 < \beta \leq \frac{\pi}{2}$.

Without loss of generality we may assume $\beta \notin \frac{\pi}{k}\mathbb{N}$, because in this case the coefficient a cancels out and what remains has at least discrete solutions – they play no role for the rest.

Obviously, the boundary of $C_{k,n}(b)$ is a subset of the set of singular points of the mapping $F_b: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (in the sense that the total differential JF_b is singular).

We introduce the notations

$$r(\beta) := \frac{\sin \beta}{\sin k\beta}, \quad s(\beta) := -b \frac{\sin n\beta}{\sin k\beta}$$

and obtain (see [11], r', s' stand for the derivative with respect to β) the following theorem.

THEOREM 7. *The boundary of $C_{k,n}(b)$ is contained in the supports of the following curves:*

$$\begin{aligned} \gamma_1(\alpha) &:= F_b(\alpha, 0), \quad \alpha \in [0, 2\pi[\\ \gamma_2(\beta) &:= F_b(\alpha(\beta), \beta), \end{aligned}$$

where

$$\begin{aligned} \cos(n-1)\alpha &= \frac{(n-k)ss' - (k-1)rr'}{(n-k)r's - (k-1)rs'} \quad \text{and} \quad \beta \in \left]0, \frac{\pi}{2}\right], \\ \gamma_3(\alpha) &:= F_b(\alpha, \beta_0), \quad \alpha \in [0, 2\pi[, \end{aligned}$$

where β_0 is a solution of the equitation system $\tan k\beta_0 = k \tan \beta_0$, $\tan n\beta_0 = n \tan \beta_0$.

If we remove from the complex plane the support points of the curves γ_1 , γ_2 and (if it at all exist) γ_3 , then $C_{k,n}(b)$ comes out as the component of this set which contains 0.

It is conjectured that the equitations $\tan k\beta_0 = k \tan \beta_0$, $\tan n\beta_0 = n \tan \beta_0$ never have a joint solution β_0 . But proofs are only known for special combinations of k and n .

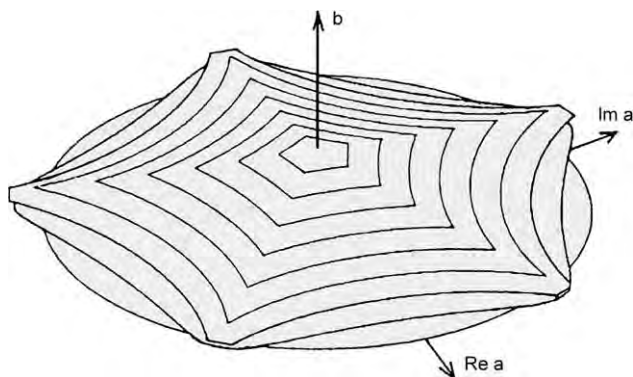
Fig. 1. The coefficient body $C_{3,11}$.

Figure 1 shows the three-dimensional coefficient body of univalent trinomials $z + az^3 + bz^{11}$ for $b \geq 0$.

Double points on the image of $|z| = 1$ occur only where $|a|$ is close to its maximum (corresponding with γ_2), while all the other boundary points come from γ_1 and the related trinomial therefore has a derivative zero somewhere on $|z| = 1$.

The description of $C_{k,n}$ leads to the following sharp coefficient estimations:

THEOREM 8. *Let $f(z) = z + az^k + bz^n$ with $a, b \in \mathbb{C}$ and $1 < k < n$. Then the following is true:*

1. *If f is locally univalent (i.e., f' is nonvanishing) in E , then*

$$|a|k \leq 1 + |b|n \leq 2.$$

2. *If f is univalent in E , then*

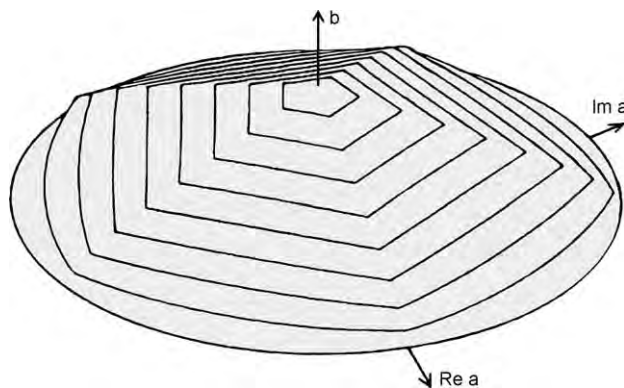
$$|a| \leq \min_{\beta \in B} \left(\left| \frac{\sin \beta}{\sin k\beta} \right| + \left| b \frac{\sin n\beta}{\sin k\beta} \right| \right).$$

3. *f is univalent in E if*

$$|a| \leq \min_{\beta \in B} \left(\left| \frac{\sin \beta}{\sin k\beta} \right| - \left| b \frac{\sin n\beta}{\sin k\beta} \right| \right).$$

We remark that the first inequality easily implies that each trinomial $1 + Az^K + Bz^N$ with complex A, B , $1 \leq K < N$ and $|A| > 2$ has a zero in E . For $K = 1$ is this due to Landau [14, Section 16] and the case of arbitrary K is a result of Fejér [9, Section 415].

Figure 1 shows the typical coefficient body $C_{k,n}$ in the case that $k - 1$ divides $n - 1$. Otherwise, for each b , the double point curves γ_2 play no role for the description of the coefficient body (of course they exist, but they are far away from $C_{k,n}$). This has been proved by Kasten and Schmieder [12] and independently by Rahman and Waniurski [18]:

Fig. 2. The coefficient body $C_{5,11}$.

THEOREM 9. Let $2 < k < n$ be integers and assume that $k - 1$ is no divisor of $n - 1$. Then for every complex trinomial $f(z) = z + az^k + bz^n$, the following statements are equivalent:

- (1) f is univalent in E ,
- (2) f is locally univalent in E ,
- (3) $f'(z) \neq 0$ for all $z \in E$.

The characteristic shape of $C_{k,n}$ in this case is shown in Figure 2 ($k = 5, n = 11$).

For such trinomials we can improve the coefficient estimates as follows [12]:

THEOREM 10. Let k, n, f be as before. If f is univalent in E then

$$|a| \leq \frac{1}{k} \quad \text{and} \quad |b| \leq \frac{1}{n} \left(1 - |a|k \cos \frac{\pi}{\mu} \right),$$

where $\frac{n-1}{k-1} = \frac{\mu}{\nu}$ (right-hand fraction in reduced form).

As an application we mention the following location of zeros [12]:

THEOREM 11. Let $p(z) = 1 + Az^K + Bz^N$ be a trinomial with complex coefficients and assume that K does not divide N .

Let x_0 be the positive zero of $-1 + |A|x^K + |B|x^N$ and x_1 be the positive zero of $-1 + |A|\cos \frac{\pi}{M}x^K + |B|x^N$, where $\frac{N}{K} = \frac{M}{L}$ (reduced). Then the following is true:

- (1) p has no zero in $|z| < x_0$,
- (2) p has at least one zero in the annulus

$$x_0 \leq |z| \leq \min\{x_1, |A|^{-1/K}\}.$$

It has been conjectured that if $k - 1$ is no divisor of $n - 1$ the locally univalent trinomials are not only univalent but, even, close-to-convex (a geometrically-motivated property

which implies univalence, cf. [8,10,17]). Schmieder [19] has proved that this is true up to $n = 25$, but it is false for $n \geq 36$.

Now we give a separate description of the real coefficient body (k, n as above)

$$\widetilde{C}_{k,n} := \{(a, b) \in \mathbb{R}^2 \mid z + az^k + bz^n \text{ is univalent in } U\}.$$

Let $\frac{k-1}{n-k} = \frac{p}{q}$ (reduced fraction). We define the following curves in $\overline{\mathbb{C}}$:

$$\Gamma_1 \quad \text{by} \quad (a(\alpha), b(\alpha)) = \left(-\frac{\sin(n-1)\alpha}{k \sin(n-k)\alpha}, \frac{\sin(k-1)\alpha}{n \sin(n-k)\alpha} \right),$$

$$\Gamma_2^m \quad \text{by} \quad ak + (-1)^{qm}bn = (-1)^{pm+1}, \quad m = 0, 1,$$

$$\Gamma_3^m \quad \text{by} \quad (a(\beta), b(\beta)),$$

where

$$a(\beta) = (-1)^{pm+1} \frac{\cos \beta}{\cos k\beta} \frac{\tan n\beta - n \tan \beta}{k \tan n\beta - n \tan k\beta}$$

and

$$b(\beta) = (-1)^{(p+q)m} \frac{\cos \beta}{\cos n\beta} \frac{\tan k\beta - k \tan \beta}{k \tan n\beta - n \tan k\beta}, \quad m = 0, 1.$$

The points (a, b) on the curves Γ_1 , Γ_2^1 and Γ_2^2 belong to trinomials $p(z) = z + az^k + bz^n$ which derivative vanishes somewhere on $|z| = 1$. The curves Γ_3^1 and Γ_3^2 correspond with certain p such that $\underline{p}(z_1) = p(z_2)$ holds for suitable z_1, z_2 on the boundary of U . These curves determine $\widetilde{C}_{k,n}$ in this way that the boundary of this coefficient body is contained in the union Γ of these five curves. The set $\widetilde{C}_{k,n}$ is the component of $\overline{\mathbb{C}} \setminus \Gamma$ containing 0.

Figure 3 shows the coefficient body $\widetilde{C}_{3,11}$. The dotted curves starting on the straight lines on the left and on the right are parts of Γ_3^1 and Γ_3^2 while the rest of the boundary comes from derivative zeros of the corresponding trinomials.

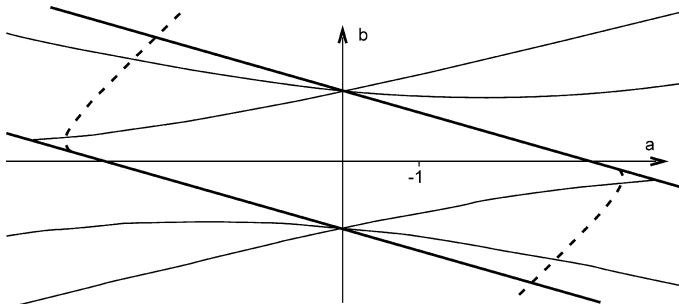


Fig. 3. Curves bounding $\widetilde{C}_{3,11}$.

References

- [1] L.V. Ahlfors, *Complex Analysis*, 3rd edn, McGraw-Hill, New York (1979).
- [2] M. Brandt, *Variationsmethoden für in der Einheitskreisscheibe schlichte Polynome*, Seminarber Humboldt-Univ. Berlin Sect. Math. **96** (1987), 1–91 (in German).
- [3] M. Brandt, *Representation formulas for the class of typically real polynomials*, Math. Nachr. **144** (1989), 29–37.
- [4] M. Brandt, *On univalent polynomials*, Complex Var. Theory Appl. **20** (1992), 93–98.
- [5] D.A. Brannan, *Coefficient regions for univalent polynomials of small degree*, Mathematika **14** (1967), 165–169.
- [6] V.F. Cowling and W.C. Royster, *Domains of variability for univalent polynomials*, Proc. Amer. Math. Soc. **19** (1968), 767–772.
- [7] J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe*, Ann. Sci. Ecole Norm. Sup. **48** (3) (1931), 247–358 (in French).
- [8] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [9] L. Fejér, *Über die Wurzel vom kleinsten Betrage einer algebraischen Gleichung*, Math. Ann. **65** (1909), 413–423 (in German).
- [10] A.W. Goodman, *Univalent Functions*, Vols 1 and 2, Mariner, Tampa, FL (1983).
- [11] V. Kasten and G. Schmieder, *Die Koeffizientenkörper schlichter Trinome*, Math. Z. **171** (1980), 269–284 (in German).
- [12] V. Kasten and G. Schmieder, *Über eine Klasse von Trinomen*, Arch. Math. **35** (1980), 374–385 (in German).
- [13] M. Kössler, *Simple polynomials*, Czech. Math. J. **76** (1951), 5–15.
- [14] E. Landau, *Über den picardschen Satz*, Vierteljschr. Naturforsch. Gesellsch. Zürich **51** (1906), 252–318 (in German).
- [15] M. Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*, Math. Surveys, Vol. 3, Amer. Math. Soc., New York (1949).
- [16] C. Michel, *Eine Bemerkung zu schlichten Polynomen*, Bull. Acad. Polon. Sci. **18** (1970), 513–519 (in German).
- [17] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
- [18] Q.I. Rahman and J. Waniurski, *Coefficient regions for univalent trinomials*, Canad. J. Math. **32** (1980), 1–20.
- [19] G. Schmieder, *Über eine Klasse von Trinomen*, Preprint series Institut für Mathematik, Nr. 136, Universität Hannover, (1982), 1–9 (in German).
- [20] T.J. Suffridge, *On univalent polynomials*, J. London Math. Soc. **44** (1969), 496–504.
- [21] G. Szegő, *Bemerkungen zu einem Satz von J.H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z. **13** (1922), 28–55 (in German).

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CHAPTER 9

Methods for Numerical Conformal Mapping

Dedicated to the memory of Dieter Gaier

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1. Introduction

Riemann formulated in his famous thesis [235] a remarkable mapping theorem which in modern language reads:

THEOREM 1. *Each simply connected region G in the extended complex plane $\bar{\mathbb{C}}$ with at least two boundary points can be mapped conformally to the unit disk D .*

Riemann left us with the problem of how to determine for a given region G such a conformal mapping from G to D , or the inverse mapping from D to G . There are many well-studied classes of analytic functions: polynomials, rational functions, the elementary transcendental functions, such as the exponential, the logarithm and the trigonometric functions, and the higher-transcendental functions, such as elliptic integrals and hypergeometric functions. One can find the known mapping properties of these functions collected in dictionaries, like those of Kober [137] or von Koppenfels and Stallmann [145]. Nehari gives an extensive collection of mapping properties of special functions in Chapter VI of his book [189]. The book of Lavrik and Savenkov [157] contains a catalog of 115 conformal mappings, accompanied by diagrams. Ivanov and Trubetskov [128] offer computer-aided visualization of numerous mapping functions.

When suitable explicit functions cannot be found the only means to determine a conformal mapping is by numerical calculation. The main textbook for numerical conformal mapping was for a long time Gaier's book [65], which is still an excellent source. One must not be discouraged by Gaier's reports on numerical experiments, where computing times of several hours are reported. Since 1964, computers have been improved, the Fast Fourier Transform has been (re)invented by Cooley and Tukey [30] and fast mapping methods have been developed such as, e.g., Wegmann's method [283]. Thus, a mapping problem can now be solved numerically in seconds – even on a small computer. The statement of Symm [253]: “When a conformal mapping, purporting to simplify solution of some problem of applied mathematics, can be obtained only by numerical means, it is often considered to have outlived its usefulness” is no longer true.

The third volume of Henrici's monumental work on *Applied and Computational Complex Analysis* [107] is now one of the main sources for theoretical and computational aspects of numerical conformal mapping.

In 1986 Trefethen edited a collection of articles about numerical conformal mapping [265]. Recent books about computational conformal mapping are those of Kythe [152] and of Schinzinger and Laura [240]. Articles by Anderson et al. [4], Opfer [199], Gutknecht [95] and DeLillo [34] give overviews of available methods. Much work on conformal mapping has been done in the former Soviet Union. Some of it is summarized in the book of Fil'čakov [57].

Conformal mapping is frequently applied for the solution of problems of fluid mechanics (see, e.g., the classic book of Lamb [153]). But it is also a useful tool in grid generation for numerical calculations (see, e.g., Thompson et al. [261,262]).

This review covers mainly the recent literature. For some of the older literature we refer to Gaier's book [65]. Interest in numerical conformal mapping began again to grow in the late seventies and culminated in 1986 in Trefethen's collection [265] of 15 articles. In the

same year Henrici's book [107] appeared. Numerical conformal mapping is still an active field.

A review article about numerical conformal mapping must discuss many (if not all) available methods with the unavoidable consequence that the bewildered reader at the end will not know what to do when he really feels the need to map a region of his choice to some standard region. Authors give, in general, an evaluation biased in favor of their own methods. Therefore, it is important to report also some experimental experience, in particular, about test calculations for the same problem with different methods.

2. Auxiliary material

2.1. Spaces

Functional analytic methods are very powerful for proving convergence of some iterative methods. Function spaces are also convenient for describing certain properties of functions, such as smoothness of various degrees, and properties of some associated functions, such as the harmonic conjugate.

We consider only spaces of (complex or real) 2π -periodic functions. The Lebesgue space L^2 consists of all functions f which are square integrable over the interval $[0, 2\pi]$; it becomes a Hilbert space with norm

$$\|f\|_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt \right)^{1/2}. \quad (1)$$

Each function $f \in L^2$ can be represented by a Fourier series

$$f(t) \sim \sum_{l=-\infty}^{\infty} A_l e^{ilt} \quad (2)$$

(the sign $\sim$ denotes that the series converges in the L^2 norm but in general not pointwise). The norm (1) can be expressed in terms of the Fourier coefficients by

$$\|f\|_2 = \left(\sum_{l=-\infty}^{\infty} |A_l|^2 \right)^{1/2}. \quad (3)$$

Related to L^2 there is the *Sobolev space* W (to be precise, $W^{1,2}$) which consists of absolutely continuous functions with derivative f' in L^2 . It becomes a Banach space when provided with the norm

$$\|f\|_W := \|f\|_2 + \|f'\|_2. \quad (4)$$

The supremum norm

$$\|f\|_{\infty} := \sup_t |f(t)| \quad (5)$$

is defined for bounded functions. The space $C^{(n)}$ of n times differentiable functions with continuous n th derivative $f^{(n)}$ is a Banach space when normed by

$$\|f\|_{C^{(n)}} = \|f\|_{\infty} + \|f^{(n)}\|_{\infty}. \quad (6)$$

For α in the interval $0 < \alpha \leq 1$, the Hölder space C^{α} consists of all functions f which are uniformly Hölder continuous with exponent α . Then the Hölder coefficient

$$[f]_{\alpha} := \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^{\alpha}} \quad (7)$$

is finite. With the norm

$$\|f\|_{\alpha} := \|f\|_{\infty} + [f]_{\alpha}, \quad (8)$$

C^{α} is a Banach space. When f is Hölder continuous with exponent $\alpha = 1$, it is also called *Lipschitz continuous*.

More generally, for an integer $n \geq 0$ the Hölder space $C^{n,\alpha}$ consists of all functions f which are n times differentiable with derivative $f^{(n)} \in C^{\alpha}$. With the norm

$$\|f\|_{n,\alpha} := \|f\|_{\infty} + [f^{(n)}]_{\alpha}, \quad (9)$$

$C^{n,\alpha}$ is a Banach space.

2.2. Conformal mapping

The typical domain G dealt by numerical conformal mapping is bounded by finitely many smooth arcs that may form corners. Parts of the boundary may be run through twice; different parts of the boundary may touch each other.

This is described by the following situation. Let G be a bounded simply connected region. Without loss of generality we assume that G contains the origin 0. The boundary Γ of G is parameterized by a 2π -periodic complex function $\eta(s)$ such that $\eta(s)$ surrounds G once in the counterclockwise direction when s increases from 0 to 2π . The function $\eta(s)$ is differentiable for all s with the exception of finitely many values s_j in the interval $[0, 2\pi]$, and the derivative $\dot{\eta}(s)$ does not vanish. (We denote by a dot always the derivative with respect to the parameter s . A prime denotes derivatives with respect to other variables $t, z, \dots$) The function η is not assumed to be one-to-one. So it can also describe cuts in the region such as shown in the example of Figure 1.

One considers two types of conformal mappings: The mapping F from a region G to a canonical region, which, in general, is the unit disk D , and the inverse mapping Φ from the disk to the region. We will consistently use F and Φ for these two types of mappings. The geometry is illustrated in Figure 2.

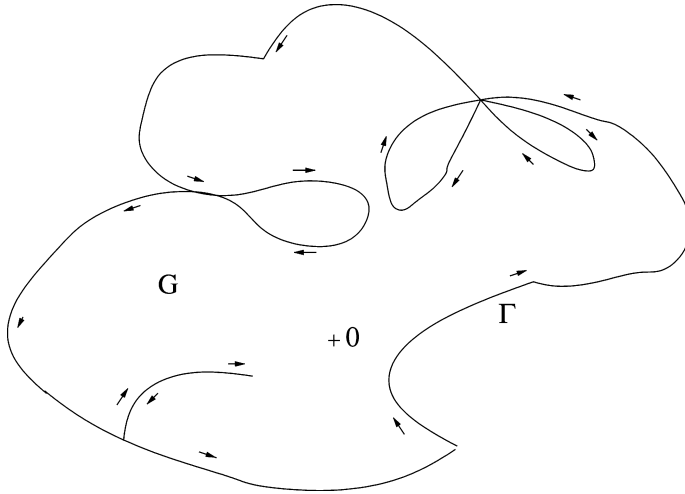


Fig. 1. A typical region G for conformal mapping. The arrows indicate the orientation of the boundary curve.

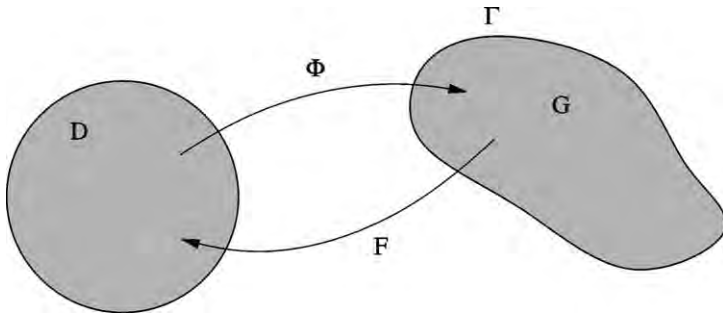


Fig. 2. The conformal mapping F from the region G to the disk and its inverse Φ .

The mapping $\Phi : D \rightarrow G$ is unique only up to conformal automorphisms of D . These are of the form

$$\Psi(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0}z} \quad (10)$$

with $\theta \in \mathbb{R}$ and $z_0 \in D$. These parameters are fixed when Φ is constrained by the conditions

$$\Phi(0) = 0, \quad \Phi'(0) > 0. \quad (11)$$

Instead of the second condition (11) one can also prescribe $\Phi(1) = w_0$ for some w_0 on the boundary $\Gamma := \partial G$. One can also replace both conditions (11) by the prescription of three boundary values $\Phi(e^{it_j}) = w_j$ for $0 \leq t_1 < t_2 < t_3 < 2\pi$ and counterclockwise ordered points w_1, w_2, w_3 on Γ .

It is very helpful to know in advance something about the mapping function. To this aim we collect here some properties of the mapping which can be inferred from the properties of the boundary Γ . For details we refer to the book of Pommerenke [224].

A homeomorphic image of the unit circle is called a *Jordan curve*. A *Jordan region* is the interior of a Jordan curve.

THEOREM 2 (Osgood, Carathéodory; see also [224, p. 18]). *The conformal mapping $\Phi : D \rightarrow G$ can be extended to a homeomorphism $\Phi : \bar{D} \rightarrow \bar{G}$ of the closed disk to the closure $\bar{G}$ of G if and only if G is a Jordan region.*

It follows, in particular, that the extended function Φ restricted to the boundary is a homeomorphism $\Phi : \partial D \rightarrow \partial G$ of the boundaries. Theorem 2 implies that for Jordan regions the inverse mapping $F : G \rightarrow D$ can also be extended to a homeomorphism $F : \bar{G} \rightarrow \bar{D}$.

Numerical conformal mappers have to deal with regions whose boundaries consist of one (or several) closed curves, i.e., of the continuous (not necessarily one-to-one) images of circles. Therefore, the following result is very useful.

THEOREM 3 [224, p. 20]. *The conformal mapping $\Phi : D \rightarrow G$ can be extended to a continuous mapping $\Phi : \bar{D} \rightarrow \bar{G}$ if and only if the boundary of G consists of a closed curve.*

Since the values of Φ in D can be constructed from the values on ∂D by Cauchy's formula

$$\Phi(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\Phi(\zeta)}{\zeta - z} d\zeta \quad \text{for } z \in D, \quad (12)$$

it is sufficient to determine only the boundary values of Φ . This reduces the two-dimensional problem to a one-dimensional one.

The next theorem guarantees that under certain circumstances the mapping function is just as smooth as the boundary curve.

THEOREM 4 (Warschawski–Kellogg [273]). *When the boundary Γ of the simply connected region G is parameterized by a function $\eta \in C^{k,\alpha}$ for some $k \geq 1$ and $0 < \alpha < 1$ then the boundary values $\Phi(e^{it})$ of the conformal mapping Φ of the unit disk D to G as a function of t are also in $C^{k,\alpha}$.*

Theorem 4 does not hold for $\alpha = 1$. It does not hold for $k = 0$ either, as the mappings to regions with corners show (see Section 2.3). When the boundary curve Γ is rectifiable and satisfies the condition that for any two points $z_1, z_2 \in \Gamma$, the length Δs of the shorter arc of Γ between z_1, z_2 satisfies $\Delta s \leq C|z_1 - z_2|$ with a constant C independent of z_1, z_2 , then the conformal mapping $F : G \rightarrow D$ is Hölder continuous with a certain exponent α which depends on C (Warschawski [278]; see also Lesley [162]).

One can assign to each complex 2π -periodic function η which parameterizes the boundary of a region G , the boundary values $\Phi(e^{is})$ of the conformal mapping $\Phi : D \rightarrow G$. The

continuity of this mapping $\eta \rightarrow \Phi(e^{i\eta})$ in certain Hölder and Sobolev spaces is investigated by Warschawski [274] and Lanza de Cristoforis [155].

A theorem of Coifman and Meyer (see Semmes [242]) says that the conformal mapping function $F: G \rightarrow U$ from the region to the upper half-plane U depends (in suitable topologies) in a real-analytic way on the boundary curve of G .

2.3. Corners

Regions with corners are beyond the range of many mapping methods. Corners deserve special attention since in any case they deteriorate the accuracy of the result. Therefore, it is often advisable to give the corners a special treatment using a priori information about the behavior of the mapping function near corners.

It is generally assumed that a corner is formed by analytic arcs. Recall that an *analytic arc* is the image of an interval $[a, b]$ by a function f which is analytic in a neighborhood of $[a, b]$. The function $f(z) = z^\alpha$, $0 < \alpha \leq 2$, maps the upper half-plane U to a wedge region with opening angle $\alpha\pi$. This mapping is representative for mappings to regions with corners, since the following theorem holds. We consider mappings to U and assume that $0 \in \Gamma$, and Γ has a corner at 0.

THEOREM 5 (Lichtenstein, Warschawski; see Henrici [107]). *Assume that the boundary Γ of G has a corner of opening angle $\alpha\pi$ at $z = 0$ formed by two analytic arcs. If F maps G conformally to the upper half-plane U so that $F(0) = 0$, then the limit*

$$c := \lim_{z \rightarrow 0} (z^{-1/\alpha} F(z)) \quad (13)$$

exists and is not equal to 0. For $n = 1, 2, \dots$,

$$\lim_{z \rightarrow 0} (z^{n-1/\alpha} F^{(n)}(z)) = c \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) \cdots \left(\frac{1}{\alpha} - n + 1 \right) \quad (14)$$

holds for unrestricted approach $z \rightarrow 0$ in G .

Therefore, at a corner the conformal mapping $F: G \rightarrow U$ behaves locally like the function $cz^{1/\alpha}$ which straightens the corner at 0. The condition that the corner is formed by analytic arcs has been somewhat relaxed by Gaier [74].

Let G and F be defined as in Theorem 5. Since the corner at 0 is formed by two analytic arcs, F can be extended by reflection to a function defined on a Riemann surface covering a neighborhood of the origin. Lehman [160] has shown that for irrational α there is an asymptotic expansion of F in the neighborhood of the origin, valid in any sector, in integral powers of z and $z^{1/\alpha}$. If α is rational there is an asymptotic expansion of F in integral powers of z , $z^{1/\alpha}$ and $\log z$. In both cases the term $z^{1/\alpha}$ occurs with nonzero coefficient. The inverse mapping $\Phi := F^{-1}$ from the upper half-plane to the region G admits near the origin an asymptotic expansion in integral powers of z , z^α and $\log z$ for rational α ,

and in integral powers of z and z^α for irrational α . In both cases the term z^α has nonzero coefficient.

The accuracy of the numerically calculated mapping depends on the smoothness of the boundary. For general regions with corners it is sometimes advisable to remove the corners by some auxiliary mappings of the type $f(z) = (z - z_0)^\alpha$ (see, e.g., Carey and Muleshkov [25]). Landweber and Miloh [154] consider a transformation which removes all corners of a simple closed curve at the same time.

Several of the methods discussed in the next sections work only for sufficiently smooth boundaries. Some methods can be adapted to treat corners. For polygons a Schwarz–Christoffel mapping is advisable (see Trefethen [264], Henrici [107], Driscoll and Trefethen [47]).

2.4. Crowding

The behavior of a conformal mapping depends on the local property of smoothness – and on the global property of shape.

On small scales a conformal mapping maps disks to disks, but on large scales a disk can be mapped to any simply-connected bounded region, however elongated and distorted it may be. But it takes some effort for a mapping which has such a strong tendency to map disks to disks, to map a disk to an elongated region. The mapping suffers, lying on a Procrustean bed, and the numerical conformal mapper must share the pains.

It was first noted by Grassman [86] that the numerical calculation of the mapping from the disk to an elongated region becomes laborious due to an effect which is now called *crowding*. The images of equidistributed points on the unit circle become very unevenly distributed on the boundary of the region.

This is nicely illustrated by the inverse hyperbolic tangent function

$$\Phi(z) = \text{Arctanh}(rz) \quad (15)$$

which, for $0 < r < 1$, maps the unit disk to an elongated region G with axes $a := 2\Phi(1) = 2 \text{Arctanh} r$, $b := 2|\Phi(i)| = 2 \arctan r$ (see DeLillo [37]). The aspect ratio of G is

$$\tau := \frac{b}{a} = \frac{\arctan r}{\text{Arctanh} r}. \quad (16)$$

The supremum norm of the derivative

$$\|\Phi'\|_D = \sup_{|z|<1} |\Phi'(z)| \quad (17)$$

measures the distortion of the mapping. The distortion $|\Phi'(z)|$ is maximal on the boundary.

For the inverse hyperbolic tangent function (15) the distortion can be calculated and expressed in terms of the aspect ratio

$$\|\Phi'\|_D = |\Phi'(1)| = \frac{r}{1-r^2} \approx \frac{b}{2\pi} \exp\left(\frac{\pi}{2\tau}\right). \quad (18)$$

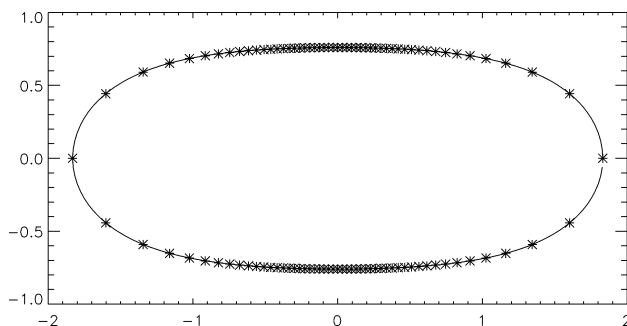


Fig. 3. The images (stars) of 100 equidistributed points on the unit circle mapped by the inverse hyperbolic tangent (15) with parameter $r = 0.95$.

The sign $\approx$ in this section means that the ratio of the left-hand and right-hand sides tends to 1 as $\tau \rightarrow 0$. Equation (18) means that the distortion increases exponentially when the image region becomes more and more elongated. Figure 3 shows the images of 100 equidistributed points on the unit circle for $r = 0.95$. These points assemble in the flat part of the boundary curve while the ends are only poorly covered by the image points.

A sort of crowding was already detected by Gaier [66] in the mapping to a rectangle. When the unit disk is mapped to a rectangle with sides $a > b$ in such a way that the corners correspond to the points $\pm \exp(\pm i\theta)$ then (see DeLillo [37])

$$\theta \approx 4 \exp\left(-\frac{\pi}{2\tau}\right) \quad \text{with } \tau = \frac{b}{a}. \quad (19)$$

This means that the preimages of the small sides of the rectangle become exponentially small when the aspect ratio τ of the rectangle tends to 0. One might suspect that this behavior is caused by the corners of the image region where the derivative Φ' becomes unbounded. Crowding occurs however also for regions with analytic boundaries, such as ellipses. Numerical experiments indicate that it depends only on the aspect ratio τ and increases exponentially with $1/\tau$.

The example of the inverse hyperbolic tangent function is typical for the distortion of conformal maps of the disk to elongated regions. Wegmann [289,290] proved the following result.

THEOREM 6. *When the region G can be enclosed in a rectangle with sides a and b , $b \leq a$, such that G touches both small sides (see Figure 4) then the conformal mapping $\Phi : D \rightarrow G$ satisfies*

$$\|\Phi'\|_D \geq b\psi(b/a) \quad (20)$$

with a function $\psi(\tau)$ which behaves for small τ like

$$\psi(\tau) \approx \frac{1}{2\pi\sqrt{e}} \exp\left(\frac{\pi}{2\tau}\right). \quad (21)$$

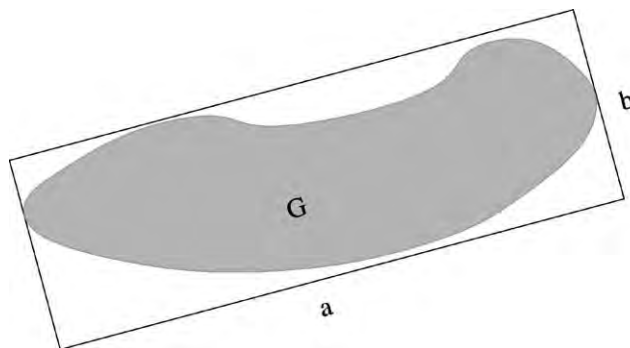


Fig. 4. The aspect ratio $b:a$ of the region G for Theorem 6.

The constant in (21) is best possible. Equality in (20) for $\tau \rightarrow 0$ is asymptotically attained by sportgrounds bounded by two long straight lines connected by two nearly circular arcs.

To give an impression of how crowding disturbs conformal mapping procedures, consider a region G of breadth $b = 1$ and length $a = 10$. Let $\zeta_j := \exp(2\pi j i/N)$ be N equidistributed points on the unit circle. The maximum distance of adjacent image points $\Phi(\zeta_j)$ by a conformal map is then about $2\pi \|\Phi'\|_D/N \geq 4 \times 10^6/N$. For a good resolution of the boundary of G one needs therefore a grid with several millions of points. The problems connected with the mapping of the disk to an ellipse with axis ratio 1:5 are nicely illustrated in Figure 7.5 and Table 7.4 of Gutknecht's paper [94].

Crowding is cumbersome for all methods which work with grid points. On the other hand, methods which approximate the mapping functions by polynomials also face severe problems when the target region is elongated. Szegő [257] proved the following sharp estimate:

THEOREM 7. *If P is a polynomial of degree n such that $P(0)$ is real and $|\operatorname{Re} P(z)| \leq 1$ for $|z| \leq 1$, then the imaginary part is estimated by*

$$|\operatorname{Im} P(z)| \leq \frac{2}{n+1} \sum_{k=1}^{[(n+1)/2]} \cot\left(\frac{(2k-1)\pi}{2n+2}\right). \quad (22)$$

The right-hand side of (22) is asymptotically equal to $(2/\pi) \ln n$ for $n \rightarrow \infty$. This means that the aspect ratio of the image $P(D)$ of the unit disk by a polynomial of degree n is asymptotically

$$\tau \geq \frac{\pi}{2 \ln n}. \quad (23)$$

It follows that, for the mapping of the disk to a region of aspect ratio 1:10 by a polynomial, the degree must be of several millions.

In any case, the number of grid points and the degree of the approximating polynomials increase both like $\exp(\pi/2\tau)$ as the aspect ratio, τ , tends to zero.

One can to some extent avoid or reduce the adverse effects of crowding by choosing as the fundamental domain not the disk but an ellipse with similar aspect ratio as the target region (see DeLillo and Elcrat [39] and Wegmann [291]).

Domain decomposition methods, such as described by Papamichael and Stylianopoulos [214], cut the region G into pieces and approximate the mapping from G to a rectangle by means of the mappings of the subregions.

DeLillo [35] used an inequality of Rengel to relate the crowding for elongated regions to the crowding (19) for rectangles. DeLillo and Pfaltzgraff [43] gave estimates for crowding in terms of harmonic measure and extremal distance. These estimates also show the exponential increase. They are more complicated than those of Theorem 6, but are more generally applicable.

DeLillo [37] has shown how crowding affects the accuracy of numerical computations. Crowding also limits the practical usefulness of conformal maps. This was demonstrated by DeLillo [36] for the Laplace equation.

Crowding has also been observed for regions with elongated sections (“fingers”). For “pinched” regions, such as the interior of an inverted ellipse, ill conditioning occurs of a less severe, algebraic nature (DeLillo [37]).

2.5. Function theoretic boundary value problems

Riemann considered in his thesis [235] conformal mapping as a special case of a more general class of boundary value problems for analytic functions, which are now called nonlinear *Riemann–Hilbert* (RH) *problems* (see Wegert [281]). One of the most effective strategies for dealing with the conformal mapping problem solves the nonlinear RH problem by a Newton iteration using in each iterative step a linear RH problem.

We consider here mainly boundary value problems for analytic functions in the unit disk D or in the exterior of the disk $D^- := \{z: |z| > 1\}$. By saying that a function is analytic in D^- we tacitly assume that it is also analytic at ∞ .

Let us start with the simplest and most basic problem.

For a given real Hölder continuous 2π -periodic function ψ , a function Ψ is to be determined as analytic in D , continuous in $\bar{D}$ and satisfying on the boundary,

$$\operatorname{Re} \Psi(e^{it}) = \psi(t). \quad (24)$$

It is well known that this problem has a solution which is unique up to an imaginary constant. The imaginary part of Ψ can be constructed by means of the operator $\mathbf{K}$ of *conjugation*, which can be defined in several equivalent ways. Usually it is defined as a singular integral operator

$$\mathbf{K}\psi(s) := \frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot \frac{s-t}{2} dt, \quad (25)$$

where the integral is understood as a Cauchy principal value integral. The operator $\mathbf{K}$ is sometimes also called *Hilbert transform* (see, e.g., Henrici [107, p. 103]).

For numerical calculations the representation in terms of Fourier series is most convenient, since it can be evaluated numerically in a very efficient way using Fast Fourier Transforms (FFT).

When the right-hand side of (24) is represented by a (real or complex) Fourier series

$$\psi(t) = \sum_{l=-\infty}^{\infty} A_l e^{ilt} = a_0 + \sum_{l=1}^{\infty} (a_l \cos lt + b_l \sin lt) \quad (26)$$

then the conjugate function $\mathbf{K}\psi$ has the (real or complex) Fourier series representation

$$\mathbf{K}\psi(s) = \sum_{l=1}^{\infty} (-iA_l e^{ils} + iA_{-l} e^{-ils}) = \sum_{l=1}^{\infty} (-b_l \cos ls + a_l \sin ls). \quad (27)$$

This means that conjugation is done in the complex Fourier series simply by multiplication of the complex Fourier coefficients by $\pm i$, and in the real trigonometric series by interchange of the coefficients and multiplication by ± 1 .

The operator $\mathbf{K}$ of conjugation maps each of the spaces L^2 , W , $C^{n,\alpha}$, $0 < \alpha < 1$, into itself and is a bounded operator in each of these spaces.

A theorem of Plessner says something about pointwise convergence: When the series (26) converges everywhere on a measurable subset E of $[0, 2\pi]$ then the series (27) converges almost everywhere on E .

Jumps in ψ generate logarithmic singularities in $\mathbf{K}\psi$. To take care of these explicitly, sometimes the representation as a Stieltjes integral

$$\mathbf{K}\psi(s) := \frac{1}{\pi} \int_0^{2\pi} \log \left| \sin \frac{s-t}{2} \right| d\psi(t) \quad (28)$$

is used which is obtained by integrating (25) by parts (see DeLillo [34]).

The property of conjugate functions most important for what follows is contained in the following theorem.

THEOREM 8. (a) *For each function $\psi \in W$, there is a unique function Ψ , which is analytic in D , continuous in $\bar{D}$ and satisfies (24) and $\text{Im } \Psi(0) = 0$. This function has the boundary values*

$$\Psi(e^{it}) = (\mathbf{I} + i\mathbf{K})\psi(t) \quad (29)$$

with the identity operator $\mathbf{I}$.

(b) *There is also a unique function Ψ , which is analytic in the exterior D^- of the unit disk (including ∞), and continuous in $|z| \geq 1$ and satisfies (24) and $\text{Im } \Psi(\infty) = 0$. This function has the boundary values*

$$\Psi(e^{it}) = (\mathbf{I} - i\mathbf{K})\psi(t). \quad (30)$$

This means that the operator $\mathbf{K}$ constructs from the boundary values of the real part of an analytic function ψ in D (or D^-) the boundary values of the imaginary part up to a constant.

For numerical calculations ψ is represented on a grid of $N = 2n$ equidistant points $t_j = (j-1)2\pi/N$ by the interpolating trigonometric polynomial of degree n

$$\psi(t_j) = \sum_{l=-n+1}^n \tilde{A}_l e^{ilt_j} = \sum_{l=0}^n \tilde{a}_l \cos lt_j + \sum_{l=1}^{n-1} \tilde{b}_l \sin lt_j \quad (31)$$

for $j = 1, \dots, N$. The conjugate function $\mathbf{K}_N \psi$ has the representation by a (real or complex) Fourier polynomial

$$\mathbf{K}_N \psi(s) = \sum_{l=1}^{n-1} (-i\tilde{A}_l e^{ils} + i\tilde{A}_{-l} e^{-ils}) = \sum_{l=1}^{n-1} (-\tilde{b}_l \cos ls + \tilde{a}_l \sin ls). \quad (32)$$

Conjugation is thus reduced to forward and inverse Fourier transform. This can be evaluated very fast by FFT (Cooley and Tukey [30]). The representation (32) is equivalent to Wittich's method which evaluates discrete conjugation by multiplication by a matrix with checkerboard structure (see Gaier [65, p. 75]). But while this matrix multiplication needs $O(N^2)$ operations, the Fourier transform with FFT needs only $O(N \log N)$ operations, and therefore is much faster (Henrici [103,104], Gutknecht [91]).

There is a qualitative difference between $\mathbf{K}$ and $\mathbf{K}_N$ which becomes important at some places. The operator $\mathbf{K}$ has defect 1. It maps the constant functions to 0. The operator $\mathbf{K}_N$ has defect 2. It maps the constant functions to 0, but also the function $\cos nt$ which on the grid is simply a sequence of alternating $+1$ and -1 .

The function $\mathbf{K}_N \psi$ is defined for all t by the trigonometric polynomial on the right-hand side of (32) with coefficients which are calculated by trigonometric interpolation (31) of ψ at the grid points. The accuracy measured by the maximum norm can be estimated in terms of the Fourier coefficients of ψ (see Gaier [67], Henrici [105])

$$\|\mathbf{K}\psi - \mathbf{K}_N \psi\|_\infty \leq |A_n| + |A_{-n}| + 2 \sum_{|l|>n} |A_l|. \quad (33)$$

With well-known estimates for the Fourier coefficients of smooth functions one obtains from (33) the following estimates: If ψ is analytic and bounded by $|\psi(z)| \leq M$ in a strip $S_a := \{z: |\operatorname{Im} z| < a\}$ around the real axis, then

$$\|\mathbf{K}\psi - \mathbf{K}_N \psi\|_\infty = 2M \coth \frac{a}{2} e^{-an}, \quad (34)$$

i.e., the error is $O(e^{-an})$ as $n \rightarrow \infty$ (Gaier [67], Kreß [148]). Gaier [67] shows that

$$\|\mathbf{K}\psi - \mathbf{K}_N \psi\|_\infty = O(n^{-\alpha+1/2}) \quad (35)$$

if ψ is in the Hölder class C^α for some $\alpha > 1/2$.

For $(k-1)$ -times differentiable functions ψ whose derivative $\psi^{(k-1)}$ is absolutely continuous and $\psi^{(k)}$ is bounded, the estimate

$$\|\mathbf{K}\psi - \mathbf{K}_N\psi\|_\infty = O(n^{-k} \log n \|\psi^{(k)}\|_\infty) \quad (36)$$

holds. The O order in (36) is best possible, in the sense that it cannot be improved by using different methods of numerical conjugation based on the same grid (Braß [22]). When ψ is in $C^{k,\alpha}$ then (36) can be improved to

$$\|\mathbf{K}\psi - \mathbf{K}_N\psi\|_\infty = O(n^{-(k+\alpha)} \log n [\psi^{(k)}]_\alpha). \quad (37)$$

These estimates are important, since the accuracy of several mapping methods which are based on function conjugation, is determined by the accuracy of the approximation $\mathbf{K}_N$ of the conjugation operator $\mathbf{K}$.

In some cases it may be a disadvantage that FFT require equidistant grid points, since some effects (e.g., crowding, see Section 2.4) may make a nonequidistant grid preferable. There have been several attempts to develop fast Fourier transform on nonequidistant grids (see, e.g., Luchini and Manzo [169], Sugiura and Torii [251], Dutt and Rokhlin [49], Beylkin [19], Anderson and Dillon Daleh [5], Steidl [249]). According to Steidl, the most efficient algorithms for the fast direct and indirect Fourier transform are those proposed by Dutt and Rokhlin [49] and by Beylkin [19].

One could instead of trigonometric interpolation (31) use another interpolation procedure, e.g., by periodic splines. The conjugate of such an interpolating function can still be calculated by FFT, but the coefficients in (32) have to be multiplied by suitable *attenuation factors* (Gautschi [82]). The calculation of the conjugate of a rational trigonometric function has been discussed by Gutknecht [93]. Li and Syngellakis [165] evaluate conjugation by a boundary element method. They introduce a *generalized conjugation* which satisfies the property $\mathbf{K}^2\psi = -\psi + \text{const}$ of the conjugation operator $\mathbf{K}$ only approximately.

One can solve the boundary value problem (24) directly by solving the Cauchy–Riemann equations in polar coordinates by difference methods. For this purpose it is convenient to use a staggered grid (see Chakravarty and Anderson [27]).

Let $A(t) \neq 0$ be a complex and $\psi(t)$ a real function, both Hölder continuous and 2π -periodic. The (linear) RH problem asks for a function Ψ analytic in D , continuous in $\bar{D}$, such that the boundary values satisfy the relation

$$\operatorname{Re}(\overline{A(t)}\Psi(e^{it})) = \psi(t). \quad (38)$$

The function A can be represented in the form

$$A(t) = r(t)e^{i\theta(t)} \quad (39)$$

with Hölder continuous functions θ and $r > 0$. Linear RH problems are studied in full generality in the book of Muskhelishvili [187]. We consider here only the case where the function A has *winding number*

$$m := \frac{1}{2\pi}(\theta(2\pi) - \theta(0)) \geq 0. \quad (40)$$

The RH problem can be solved in closed form using the operator of conjugation. The function

$$v(t) := \theta(t) - mt \quad (41)$$

is 2π -periodic. Let $w := \mathbf{K}v$ be its conjugate.

THEOREM 9. *The general solution of (38) is obtained by*

$$\Psi(e^{it}) = \frac{e^{i\theta}}{e^w} \left[(\mathbf{I} + i\mathbf{K}) \left(\frac{e^w}{r} \psi \right) + P_m(e^{it}) \right] \quad (42)$$

with a Laurent polynomial

$$P_m(z) = ip_0 + \sum_{j=1}^m (p_j z^j - \overline{p_j} z^{-j}) \quad (43)$$

with a real number p_0 and complex coefficients p_j .

Hence the general solution of (38) contains $2m + 1$ free real parameters $p_0, \operatorname{Re} p_j, \operatorname{Im} p_j$, $j = 1, \dots, m$.

We note the special case which is most important for conformal mapping in the following corollary. We define the averaging operator

$$\mathbf{J}\psi := \frac{1}{2\pi} \int_0^{2\pi} \psi(t) dt. \quad (44)$$

COROLLARY 1. *When $m = 1$ and the angle $\alpha := \mathbf{J}(v)$ is not congruent $\pi/2$ modulo π then the RH problem*

$$\operatorname{Im}(\overline{A(t)}\Psi(e^{it})) = \psi(t) \quad (45)$$

with the constraints

$$\Psi(0) = 0, \quad \operatorname{Im} \Psi'(0) = 0 \quad (46)$$

has a unique solution Ψ . The solution is given by

$$\Psi(e^{it}) = \frac{e^{i\theta}}{e^w} \left[(\mathbf{I} + i\mathbf{K} + \cot \alpha \cdot \mathbf{J}) \left(\frac{e^w}{r} \psi \right) \right]. \quad (47)$$

For the exterior problem we note only the following special result:

There is a function Ψ analytic in D^- except for a pole of order m at ∞ which satisfies the boundary condition (38) on the unit circle. The function is unique up to a real parameter. The general form of this function is given by

$$\Psi(e^{it}) = e^{w+i\theta} \left[(\mathbf{I} - i\mathbf{K}) \left(\frac{\psi}{e^{wr}} \right) + ip_0 \right] \quad (48)$$

with a real number p_0 .

2.6. The operator $\mathbf{R}$

In some conformal mapping methods, boundary value problems

$$\Psi(e^{it}) = B(t) + A(t)U(t) \quad (49)$$

occur with given complex functions A, B . The real function U and the analytic function Ψ in D must be determined so that (49) is satisfied. By multiplication by $\overline{A}$ the real function U can be eliminated from (49) and an RH problem

$$\operatorname{Im}(\overline{A(t)}\Psi(e^{it})) = \operatorname{Im}(\overline{A(t)}B(t)), \quad (50)$$

for the analytic function Ψ , remains.

Iterative methods need the function U in the first place, not Ψ . Therefore, it is desirable to eliminate Ψ from (49) in order to get an equation for U instead. We will show in this section that U must satisfy a second kind Fredholm integral equation. Since the kernel of this equation has very nice properties, methods based on this equation are very efficient (see Section 4.4).

In the conformal mapping problem the function A has winding number 1. Therefore, we consider functions of the form

$$A(t) = \exp(i\beta(t)) \quad (51)$$

with a Hölder continuous real function β such that $\beta(t) - t$ is 2π -periodic. We define the integral operator

$$\mathbf{R}_\beta f(t) := \int_0^{2\pi} R_\beta(t, s) f(s) ds \quad (52)$$

with the symmetric kernel

$$R_\beta(t, s) := -\frac{1}{2\pi} \frac{\sin(\beta(t) - \beta(s) - (t - s)/2)}{\sin((t - s)/2)}. \quad (53)$$

The operator can be expressed by

$$\mathbf{R}_\beta U = \operatorname{Re}(e^{-i\beta}(\mathbf{J} - i\mathbf{K})(e^{i\beta}U)) \quad (54)$$

in terms of the conjugation operator $\mathbf{K}$ and the averaging operator $\mathbf{J}$ defined in (44).

The operator $\mathbf{R}_\beta$ plays a role in the solution of the problem (49) due to the following fact (Wegmann [284]):

THEOREM 10. *There exists a function Ψ analytic in D with $\Psi(0) = 0$ satisfying the boundary problem (49) if and only if U is a solution of the Fredholm integral equation of the second kind*

$$(\mathbf{I} + \mathbf{R}_\beta)U = g \quad (55)$$

with the right-hand side

$$g := -\operatorname{Re}(e^{-i\beta}(\mathbf{I} - i\mathbf{K} + \mathbf{J})B). \quad (56)$$

Multiplication of (55) from the left by $\mathbf{I} - \mathbf{R}_\beta$ yields the equation

$$(\mathbf{I} - \mathbf{R}_\beta^2)U = (\mathbf{I} - \mathbf{R}_\beta)g. \quad (57)$$

We first note that both the equations (55) and (57) have a solution when the function g on the right-hand side is given by (56). The general solution is given by

$$U = U_0 + c \exp(-w) \quad (58)$$

with a particular solution U_0 , the conjugate $w := \mathbf{K}v$ of the function $v(t) := \beta(t) - t$ and an arbitrary real number c . Therefore, uniqueness of the solution can be enforced by prescribing the value $U(t_0) = a_0$ of U at a specified point t_0 .

When β is Hölder continuous with exponent μ , the kernel has along the diagonal a weak singularity of order $|t - s|^{\mu-1}$. Hence, for $\mu > \frac{1}{2}$, the operator $\mathbf{R}_\beta$ is compact in L^2 . Since the kernel R_β is symmetric, all eigenvalues of $\mathbf{R}_\beta$ are real. It follows from (54) that the norm of $\mathbf{R}_\beta$ in L^2 is ≤ 1 . Hence, all eigenvalues of $\mathbf{R}_\beta$ are in the interval $[-1, +1]$. Therefore, the symmetric operator $\mathbf{I} - \mathbf{R}_\beta^2$ is positive semidefinite. This makes equation (57) amenable to conjugate gradient methods. These are very efficient due to the favorable eigenvalue distribution of $\mathbf{R}_\beta$. This was noted in numerical experiments by Fornberg [62], and proved by Wegmann [284].

THEOREM 11. (a) $\lambda_0 = -1$ is a simple eigenvalue of $\mathbf{R}_\beta$. All other eigenvalues λ satisfy $|\lambda| < 1$.

(b) If 0 is an eigenvalue of $\mathbf{R}_\beta$, then its multiplicity is either an even number or infinity.

(c) If $\lambda \in (0, 1)$ is an eigenvalue of $\mathbf{R}_\beta$, then $-\lambda$ is also an eigenvalue with the same multiplicity as λ .

(d) When the k th derivative of β is in C^α then at most $2n + 1$ eigenvalues λ of $\mathbf{R}_\beta$ satisfy

$$|\lambda| > C \|A\|_{k,\alpha} n^{-k-\alpha}. \quad (59)$$

(e) When $A(s) = F(e^{is})$ with a function F analytic in the annulus $1/r < |z| < r$ and bounded by $|F(z)| \leq M$ then at most $2n + 1$ eigenvalues of $\mathbf{R}_\beta$ satisfy

$$|\lambda| > CMr^{-n}. \quad (60)$$

In (d) and (e) the constant C is independent of n and A .

The eigenvalue -1 has the corresponding eigenfunction $\exp(-w)$ which occurs in (58). The properties (d) and (e) say that the eigenvalues of $\mathbf{I} - \mathbf{R}_\beta^2$ cluster at 1 when A is sufficiently smooth, and that only very few differ from 1 by an appreciable amount. Therefore, conjugate gradient methods converge very fast. Their efficiency is also due to the fact that in view of the representation (54) $\mathbf{R}_\beta U$ can be evaluated by FFT. Also, the function g in (56) and the right-hand side of (57) can be calculated by FFT.

3. Mapping from the region to the disk

Let G be a simply-connected region as described in Section 2.2 with boundary parameterization $\eta(s)$, and let F be the conformal mapping from G to the unit disk D normalized by

$$F(0) = 0, \quad F'(0) > 0. \quad (61)$$

The number $1/F'(0)$ is called the *conformal radius* of G at 0. It follows from Theorem 3 that there is a continuous function $T(s)$ such that $T(s) - s$ is 2π -periodic and

$$F(\eta(s)) = \exp(iT(s)). \quad (62)$$

The mapping is completely described by the function $T(s)$ which is called (*interior*) *boundary correspondence function*.

The values of $F(z)$ for interior points $z \in G$ can be calculated from the boundary values (62) by Cauchy's formula

$$F(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{F(\eta(s))}{\eta(s) - z} \dot{\eta}(s) ds. \quad (63)$$

3.1. Potential theoretic methods

Several methods for conformal mapping of a region G to the unit disk D are based on the following simple observation. The function

$$H(z) := \log(F(z)/z) \quad (64)$$

is analytic in G and has boundary values

$$H(\eta(s)) = -\log|\eta(s)| + i[T(s) - \arg \eta(s)]. \quad (65)$$

Its real part $u := \operatorname{Re} H$ is a harmonic function in G with boundary values

$$u(\eta(s)) = -\log|\eta(s)| \quad (66)$$

on ∂G . This means that the harmonic function u solves a *Dirichlet problem*. A boundary problem of the kind (66) occurs in the construction of the *Green's function* $g(z, 0)$ of the region G with respect to the point 0. This gives the relation $u(z) = \log|F(z)| - \log|z| = -g(z, 0) - \log|z|$ (see, e.g., Nehari [189]). (Note that the singularity of the Green's function is sometimes defined with a factor $1/2\pi$.)

We assume for this section that G is a Jordan region and the boundary parameterization η is differentiable with continuous derivative $\dot{\eta}(s) \neq 0$. The Dirichlet problem of potential theory has a solution and this solution is unique. There are several methods for the calculation of the solution u .

An integral equation of the second kind can be derived starting from Cauchy's integral formula (63) restricted to the boundary

$$F(\eta(s)) = \frac{1}{\pi i} \int_0^{2\pi} \frac{F(\eta(t)) \dot{\eta}(t)}{\eta(t) - \eta(s)} dt. \quad (67)$$

This integral with a singularity at $t = s$ must be interpreted as a Cauchy principal value integral.

The Cauchy kernel can be split into its real and imaginary parts

$$\frac{1}{\pi i} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} = K_1(s, t) + iK_2(s, t). \quad (68)$$

Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the integral operators with kernels K_1 and K_2 , respectively.

With the components $\eta(s) = x(s) + iy(s)$ of η we get the explicit representation

$$K_1(s, t) = \frac{1}{\pi} \frac{(x(t) - x(s))\dot{y}(t) - (y(t) - y(s))\dot{x}(t)}{(x(t) - x(s))^2 + (y(t) - y(s))^2}, \quad (69)$$

$$K_2(s, t) = -\frac{1}{\pi} \frac{(x(t) - x(s))\dot{x}(t) + (y(t) - y(s))\dot{y}(t)}{(x(t) - x(s))^2 + (y(t) - y(s))^2}. \quad (70)$$

The representation (68) is very useful, since for sufficiently smooth curves, only the kernel K_2 is singular. If the second derivative of η at s exists, then it follows from Taylor's formula that

$$\lim_{t \rightarrow s} K_1(s, t) = \frac{1}{2\pi} \kappa(s) |\dot{\eta}(s)| \quad (71)$$

with the curvature $\kappa(s)$ of the curve Γ at the point $\eta(s)$. Hence the kernel K_1 is bounded. When η is only differentiable with $\dot{\eta} \in C^\alpha$ then it follows from $|K_1(s, t)| \leq C|s - t|^{\alpha-1}$ that K_1 has a weak singularity on the diagonal. This has the consequence that Fredholm's theorems are valid for the operator $\mathbf{K}_1$ (see, e.g., Mikhlin [181, p. 59]).

The kernel K_1 is well known in potential theory. This is due to the fact that the boundary values of a double layer potential with density μ and the normal derivative of a single layer potential with density σ can both be expressed in terms of the operator $\mathbf{K}_1$ and its adjoint.

It is well known, that all eigenvalues λ of $\mathbf{K}_1$ are real and contained in the interval $(-1, 1]$. The eigenvalue $\lambda_1 = +1$ is simple and the corresponding eigenfunction is $f_1 \equiv 1$. Let λ_2 be the eigenvalue $\neq 1$ of $\mathbf{K}_1$ with largest modulus $|\lambda_2|$. Ahlfors [1] gives an estimate for $|\lambda_2|$.

Integral equations with kernel K_1 were first studied by Carl Neumann in 1877 long before Fredholm theory was developed. Therefore, K_1 is called *Neumann kernel*, or to be more specific the *parametric Neumann kernel* (Henrici [107, p. 394], where the Neumann kernel is defined with s, t interchanged. The usual definition of the Neumann kernel is for curves with parameterization by arclength. In the general definition (69) the factor $|\dot{\eta}(t)|$ occurs).

By taking the real and imaginary parts of (67), the integral equations

$$(\mathbf{I} - \mathbf{K}_1)F_r = -\mathbf{K}_2F_i \quad \text{and} \quad (\mathbf{I} - \mathbf{K}_1)F_i = \mathbf{K}_2F_r \quad (72)$$

are obtained which connect the real and imaginary parts of the boundary values $F(\eta(s)) = F_r + iF_i$ of F . For Hölder continuously differentiable η , the equations (72) are Fredholm integral equations of the second kind for F_r when F_i is given, or for F_i , when F_r is known. The right-hand sides of (72) are calculated by applying the singular integral operator $\mathbf{K}_2$ to the known function F_i or F_r .

The second of the equations (72) applied to the boundary values (65) of the function H defined in (64) gives the *integral equation of Lichtenstein*

$$(\mathbf{I} - \mathbf{K}_1)g = \phi(s) := -\mathbf{K}_2(\log |\eta|) \quad (73)$$

for the difference $g(s) := T(s) - \arg \eta(s)$ of the arguments $T(s)$ of the image and $\arg \eta(s)$ of the preimage. Since a conformal mapping exists, it is clear that equation (73) has a solution. Since $+1$ is a simple eigenvalue of $\mathbf{K}_1$ with eigenfunction $g \equiv 1$, the solution of (73) is unique up to a constant. This constant corresponds to a rotation of the unit disk. It can be fixed by prescribing the boundary correspondence at a specified point

$$T(s_0) = t_0. \quad (74)$$

There are several ways to transform the right-hand side of (73) (see, e.g., Gaier [65, p. 11]). But in any case ϕ must be calculated by an integral transform.

Due to the eigenvalue distribution of $\mathbf{K}_1$, integral equations with Neumann kernel, such as Lichtenstein's, can be conveniently solved by iteration. Starting with $g_0 := \phi$, the iterative step from g_k to g_{k+1} is performed by

$$g_{k+1}(s) = \int_0^{2\pi} K_1(s, t)g_k(t) dt + \phi(s). \quad (75)$$

The iterates g_k converge uniformly to a solution of (73). Convergence is geometric with a rate $|\lambda_2|$ (Warschawski [277], Gaier [65, p. 32]).

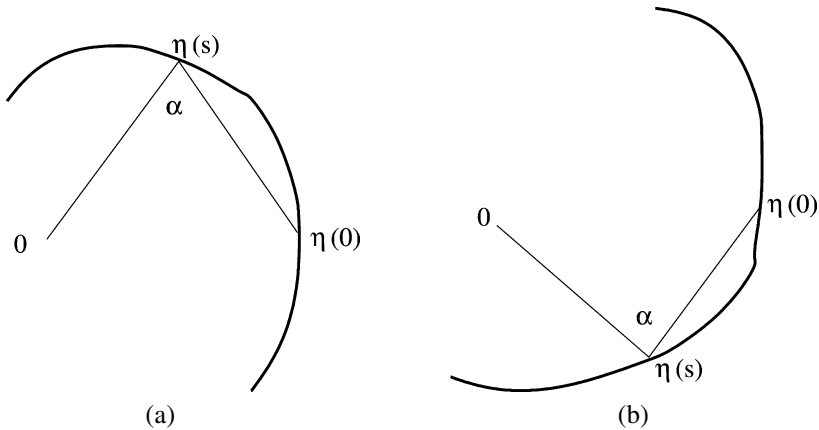


Fig. 5. Geometric interpretation of the right-hand side of Gershgorin's equation (76).

The *integral equation of Gershgorin* (see Gaier [65, p. 8] or Henrici [107, p. 395])

$$(\mathbf{I} - \mathbf{K}_1)T = \beta(s) := 2(\arg \eta(s) - \arg[\eta(s) - \eta(0)]) \quad (76)$$

gives directly the parameter mapping function $T(s)$. Equation (76) has the advantage that the right-hand side can be calculated very easily from the parameterization of the curve. A simple geometric consideration shows that $\beta(s) = \pm 2\alpha$, where α is the angle α at the corner $\eta(s)$ in the triangle formed by the points $0, \eta(s), \eta(0)$, counted negative (positive) when the points are in negative (positive) orientation (see Figures 5(a) and 5(b)).

One can represent the function H from (64) in a unique way as

$$H(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\mu(s)\dot{\eta}(s)}{\eta(s) - z} ds + Ci \quad (77)$$

with a real function μ and a real number C (see Muskhelishvili [187, p. 172]). After calculating the boundary values with (67), taking real parts and using the boundary condition (66), the *integral equation of Mikhlin*

$$(\mathbf{I} + \mathbf{K}_1)\mu = -2 \log|\eta(s)| \quad (78)$$

is obtained (Mikhlin [181], Mayo [173]). It is a Fredholm integral equation of the second kind with Neumann kernel.

The numerical solution of integral equations of the second kind in potential theory is a well-developed field (see, e.g., Atkinson [6] for an overview). For smooth boundary curves “essentially any numerical method will work well, and Nyström methods with the trapezoidal rule probably work best” [6, p. 229]. Stenger and Schmidtlein [250] show that Mikhlin's equation can be solved very efficiently by Sinc methods.

When for u an ansatz as a double-layer potential is made

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \log |z - \eta(s)|}{\partial n_s} \mu(s) ds \quad (79)$$

with density μ (n_s is the inner normal at the point $\eta(s)$) then u assumes the boundary values (66) if and only if Mikhlin's equation (78) is satisfied.

When instead u is represented as a single-layer potential

$$u(z) = -\frac{1}{2\pi} \int_0^{2\pi} \log |z - \eta(s)| \sigma(s) ds \quad (80)$$

with density σ then the boundary values $-\log |\eta|$ are attained when Symm's integral equation

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\eta(t) - \eta(s)| \sigma(s) ds = \log |\eta(t)| \quad (81)$$

is satisfied (Symm [253]). With a solution σ of this equation, the function H can be evaluated

$$H(z) = -\frac{1}{2\pi} \int_0^{2\pi} \log(z - \eta(s)) \sigma(s) ds + i\alpha. \quad (82)$$

The real constant α effects a rotation by an angle α in the conformal mapping function $F(z) = z \exp(H(z))$. It can be fixed by prescribing the image $F(z_0)$ of a boundary point $z_0 \in \Gamma$.

Gaier [68] thoroughly investigated Symm's equation. We quote his main result:

THEOREM 12. (a) *If the boundary curve Γ has capacity $\gamma \neq 1$ (see definition in Section 8) then Symm's equation (81) has the unique solution*

$$\sigma(s) = T'(s), \quad (83)$$

where T' is the derivative of the (interior) boundary correspondence function T defined in (62).

(b) *If Γ has capacity $\gamma = 1$, then the general solution of (81) is*

$$T'(s) + cT'_e(s) \quad (84)$$

with the exterior boundary correspondence function T_e (defined by (222) in Section 8) and an arbitrary real number c .

In view of this theorem it is important to know when the derivative T' exists. We quote the result: If Γ is a rectifiable Jordan curve, then T is absolutely continuous (see Priwalow [226] or Gaier [68, p. 121]).

It is somewhat unexpected that the unique solvability of Symm's equation depends on the size of the region (measured by the capacity). This is a special feature of the logarithmic kernel which can be exemplified in the following way: For a circle of radius R the integral operator with logarithmic kernel on the left-hand side of (81) has eigenfunction $\sigma_0 \equiv 1$ to the eigenvalue $\lambda_0 = \log R$, which becomes 0 for the unit circle, $R = 1$, with capacity $\gamma = 1$.

One can easily remove the ambiguity by a suitable scaling of the curve Γ by a factor λ in such a way that the capacity $\lambda\gamma$ of $\lambda\Gamma$ is different from 1.

Equation (81) is more difficult than the more popular Fredholm equations of the second kind. This is due to the fact that the operator with logarithmic kernel has a smoothing effect. It maps a function space typically into a proper (dense) subspace and has no bounded inverse. As a result, analysis of this equation in a single function space will result in solutions failing to exist for some right-hand sides, and hence instability (see, e.g., Yan and Sloan [295, p. 550]).

There are standard methods available for the numerical solution of integral equations of the form (81) (see, e.g., Atkinson [6]). The logarithmic singularity in the integrals in formulas (81) and (82) requires special treatment. There is now an extensive literature about how to treat Symm's equation numerically. Symm [253] approximated σ by a step function. The convergence of the collocation solution can be improved by a simple postprocessing, the so-called *Sloan iteration* (Graham and Atkinson, [85]). Hayes et al. [101] represent σ by a piecewise quadratic polynomial, and Hough and Papamichael [115] by spline functions of various degrees.

Symm's method does not require the boundary of the region be smooth. If G has corners, however, the function σ may have singularities. In particular, at a re-entrant corner σ becomes unbounded, and this singularity has a serious damaging effect on the accuracy of the approximate mapping function (see Hough and Papamichael [115, p. 135]). To overcome this problem these authors approximate σ near corners by functions which reflect the main singular behavior. A priori information about the behavior of σ near a corner of a polygonal domain with interior angle α is detailed in [115, p. 136]. The experiments of [115] suggest that a reasonable strategy for solving Symm's equation is to use cubic splines with three singular terms for each corner. For the use of singular density functions see also Papamichael [220].

In a later paper Hough and Papamichael [116] give a unified treatment of Symm's equation for interior, exterior and doubly connected domains and again emphasize the importance of including appropriate "singular" functions to cope with corner singularities.

Hoidn [110] enforces by a reparameterization of the boundary curve that the boundary correspondence has certain required smoothness properties. McLean [174] uses a spectral Galerkin method suitable for smooth boundaries. There is rapid convergence of the approximate solution to the Dirichlet problem and all its derivatives uniformly up to the boundary. Hough et al. [114] and Levesley et al. [163] use expansion in terms of Chebyshev polynomials. This approach is also suitable for regions with corners. A convergence analysis is given.

All these methods need $O(N^3)$ operations when discretized with N boundary points. Henrici [104] and Reichel [234] use iteration procedures with FFT. This reduces the com-

putational cost to $O(N^2 \log N)$. Berrut and Trummer [18] show that the Fourier method is equivalent to the Nyström method.

Elschner and Stephan [52] propose a collocation method on curves with corners. When the corner singularities are smoothed by a mesh grading which accumulates grid points near the corners, fast convergence of the approximate solutions can be obtained. Graded meshes, however, produce ill-conditioned linear systems to be solved. This limits the achievable accuracy (see Monegato and Scuderi [183]).

Elschner and Graham [53] apply near corners a smoothing change of variables. A collocation method with splines on a uniform grid leads to optimal order of approximation. Monegato and Scuderi [183] approximate the solution of the transformed equation globally by algebraic polynomials. This leads to well-conditioned systems of equations.

Berrut [17] derived from Symm's equation a Fredholm integral equation of the second kind for the derivative T' of the parameter mapping function. A second kind integral equation for T' with Neumann kernel has been derived by Warschawski (see [107, p. 395]).

Ellacott [50] uses a more general form of the function H , namely $H(z) = \log(F(z)/g(z))$ with a function g satisfying $g'(0) > 0$, $g(0) = 0$ and $g(z) \neq 0$ elsewhere. Then $u = \operatorname{Re} H$ is a harmonic function with boundary values $-\log|g(\eta(s))|$. Ellacott approximates these boundary values in the uniform norm by the real part $\operatorname{Re} p_n(z)$ of polynomials of degree n . Then $F_n(z) = g(z) \exp(p_n(z))$ is an approximation for F .

Saranen and Vainikko [238] propose a two-grid method where inversions on a coarse grid and iterations on the fine grid are alternated. With appropriate solvers the computational cost varies between $O(N^2)$ and $O(N \log N)$ arithmetic operations with the number N of grid points.

Driscoll [46] proposes a domain decomposition method to solve Symm's equation on regions with a long narrow channel, or with structures on different scales.

For the calculation of conformal maps via Symm's equation the public-domain software package CONFPACK described by Hough [113] is available.

The methods discussed so far all solve the Dirichlet problem (66) by an integral equation. These methods are also called *integral equation methods*. These integral equations are linear but they must be solved on the (possibly complicated) boundary of the region G .

Hayes et al. [102] compared the integral equation methods of the first and of the second kind in a series of test calculations. Their conclusion is that Symm's method can compete with Lichtenstein's method. Symm's method is more robust in that it can deal with more distorted domains and with domains with corners. Lichtenstein's method "is clearly better if the domain to be mapped is not excessively distorted from the circle". Even if the domain is distorted Lichtenstein's method may be better if the mesh is sufficiently fine and special care is used in computing the right-hand side $\phi(s)$ of (73) ([102, p. 521]).

Closely related to Symm's method is the *charge simulation method* (Amano [2]). While in (80) or its discretized versions the harmonic function is represented as potential of charges distributed along the curve Γ , the charge simulation method approximates u as a linear combination of N charges Q_i at points ζ_i outside Γ . The charges Q_i are determined from the Dirichlet boundary condition $u(z_j) = -\log|z_j|$ at N collocation points $z_j \in \Gamma$. This resembles one of the methods discussed by Christiansen [28], where the Dirichlet problem is solved by an ansatz with charges outside the boundary. The results of [28, p. 383] indicate that the ensuing system of equations for the determination of the

charges has worse condition than for the discrete Symm's equation, where the charges lie on the boundary.

The charge simulation method has been used by Inoue [124] to calculate the inverse mapping Φ . It is still an open problem how to find the optimal arrangement of the charges. However, it is empirically known that the method can give numerical results of high accuracy if the charge point ζ_i is arranged on the outward normal of the boundary curve at the corresponding collocation point z_i and closer to the boundary where collocation points are dense (Amano [3, p. 1178]). Corners do not present severe difficulty if they are convex, but it seems difficult to obtain accurate results for domains with concave corners (Amano [2, p. 368]). In particular, for slit domains there is no space to place the charges outside the region but close to the boundary. For such difficult cases a premap is recommended (Okano et al. [196]).

Gillot [83] approximated directly F by a polynomial ansatz

$$F_n(z) = C_1 z + \cdots + C_n z^n \quad (85)$$

and determined the coefficients C_j from the condition $|F_n(z_j)|^2 = 1$ for n points $z_j \in \Gamma$. These conditions give quadratic equations for the coefficients, which are convenient to handle numerically. This is closely related to a method described by Kantorowitsch and Krylow [132, p. 360].

Curtiss [33] discusses the solution of the Dirichlet problem by interpolating harmonic polynomials (see also Gaier [65, p. 154]). Volkov [271] approximates the solution of the Dirichlet problem by means of the block method.

When the parameterization $\eta_\lambda(s)$ of the boundary curves of a family G_λ of regions depends analytically on a parameter λ , one may expect that the conformal mapping functions F_λ from G_λ onto the unit disk can be represented as a series

$$F_\lambda(z) = F_0(z) + \sum_{k=1}^{\infty} \lambda^k g_k(z) \quad (86)$$

in powers of λ . This approach is described by Kantorowitsch and Krylow [132, p. 359], where also hints are given on how to calculate the coefficient functions g_k .

The integral equation methods give only the boundary values of the mapping function. The values in the interior must be calculated by Cauchy's formula (77) or the integral (82). This may be a tedious task when the values at many points are required. Another difficulty arises from the fact that the kernels in (77) and (82) become unbounded as z approaches the boundary curve.

Mayo [172, 173] proposes the following procedure. The region G is embedded into a larger one, say a rectangle R , for which fast Poisson solvers are available. The double layer potential u of (79) is a harmonic function inside and outside Γ . The normal derivative is continuous on Γ , but the function u itself has a jump of μ on Γ . The idea is now to represent u as the solution of a Poisson equation

$$\Delta u = q \quad \text{in } R, \quad u = u_0 \quad \text{on } \partial R. \quad (87)$$

The boundary values u_0 are calculated with the integral (79). The volume source q is concentrated in a region of one-grid size width around Γ . It can be calculated from the density μ and the parameterization η of the curve. The complex conjugate function v can be calculated with the same method at little extra cost. It satisfies also a Poisson equation like u . The source term for v can also be calculated directly from μ and η and the boundary values are obtained from the Cauchy integral (77).

The harmonic measure is invariant under conformal mapping. It has a very simple structure in the unit disk: The level lines are segments of circles. Based on these observations one can construct the conformal mapping function F when only the harmonic measures of two segments of the boundary curve Γ are known (Hofmann [108]).

3.2. Extremum principles

The standard proof of the Riemann mapping theorem relies on the following extremum principle:

Among all functions f which are analytic and univalent in G and satisfy

$$f(0) = 0, \quad f'(0) = 1 \quad (88)$$

there is a unique function F which minimizes the supremum norm

$$\|f\|_\infty := \sup_{z \in G} |f(z)| \quad (89)$$

on G . This function F maps G conformally to a disk of a certain radius R , the conformal radius introduced at the beginning of Section 3.

In view of (88) the admissible functions are of form $f(z) = z - g(z)$ with functions g , analytic in G such that $g(0) = g'(0) = 0$. The minimum in (89) is attained when

$$\|z - g\|_\infty := \sup_{z \in G} |z - g(z)| \quad (90)$$

is as small as possible. In this formulation the function g is an *approximation of the identity* z (see, e.g., Opfer [200]).

With a polynomial ansatz for g , this leads to a problem of uniform approximation (Opfer [200]). Krabs and Opfer [147] describe an algorithm for the numerical solution of this kind of approximation problem. The experiments by Hartmann and Opfer [100] show that for some regions (such as a “dented square”) the polynomial ansatz does not yield useful approximations. This is due to the fact that polynomials are analytic at some boundary points where the mapping function has singularities. As a remedy, singular ansatz functions should be included which reflect the singular behavior of the mapping function at corners.

One can use instead of the modulus $|z|$ a more general norm $N(z)$ on $\mathbb{C}$, and define instead of (89) the norm

$$\|f\|_N := \sup_{z \in G} N(f(z)). \quad (91)$$

There is a unique function F_N which gives among all analytic univalent functions f on G with side condition (88) minimum norm $\|f\|_N$. This function F_N maps G conformally onto a “disk” $\{z: N(z) < R\}$ with a suitable $R > 0$ (Opfer [201]). The standard case is included in this general framework for $N(x+iy) = \sqrt{x^2 + y^2}$. By using instead of a norm a suitable positively homogeneous functional, one can also characterize conformal mappings from G onto any star-shaped region S by extremal properties. If the region S is a convex polygon, the extremal problem can be treated numerically by linear programming methods (Opfer [202]).

Samli [237] shows that for a star-shaped region G there are univalent polynomials P_n on G such that $|P_n|$ approximates on the boundary the constant $\equiv 1$ best in the uniform norm. He describes also how to construct such a P_n which then can be used as an approximation to the conformal mapping $F: G \rightarrow D$.

Problems of uniform approximation, however, are not very convenient from the computational point of view. Much easier to handle are approximation problems in Hilbert spaces. There are two major ways to make a Hilbert space of analytic functions in G .

The set $B(G)$ of all functions f analytic in G such that the integral (in the Lebesgue sense)

$$\|f\|_B^2 := \int_G |f(z)|^2 dx dy \quad (92)$$

is finite is called the *Bergman space* of G . It is a Hilbert space when provided with the scalar product

$$(f, g)_B := \int_G f(z) \overline{g(z)} dx dy. \quad (93)$$

This space was first studied by Bergman in 1922 (see [13]). Bergman gives in his book [14] a comprehensive presentation of this space of analytic functions and its relation to conformal mapping.

$B(G)$ is a closed linear subspace of the Lebesgue space $L^2(G)$ of all square integrable functions on G . Let $f_1, f_2, \dots$ be a complete orthonormal system of functions in $B(G)$; then the projection of $L^2(G)$ onto $B(G)$ is determined by the bilinear series

$$k_B(z, w) := \sum_{l=1}^{\infty} f_l(z) \overline{f_l(w)} \quad (94)$$

for $z, w \in G$. This series converges for each fixed $w \in G$ as a function of z in $L^2(G)$, but it converges also pointwise and uniformly for z in every compact subset of G . Therefore, the sum on the right-hand side of (94) defines a function k_B of two variables, the *Bergman kernel function* of the region G . The integral operator $\mathbf{k}_B$ with kernel k_B is called the *Bergman projection*. It projects $L^2(G)$ onto $B(G)$.

In view of the definition of the inner product (93), the integral operator $\mathbf{k}_B$ applied to a function $f \in B(G)$ gives the representation of f in terms of the orthonormal system $f_1, f_2, \dots$. This proves the *reproducing property* of the kernel

$$f(w) = \int_G \overline{k_B(z, w)} f(z) dx dy = (f, k_B(\cdot, w))_B \quad (95)$$

for all functions $f \in B(G)$ and points $w \in G$.

The interest for the Bergman kernel for conformal mapping comes from the following fact. The conformal mapping $F_w(z)$ from G to the unit disk normalized by

$$F_w(w) = 0, \quad F'_w(w) > 0 \quad (96)$$

is related to the kernel k_B by the equation

$$k_B(z, w) = \frac{1}{\pi} F'_w(z) \overline{F'_w(w)}. \quad (97)$$

This equation can be used to represent F'_w in terms of the kernel k_B .

THEOREM 13. *The conformal mapping $F_w : G \rightarrow D$ normalized by the conditions (96) is related to the Bergman kernel by the equation*

$$F'_w(z) = \sqrt{\frac{\pi}{k_B(w, w)}} k_B(z, w) \quad (98)$$

for $z \in G$.

The mapping function F_w can be calculated from (98) by integration. Kerzman and Trummer [134], however, noted that the boundary values of F_w can be retrieved from (98) without integration by

$$F_w(\eta(t)) = -i \frac{\dot{\eta}(t) F'_w(\eta(t))}{|\dot{\eta}(t) F'_w(\eta(t))|}. \quad (99)$$

These results suggest the following procedure for approximating the mapping function F . Given a complete set of functions $v_1, v_2, \dots$ in $B(G)$, the finite subset $v_1, \dots, v_N$ is orthonormalized by means of the *Gram–Schmidt process* to give a set of orthonormal functions $f_1, \dots, f_N$. Henrici [107, p. 545] gives hints for the numerical treatment of the Gram–Schmidt orthonormalization. The double integrals are transformed to line integrals using the equation

$$(f, g')_B := \int_G f(z) \overline{g'(z)} dx dy = \frac{1}{2i} \int_\Gamma f(z) \overline{g(z)} dz \quad (100)$$

which can be derived from Green's formula. The inner products must be calculated as accurate as possible, since the Gram–Schmidt procedure is extremely sensitive to errors in the scalar products.

The series (94) is truncated to give the approximation

$$k_{B,N}(z, w) := \sum_{l=1}^N f_l(z) \overline{f_l(w)}. \quad (101)$$

This approximation procedure is called the *Bergman kernel method* (BKM) with basis $f_1, \dots, f_N$. The BKM has as a major shortcoming that the Gram–Schmidt process is usually numerically unstable. This means that only a limited number of functions can be orthonormalized. Therefore, it is of great practical importance to choose an appropriate set of basis functions $v_1, \dots, v_N$.

It is not always possible to use polynomial basis functions, since there are regions G for which the functions z^j , $j = 0, 1, 2, \dots$, are not complete in $B(G)$. The disk with a slit $G = D \setminus [1/2, 1)$ is an example. But for regions G whose boundary ∂G is contained in the boundary ∂E of a region E which is disjoint to G the polynomials are dense in $B(G)$ (see Bieberbach [20]). All Jordan regions have this property since ∂G is also the boundary of the complement of G in $\mathbb{C}$.

The convergence of the series (94) calculated from orthonormalized polynomials z^j is often very slow. This is due to the presence of singularities of $k_B(z, w)$ in the complement of G , close to or on the boundary ∂G . Therefore, it is advisable to augment the polynomial basis by suitable singular functions. In many cases sufficient information about the singular behavior of $k_B(z, w)$ is available.

Levin et al. [164] take into account two types of singularities:

1. *Poles*: The damaging effect of poles on the convergence of the series (94) is exemplified by the mapping $F_w(z) := (z - w)/(1 - \bar{w}z)$ of the unit disk onto itself with $F_w(w) = 0$. This mapping has a pole at $z = 1/\bar{w}$. The polynomial series for the Bergman kernel function

$$k_B(z, w) = \frac{1}{\pi} \sum_{l=1}^{\infty} l(\bar{w}z)^{l-1} \quad (102)$$

converges rapidly when $|w|$ is small. It converges very slowly, when $|w|$ is close to 1. In general, the damaging influence of poles on the numerics of the BKM can be removed by introducing appropriate rational functions into the basis set. When F has a pole at $z = p$ outside G , then the polynomial basis should be augmented by the function $(z/(z - p))'$ (see [164, p. 175] for more details).

2. *Branch points*: When the boundary of G has a corner at z_0 of interior angle $\alpha\pi$, formed by two analytic arcs, the mapping in the neighborhood of z_0 is represented by a series in fractional powers $(z - z_0)^{k/\alpha}$, $k = 0, 1, 2, \dots$. Therefore, if $1/\alpha$ is not an integer, F and k_B have a branch point singularity at z_0 which affects the rate of convergence of the polynomial series at least in the neighborhood of z_0 . The polynomial basis should be augmented by the function $(z - z_0)^{k/\alpha-1}$ [164, p. 175].

When the boundary curve Γ consists of analytic arcs then the position of the poles can be determined in favorable cases by a symmetry principle which is a generalization of Schwarz's reflection principle (Papamichael et al. [219]). The use of a basis augmented by singular functions can deteriorate the stability of the orthonormalization process but it can also speed up the convergence of the numerically calculated approximations to F (Papamichael and Warby [218]).

For ellipses the Chebyshev polynomials of the second kind form a set of orthogonal basis functions, and the kernel function can be calculated explicitly (see Nehari [189, p. 258]). Burbea [23] calculates the mapping from ellipses and squares to the disk using the orthonormalized polynomials z^j and compares the result with theoretical values.

In view of the numerical difficulties of the calculation of the Bergman kernel via an orthonormalization procedure, it is interesting to note that the boundary values of k_B satisfy an integral equation of the second kind with Neumann kernel (Razali et al. [231]). Assume that the boundary of G has a parametric representation by a twice differentiable function η and fix a point $w \in G$. Then the function $\phi(t) := \dot{\eta}(t)k_B(\eta(t), w)$ satisfies the equation [231, p. 343]

$$\phi(t) + \int_0^{2\pi} K_1(s, t)\phi(s) ds = -\frac{1}{\pi} \frac{\overline{\dot{\eta}(t)}}{(\overline{\eta(t)} - \bar{w})^2} \quad (103)$$

with the Neumann kernel K_1 defined in (69). Equation (103) has a unique solution. It can be solved numerically by the Nyström method. Since the functions involved are all 2π -periodic, the integrals are best evaluated by the trapezoidal rule on an equidistant grid. For regions with m -fold symmetry, the integral equation can be restricted to $1/m$ of the boundary (Razali et al. [231]). When the equations arising from the Nyström discretization with N grid points are solved by the generalized minimum residual method, the computational cost can be reduced from $O(N^3)$ to $O(N^2)$ (Razali et al. [232]).

If $G = G_1 \cup \dots \cup G_m$ is the union of a finite number of regions G_j , then the Bergman projection (95) of the region G can be described in terms of the Bergman projections of the subregions G_j (Skwarczynski [243]). This representation uses the principle of alternating projections (von Neumann [190], Skwarczynski [244]).

For a region G bounded by a rectifiable Jordan curve Γ one can consider the set $S(G)$ of all functions f analytic in G , such that, for almost all (in the Lebesgue sense) points $\zeta \in \Gamma$, the nontangential limit $f(\zeta) = \lim_{z \rightarrow \zeta} f(z)$ exists and the line integral

$$\|f\|_S^2 := \int_{\Gamma} |f(\zeta)|^2 |d\zeta| \quad (104)$$

($|d\zeta|$ is the differential of arclength) is finite. With the inner product

$$(f, g)_S := \int_{\partial G} f(\zeta) \overline{g(\zeta)} |d\zeta| \quad (105)$$

$S(G)$ becomes a Hilbert space, which is sometimes called *Szegő space* in honour of Szegő who first studied this space and its bearing on conformal mapping in 1921 (see [256]).

The Szegő space $S(G)$ is a closed linear subspace of the Lebesgue space $L^2(\partial G)$. With a complete orthonormal system of functions $g_1, g_2, \dots$ in $S(G)$ one can define the bilinear series

$$k_S(z, w) := \sum_{l=1}^{\infty} g_l(z) \overline{g_l(w)} \quad (106)$$

for $z, w \in G$. This series converges for each fixed $w \in \partial G$ as a function of z in $L^2(\partial G)$, but it also converges pointwise and uniformly for z in every compact subset of G (see, e.g., Gaier [65, p. 134]). Therefore, the sum on the right-hand side of (106) defines a function k_S of two variables, the *Szegő kernel function* of the region G . The integral operator $\mathbf{k}_S$ with this kernel defines the *Szegő projection*, that is the projection of $L^2(\partial G)$ onto $S(G)$. This implies the reproducing property

$$g(w) = \int_{\partial G} \overline{k_S(\zeta, w)} g(\zeta) |d\zeta| = (g, k_S(\cdot, w))_S \quad (107)$$

of the Szegő kernel function for functions $g \in S(G)$ and points $w \in G$.

The Szegő kernel is connected with the conformal mapping function in a similar way as the Bergman kernel.

THEOREM 14. *The conformal mapping $F_w : G \rightarrow D$ normalized by the conditions (96) is related to the Szegő kernel by the equations*

$$F'_w(z) = \frac{2\pi}{k_S(w, w)} k_S^2(z, w) \quad (108)$$

for $z \in G$.

It follows from (98) and (108) that the Bergman and Szegő kernels are related by the equation

$$k_B(z, w) = 4\pi k_S^2(z, w). \quad (109)$$

The Szegő kernel has for a long time not been very popular for numerical conformal mapping, since it requires the tedious orthogonalization of polynomials on the boundary curve. This situation has changed, however, since it has been noticed that the Szegő kernel can be conveniently calculated from a Fredholm integral equation of the second kind.

Both the Cauchy and the Szegő kernels map functions in $L^2(\partial G)$ to boundary values of analytic functions in G . Both are projectors, the Szegő projector is orthogonal in $L^2(\partial G)$, the Cauchy projector is not. The Cauchy kernel can be written down very easily, the Szegő kernel can be written down only if a conformal mapping $F : G \rightarrow D$ is known. Kerzman and Stein [133] found that the Szegő projector can be obtained by “orthogonalizing” the Cauchy projector. This leads to an integral equation for k_S .

We present here the version of Trummer [266] which is convenient for numerical treatment. Let

$$H(s, t) := \frac{1}{2\pi i} \frac{\dot{\eta}(t)}{|\dot{\eta}(t)|} \frac{1}{\eta(t) - \eta(s)} \quad (110)$$

be the Cauchy kernel. Then the nonsymmetric part is defined by

$$A(s, t) := \overline{H(t, s)} - H(s, t) \quad (111)$$

for $s \neq t$, and by $A(s, s) = 0$ on the diagonal. When the boundary parameterization η is in C^2 then A is a continuous function. It is skew-symmetric: $A(s, t) = -\overline{A(t, s)}$. The Szegő kernel satisfies the Kerzman–Stein integral equation [133]

$$\begin{aligned} k_S(\eta(t), w) + \int_0^{2\pi} A(t, s) k_S(\eta(s), w) |\dot{\eta}(s)| ds \\ = -\frac{1}{2\pi i} \frac{\overline{\dot{\eta}(t)}}{|\dot{\eta}(t)|(\overline{\eta(t)} - \bar{w})} \end{aligned} \quad (112)$$

for any fixed $w \in G$. This is a second kind Fredholm integral equation for $k_S(\eta(\cdot), w)$ with continuous kernel A . Equation (112) is similar to (103) for the Bergman kernel.

Equation (112) has been used by Kerzman and Trummer [134] for numerical purposes. Trummer [266] solves (110) by Nyström's method using equidistant grid points and the trapezoidal rule for integration. The ensuing linear system of equations is solved by a conjugate gradient method.

Each iterative step of the conjugate gradient (CG) method requires multiplication of a vector by a matrix. The operation count for this scales like $O(N^2)$ with the number N of grid points. O'Donnell and Rokhlin [194] give a variant of this method where by means of the Fast Multipole Method the operation count for this matrix multiplication is decreased to $O(N)$. But the constant of this $O(N)$ is quite large, so that methods based on FFT with their $O(N \log N)$ operation count are likely to be considerably faster. On the other hand, the performance of this method is not affected by crowding, making it a method of choice for elongated regions [194, p. 476]. Lee and Trummer [159] have improved the numerics of this method further by using a multi-grid approach. The multi-grid seems to outperform the CG for elongated regions where a large number of grid points is needed in view of crowding. But CG seems to be preferable when the boundary contains some sharp changes [159, p. 43]. Lee and Trummer [159] calculated successfully examples of regions with nonsmooth boundaries with corners and cusps. Thomas [260] provided the theoretical framework for this experimental finding by extending equation (112) to domains with piecewise C^∞ boundaries. The Szegő kernel for such domains is defined as the limit of the Szegő kernels of certain subdomains with smooth boundaries.

Murid et al. [186] give a unified derivation of the integral equations for the Bergman and the Szegő kernels.

Both the Bergman and the Szegő norms are very useful for characterizing the conformal mapping from G to a disk by extremum principles:

THEOREM 15. *Let F be the conformal mapping from G to a disk normalized by (88). Then*

(a) F' is the unique function which minimizes $\|f\|_B$ among all functions $f \in B(G)$ satisfying $f(0) = 1$.

(b) $\sqrt{F'}$ is the unique function which minimizes $\|f\|_S$ among all functions $f \in S(G)$ satisfying $f(0) = 1$.

The dilatation of a function f is equal to $|f'|$. Therefore, the image of the region G under the mapping f has area $\|f'\|_B^2$. Statement (a) is called the *principle of minimum area*. For the same reason $\|\sqrt{f'}\|_S^2$ is the length of the boundary of the image region $f(G)$. Statement (b) is called the *principle of minimum length* (see Bieberbach [20]).

In numerical calculations one solves the minimization problems of the theorem not in the whole space $B(G)$ but in a finite-dimensional subspace generated by basis functions $g_1, \dots, g_n$, i.e., by a Ritz ansatz as is generally applied in the calculus of variations (see, e.g., Kantorowitsch and Krylow [132]). Methods based on this approach are called *Ritz methods*. Usually one takes the space Π_n of polynomials of degree $\leq n$. Once again, however, one has to keep in mind that for general regions it can happen that the polynomials are not dense in $B(G)$.

The Ritz method applied with functions $g_1, \dots, g_n$ gives the approximation F'_n for F' as an expansion

$$F'_n(z) = \sum_{j=1}^n a_j g_j(z) \quad (113)$$

in terms of given basis functions. The coefficients a_j are determined by the minimization. Therefore, these methods are also called *expansion methods* (Papamichael [220]).

When the minimization problem is to be solved in the space Π_n of n th degree polynomials, systems of linear equations arise which are in general ill-conditioned (Švecová [252]). The minimizing polynomials in general do not give conformal mappings. Opfer [198] gives several illustrative examples.

There is an analogy to statement (a) of Theorem 15: There is a unique polynomial p_n which minimizes $\|p\|_B$ among all $p \in \Pi_n$ with $p(0) = 1$. The primitives π_n of these polynomials p_n normalized by $\pi_n(0) = 0$ are called *Bieberbach polynomials*. The importance for conformal mapping stems from the fact that, when F is the conformal mapping normalized by (88), the p_n also solve the problem of minimizing $\|F' - p\|_B$ among all $p \in \Pi_n$ (see Gaier [65, p. 122]).

The Bieberbach polynomials also give nearly best approximations in the uniform norm. When Γ is a Jordan curve with parametric representation $\eta \in C^\alpha$ with $\alpha \geq 3/4$ then

$$\|F - \pi_n\|_B = O(\log n) E_n(F, G), \quad (114)$$

where $E_n(F, G)$ denotes the minimal error of uniform approximation of F on G by polynomials of degree $\leq n$. For piecewise analytic curves, however, with exterior angles $\lambda_j\pi$ ($0 < \lambda_j < 2$) and without cusps, the estimate

$$\|F - \pi_n\|_B = O(\log n)n^{-\gamma} \quad (115)$$

holds with

$$\gamma := \min_j \frac{\lambda_j}{2 - \lambda_j}, \quad (116)$$

and this exponent is the best possible for general regions (Gaier [73]). For corners of a special type, however, one can get an improvement: For corners with interior angles of form π/N for some $N = 1, 2, \dots$ one can insert into the right-hand side of (116), $2\lambda_j/(2 - \lambda_j)$ (Gaier [76]).

Gaier [72] generalized a lemma of Andrievskii which is useful for transforming estimates for the norm $\|\cdot\|_B$ into those for the supremum norm $\|\cdot\|_\infty$ on G .

Assume that the conformal mapping $\Phi: D \rightarrow G$ is Hölder continuous. Then there is a constant $c(G)$ which depends on G only such that for every polynomial P of degree $n \geq 2$ with $P(0) = 0$ the estimate

$$\|P\|_\infty \leq c(G)\sqrt{\log n}\|P'\|_B \quad (117)$$

holds. The order $\sqrt{\log n}$ is best possible even in the case of the unit disk.

Maymeskul et al. [171] consider *augmented Bieberbach polynomials* which contain in addition to powers of z also suitable fractional powers and logarithmic functions such as occur in the asymptotic expansion of the mapping function near corners (see Section 2.3).

Papamichael and Kokkinos [208] compare the Ritz method and the Bergman kernel method. They find that both methods are extremely efficient provided the set of polynomial basis functions is suitably augmented by singular functions. Both methods produce results of comparable accuracy and need about the same computational effort. In the BKM the basis can be easily enlarged by new basis functions. But in the Ritz method after each change of the basis one has to start from the scratch again.

For each fixed $w \in G$, the approximate kernel $k_{B,N}(\cdot, w)$ converges to $k_B(\cdot, w)$ in the L^2 norm of G as $N \rightarrow \infty$. The speed of convergence depends on the smoothness of the boundary curve. Let H be a subregion of G . Then $k_{B,N}$ converges also in the L^2 norm of H . When $\bar{H} \subset G$ then the convergence in $L^2(H)$ can be much faster than in $L^2(G)$. When the boundary ∂H contains a subarc of ∂G , however, the rates of convergence in $L^2(H)$ are not substantially different from those in $L^2(G)$ (Papamichael and Saff [213]).

3.3. Osculation methods

The osculation method (Schmiegunungsverfahren) of Koebe [140] approximates F by a composition of elementary maps. It is universally applicable, since it requires no hypotheses at all concerning the boundary ∂G of the region G .

Briefly, the idea is the following: When $0 \in G$ and G is contained in the unit disk D , the quantity

$$\rho(G) = \min_{\zeta \in \partial G} |\zeta| \leq 1 \quad (118)$$

measures how far the region deviates from the disk which has $\rho(D) = 1$. Starting with $G_1 = G$, a sequence of regions G_n is constructed recursively by applying mappings h_n from a certain set of osculating functions (Schmiegunsfunktionen) in such a way that $\rho(G_n)$ increases monotonically to 1. Ostrowski [205] and Henrici [106] have shown that the speed of convergence is $1 - \rho(G_n) = O(1/n)$.

Grassmann [86] has made some experiments which gave unexpectedly good results. We cite some of his findings: It is very important that the osculating functions are chosen carefully. The experience gained indicates that the method converges asymptotically very slowly, but it works surprisingly fast at the beginning. Therefore, it may well be competitive with other methods if the required accuracy is not higher than 1%.

Grassmann's experiments led him to the conclusion [86, p. 883] that "... the methods of successive approximation [= osculation methods] are competitive and deserve more attention than they obtain at present".

Gaier [65, p. 173] gives an overview of several classes of Schmiegunsfunktionen used in the literature. Grassman [86] describes three classes of Schmiegunsfunktionen particularly suitable for automatic calculations. Henrici [106] gives a general theory of osculation methods which covers several variants. He gives also some bounds for $\rho(G_n)$ which explain why the method converges faster initially.

Osculation methods are applicable without any requirements about shape or smoothness of the region. The more efficient rapidly convergent methods for numerical conformal mapping often require smoothness of the boundary and work best for nearly circular regions. Therefore, osculation methods are sometimes applied for preprocessing a region with the aim to make it nearly circular and to smooth the boundary. In such a way one can combine the advantages of the osculation method (fast initial convergence) with the advantages of other methods (fast convergence for nearly circular regions). Wahl [272] discusses the construction of suitable rational functions for preprocessing.

Rabinovich and Tyurin [229,230] use the original region G to construct slightly perturbed disks and then map these nearly circular regions to the unit disk using Lavrentev's variational principle (see (128) in Section 4.1).

Marshall maps first the region to a region bounded by the negative real axis plus a curve which connects 0 to infinity. Then he uses explicit mappings from the exterior of the negative real axis plus an attached segment of a circle to force one pair of grid points after the other to lie on the negative real axis. He offers a program with the illustrative title "zipper" on the Internet. (See Marshall and Rohde [170].)

3.4. Accuracy

When an approximation $\tilde{F}$ for F is calculated, then it is easy to get a bound ε for the accuracy of the modulus of $\tilde{F}$. However, it is in general impossible to get from ε a bound

for the error in the argument of $\tilde{F}$. But under an additional assumption such an estimate can be obtained as the following theorem of Grinshpan and Saff [87, Corollary 2.4] shows.

THEOREM 16. *Let $\tilde{F}$ be analytic and univalent in G with $\tilde{F}(0) = 0$, $\tilde{F}'(0) > 0$, and assume that the image $\tilde{F}(G)$ is starlike with respect to 0. If*

$$||\tilde{F}(z)| - 1| \leq \varepsilon \leq \frac{1}{2} \quad \text{for } z \in \Gamma, \quad (119)$$

then

$$|\arg \tilde{F}(z) - \arg F(z)| \leq 4\varepsilon^{1/2} \quad \text{for } z \in G. \quad (120)$$

4. Mapping from the disk to the region

Let G be a bounded simply connected region with $0 \in G$. There is a unique conformal mapping $\Phi: D \rightarrow G$ normalized at zero by the conditions (11). In what follows it will be more convenient to relax this condition slightly to

$$\Phi(0) = 0, \quad \operatorname{Im} \Phi'(0) = 0. \quad (121)$$

When the region G is bounded by a curve Γ then in view of Theorem 3 the conformal mapping $\Phi: D \rightarrow G$ can be extended continuously to the closure $\bar{D}$. When the curve Γ is parameterized by a 2π -periodic complex function $\eta(s)$ in such a way, that G is to the left of the curve when it is traversed with increasing parameter values s , then the boundary correspondence can be expressed by

$$\Phi(e^{it}) = \eta(S(t)) \quad (122)$$

with the (inverse) boundary correspondence function $S(t)$. The reduced boundary correspondence function $S(t) - t$ is 2π -periodic. This condition guarantees that the right-hand side of (122) goes once around G in the counterclockwise direction when t increases from 0 to 2π .

In view of equation (122) the boundary can be parameterized by the boundary values $\Phi(e^{it})$ of the conformal mapping function. Kantorowitsch and Krylow [132] call this the *normal representation*.

The boundary correspondence equation (122) and its variants are the basis of all methods which calculate the mapping from the disk to the region. It is interpreted as an equation which determines at the same time the conformal mapping Φ and the boundary correspondence function $S(t)$. The equation is nonlinear in S . Therefore, any solution method must be iterative.

4.1. Mapping to nearby regions

There are many results about how the conformal mapping Φ from the disk to a region G depends on the region. Generally speaking, one can say that the mapping function varies continuously with the region. Warschawski [275] reviews these results, which are useful to obtain error estimates for numerically calculated conformal mappings.

One can use the implicit mapping theorem to infer from the boundary correspondence equation (122) how the conformal mapping depends on the boundary curve of G . Particularly useful is information about how the mapping function reacts on small changes of the boundary.

The main problem for the implicit mapping theorem consists in the proof of the Fréchet differentiability of the right-hand side of (122). We quote a result of Wegmann [291] about differentiability in the Sobolev space W . When η is three times differentiable and the third derivative is Lipschitz continuous, then the mapping B defined by

$$B : W \ni S(t) - t \rightarrow \eta \circ S \in W \quad (123)$$

is Fréchet differentiable with derivative

$$DB(U)(t) = \dot{\eta}(S(t))U(t), \quad (124)$$

where on the right-hand side of (124) the function $U \in W$ is multiplied by $\dot{\eta}(S(t))$.

Assuming that the conditions for differentiability are satisfied we get from (122) the equation

$$D\Phi(e^{it}) = \dot{\eta}(S(t))DS(t) + D\eta(S(t)) \quad (125)$$

which connects the changes $D\Phi$ of the conformal mapping function and DS of the boundary correspondence function with the change $D\eta$ of the boundary curve. Therefore, equation (125) describes how Φ and S react to small changes of the region G .

Since DS is a real function the relation (125) can be transformed to an RH problem

$$\operatorname{Im} \frac{D\Phi(e^{it})}{\dot{\eta}(S(t))} = \operatorname{Im} \frac{D\eta(S(t))}{\dot{\eta}(S(t))} \quad (126)$$

which relates directly the change $D\Phi$ of the mapping function Φ to the change $D\eta$ of the curve. The analytic function $D\Phi$ must satisfy the normalization $D\Phi(0) = 0$ and $\operatorname{Im} D\Phi(0) = 0$. It follows from Corollary 1 that $D\Phi$ is uniquely determined by (126).

The right-hand side of (126) can be written in the form $(D\eta)_n/|\dot{\eta}|$ with the component $(D\eta)_n$ of the shift of the boundary curve with respect to the inner normal of Γ . The tangential component of the shift has (in first order) no influence on the mapping Φ . It affects only a change DS of the parameterization.

Most important is the special case, when G is close to the unit disk. When the boundary is parameterized by

$$\eta(s) = (1 + \tau\rho(s))e^{is} \quad (127)$$

with a real function ρ , the boundary values of the conformal mapping Φ are in first order of τ :

$$\Phi(e^{it}) = e^{it} (1 + \tau [\rho(t) + i\mathbf{K}\rho(t)]), \quad (128)$$

with the operator $\mathbf{K}$ of conjugation defined in Section 2.5 (see Nehari [189, p. 265]). This formula is closely related to *Hadamard's variational formula* for the Green's function (see Nehari [189, p. 263]). The boundary correspondence function for the disturbed disk region (127) is in first order of τ :

$$S(t) = t + \tau \mathbf{K}\rho(t). \quad (129)$$

This is *Lavrentev's principle* [158].

When ρ is in the Hölder space $C^{n,\alpha}$ with $0 < \alpha < 1$ then there is an expansion

$$\log \frac{\Phi(z)}{z} = \sum_{\nu=1}^n \Phi_{\nu}(z) \tau^{\nu} + O(|\tau| |\tau \log \tau|^n) \quad (130)$$

with functions Φ_{ν} analytic in D and continuous in $\bar{D}$ independently of τ (see Yoshikawa [296]). The first function is determined by $\operatorname{Re} \Phi_1(e^{it}) = \rho(t)$.

Kantorovich and Krylow [132] consider the case when the boundary of G is given in implicit form by an equation

$$\Gamma = \{z = x + iy: H(x, y) = 0\} \quad (131)$$

with an analytic function H in two real variables. One can insert the ansatz

$$\Phi(e^{it}) = \sum_{k=1}^{\infty} A_k e^{ikt} \quad (132)$$

into the equation (131) for the boundary. The condition $\Phi(e^{it}) \in \Gamma$ is equivalent to the condition that the Fourier coefficients of the 2π -periodic function $H(\operatorname{Re} \Phi(e^{it}), \operatorname{Im} \Phi(e^{it}))$ all vanish. This gives an infinite system of equations for the coefficients A_k in (132). These equations become particularly simple in the case when the region is close to a disk [132, p. 381].

4.2. Projection

The boundary correspondence equation (122) says that the conformal mapping Φ is characterized by two properties:

1. Φ is an analytic function in the disk normalized by the constraints (121),
2. Φ maps the unit circle onto the boundary ∂G of the region G .

This interpretation offers the possibility of constructing the mapping Φ by an iterative procedure, which alternately calculates functions which have either the first property (normalized analytic functions) or the second property (the boundary values are in ∂G). The functions are constructed by a sort of projection onto suitable linear spaces or manifolds. This approach resembles closely the well-known procedure of determining the intersection of two subspaces of a Hilbert space by alternating projection. This method was first applied by von Neumann [190]. Therefore, it is appropriate to call methods based on this idea *alternating projection* (AP) methods.

For regions with smooth boundaries alternating projection can be applied in the following simple way:

The iteration starts with a function S_0 such that $S_0(t) - t$ is 2π -periodic. The simplest choice is $S_0(t) \equiv t$.

When S_k is determined for some $k \geq 0$ then the Fourier coefficients B_l of the boundary function

$$\eta(S_k(t)) = \sum_{l=-\infty}^{\infty} B_l e^{ilt} \quad (133)$$

are calculated. The function

$$f_k(t) = (\operatorname{Re} B_1) e^{it} + \sum_{l=2}^{\infty} B_l e^{ilt} \quad (134)$$

represents the boundary values of an analytic function

$$\Phi_k(z) = (\operatorname{Re} B_1) z + \sum_{l=2}^{\infty} B_l z^l \quad (135)$$

which obviously satisfies the conditions (121). This is the first projection step. The second step calculates a new boundary correspondence function by

$$S_{k+1}(t) := S_k(t) - \operatorname{Re} \frac{g_k(t)}{\dot{\eta}(S_k(t))} \quad (136)$$

with the nonanalytic part

$$g_k(t) = \sum_{l=-\infty}^0 B_l e^{ilt} + i(\operatorname{Im} B_1) e^{it} \quad (137)$$

of the boundary function (133). Equation (136) means that g_k is projected onto the tangent at the point $\eta(S_k(t))$ and the shift along the tangent is then replaced by a shift along the curve induced by a shift in the function S . This gives a new boundary correspondence $e^{it} \rightarrow \eta(S_{k+1}(t))$ which completes the second step of the alternating projection.

It has been proved by Wegmann [287] that this iteration converges when the boundary parameterization η is in the Hölder space $C^{3,\alpha}$ for some $\alpha \in (0, 1]$ whenever the initial approximation S_0 is sufficiently close to the correct function $S(t)$. Convergence is linear. It has been noted by Gaier [65, p. 110], that convergence of the AP method in this simple version is slow. This can easily be seen for nearly circular regions with boundary parameterization (127). Starting from $S_0(t) = t$ the next iterate gives in first order $S_1(t) = t + \frac{1}{2}\tau \mathbf{K}\rho$. Therefore, only half of the first order change $\tau \mathbf{K}\rho$ according to (129) is recovered in the first step. For nearly circular regions the rate of convergence is $1/2$. This is caused by the general fact that alternating projection methods in Hilbert spaces approach the solution always from one side.

Wegmann [287] observed that this drawback of slow convergence can be remedied by overrelaxation in the following way: The new function S_{k+1} is calculated by (136) with the function

$$g_k(t) = 2 \sum_{l=-\infty}^0 B_l e^{ilt} + i(\operatorname{Im} B_1) e^{it} \quad (138)$$

instead of (137). Note that overrelaxation with a factor 2 is applied to all terms $B_l e^{ilt}$ in (133) with $l \leq 0$ but not to the term $i(\operatorname{Im} B_1) e^{it}$. We call this overrelaxed method the *OAP method*. Analysis for nearly circular regions shows that in the first iteration the full first-order term is recovered by the OAP method. Convergence is linear. It has been shown by Wegmann [287, p. 304], that for regions with parameterization (127) the rate q of convergence of the OAP method can be estimated by

$$q \leq 2\tau \|\dot{\rho} - \mathbf{K}\rho\|_{\infty} + o(\tau). \quad (139)$$

Although the method converges linearly, convergence can be very fast for some regions.

The method can be applied numerically on a grid with $N = 2n$ equidistant grid points $t_j = (j - 1)2\pi/N$. Fourier analysis of the boundary function at the grid points

$$\eta(S_k(t_j)) = \sum_{l=-n}^{n-1} B_l e^{ilt_j} \quad (140)$$

gives the coefficients B_l . With these coefficients the polynomial

$$g_k(t) = \sum_{l=-n}^0 B_l e^{ilt} + i(\operatorname{Im} B_1) e^{it} \quad (141)$$

is formed which is inserted (with a factor 2 in front of the sum for the OAP method) into (136) to calculate the values of the new function S_{k+1} at the grid points.

This AP method needs only a subroutine which calculates the Fourier analysis (140) of the boundary mapping and the Fourier synthesis (141) of the nonanalytic part. This can be

very efficiently done by FFT. The AP method is certainly one of the simplest numerical methods for conformal mapping. It is easy to apply and very robust.

Convergence of the AP method means that the curves parameterized by the functions f_k defined in analogy to (134) by

$$f_k(t) = (\operatorname{Re} B_1)e^{it} + \sum_{l=2}^{n-1} B_l e^{il t} \quad (142)$$

approach the curve Γ as $k \rightarrow \infty$.

We show this at an example.

EXAMPLE 1. An ellipse with axes a and b rotated by an angle α and shifted by z_0 is parameterized by

$$\eta(s) = z_0 + e^{i\alpha}(a \cos s + ib \sin s). \quad (143)$$

Figure 6 shows the mapping Φ from the disk to the ellipse of Example 1 for parameters $a = 1$, $b = 0.7$, $z_0 = -0.4 - 0.2i$, $\alpha = 0$. The calculation is done on a grid with $N = 256$ points with the OAP method starting with $S_0(t) = t$. The dotted lines are the curves parameterized by the functions f_k defined in (142) for $k = 0, 1, 2, \dots$. These curves approximate the ellipse better and better as k increases. Only the first two functions f_0 and f_1 differ significantly from the ellipse.

We use several indicators for convergence and accuracy. First, we take the maximum change in each step over all grid points t_j :

$$\delta_k := \max_j |S_{k+1}(t_j) - S_k(t_j)|. \quad (144)$$

As a measure of analyticity of the boundary values $\eta(S_k(t))$ one can use the L^2 norm of the nonanalytic part

$$\alpha_k := \|g_k\|_2 = \left(\sum_{l=-n}^0 |B_l|^2 + (\operatorname{Im} B_1)^2 \right)^{1/2}. \quad (145)$$

As a measure for the accuracy achieved one can take the distance of the boundary values of the polynomial f_k to the curve Γ . To this aim we evaluate f_k on a refined grid $t_j^* = (j-1)\pi/N$, $j = 1, \dots, 2N$. We define $S_k(t_j^*)$ for odd values of j by trigonometric interpolation of $S_k(t_j) - t_j$, and define

$$\varepsilon_k = \max_j |f_k(t_j^*) - \eta(S_k(t_j^*))|. \quad (146)$$

The right panel of Figure 6 shows the maximum change δ_k in S_k in each iterative step k , and the measures α_k and ε_k for analyticity and accuracy. All three of these quantities decrease geometrically like q^k with a convergence factor of $q \approx 0.85$. The convergence is

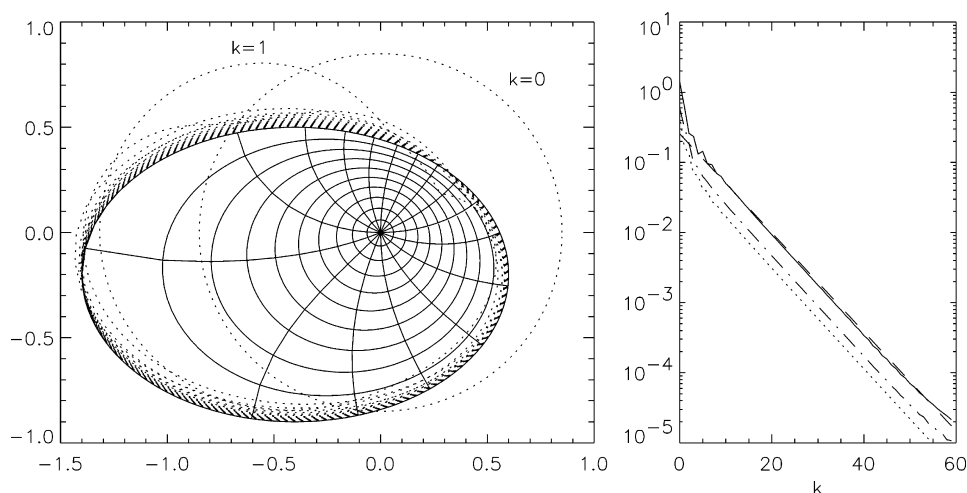


Fig. 6. Conformal mapping from the unit disk to an eccentric ellipse with parameterization (143). The solid lines show images of 10 concentric circles and 16 spokes. The dotted lines are the curves parameterized by the functions f_k defined in (142) during iteration with the OAP method. The right panel shows δ_k (solid), α_k (dotted), ε_k (dash-dotted) and Cq^k (dashed) for each iterative step k .

much faster in the first two iterations, where the f_k approach the ellipse. The AP method converges at a rate $q \approx 0.895$.

Convergence of the AP method is proved only for smooth curves. But the iteration, consisting of the steps (140), (141) and (136), can be performed also for piecewise smooth curves. Numerical experiments show that the AP method works also for such cases. We demonstrate this with two examples.

EXAMPLE 2. Square, shifted by z_0 , with boundary parameterization

$$\eta(s) = z_0 + \begin{cases} \frac{\pi}{4}(1-i) + si & \text{for } 0 \leq s \leq \frac{\pi}{2}, \\ \frac{\pi}{4}(3+i) - s & \text{for } \frac{\pi}{2} \leq s \leq \pi, \\ \frac{\pi}{4}(-1+5i) - si & \text{for } \pi \leq s \leq \frac{3\pi}{2}, \\ \frac{\pi}{4}(-7-i) + s & \text{for } \frac{3\pi}{2} \leq s \leq 2\pi. \end{cases} \quad (147)$$

The mapping from the disk to the square of Example 2 with shift $z_0 = -0.2 + 0.2i$ is calculated with the AP method. The iteration is started with $S_0(t) = t$. The calculation is done with $N = 1024$ grid points. The curves parameterized by the f_k converge to the square. The rate of convergence is about $q = 0.92$. The speed of convergence depends to some extent on the number of grid points. Convergence is slower for a finer grid, when more and more of the corner regions becomes resolved. The maximum δ_k of the shift is a little jumpy (see Figure 7), since the projection to the curve changes when a grid point is pushed from one edge to the next across a corner. The measure α_k for analyticity decreases only up to a certain point, since on 1024 grid points the slowly decaying Taylor coefficients

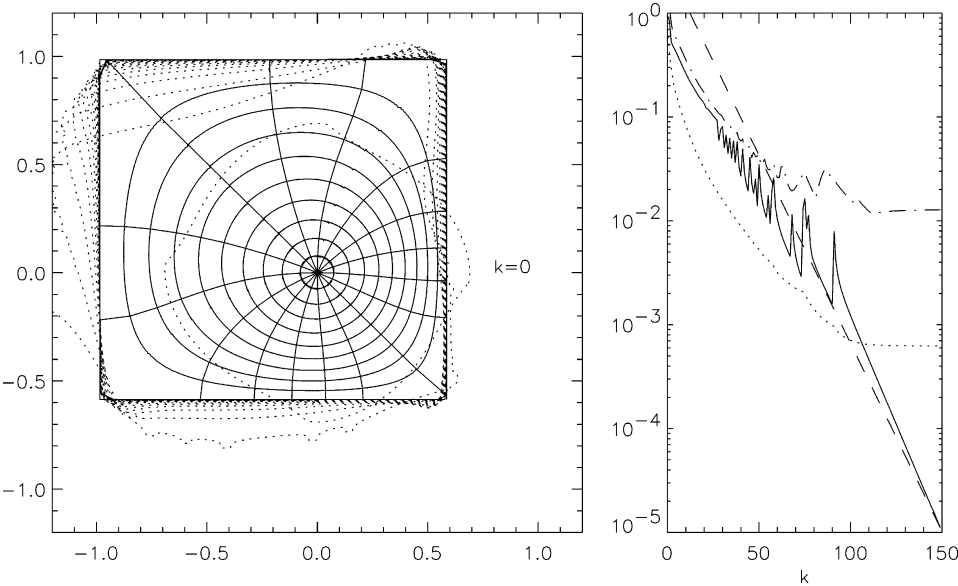


Fig. 7. The same as Figure 6 for the conformal mapping from the unit disk to the shifted square of Example 2 calculated with the AP method. The functions f_k defined in (142) are shown only for every fifth step.

of Φ cannot all be represented to machine precision. The accuracy achieved is about 0.01 as measured in ε . It cannot be further improved on this grid.

One can avoid the problems with the discontinuous $\dot{\eta}$ by replacing it by a smooth function ξ which retains to some extent the orientation of $\dot{\eta}$. Instead of the orthogonal projection onto the tangent an oblique projector is applied. The second step in this method (we call it *smoothed AP method*) is

$$S_{k+1}(t) := S_k(t) - \operatorname{Re} \frac{g_k(t)}{\xi(S_k(t))} \tag{148}$$

instead of (136).

EXAMPLE 3. Eccentric heart with boundary is parameterized by

$$\eta(s) = z_0 + \begin{cases} 1 + 2(-1 + i)s/\pi & \text{for } 0 \leq s \leq \frac{\pi}{2}, \\ 0.5(-1 + i) + \exp(i(2s - 3\pi/4))/\sqrt{2} & \text{for } \frac{\pi}{2} \leq s \leq \pi, \\ -0.5(1 + i) + \exp(i(2s - 5\pi/4))/\sqrt{2} & \text{for } \pi \leq s \leq \frac{3\pi}{2}, \\ -i + 2(1 + i)(s - 3\pi/2)/\pi & \text{for } \frac{3\pi}{2} \leq s \leq 2\pi. \end{cases} \tag{149}$$

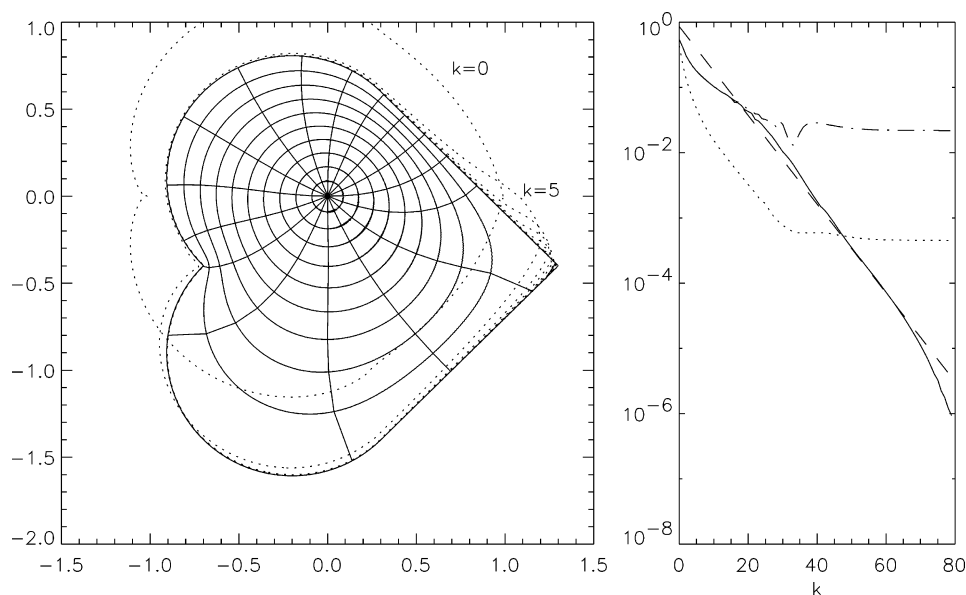


Fig. 8. The same as Figure 7 for the conformal mapping from the unit disk to the heart of Example 3 calculated by the smoothed AP method.

Figure 8 shows the result of a calculation with the smoothed AP method with $\xi(s) = ie^{is}$ for the eccentric heart of Example 3 with $z_0 = 0.3 - 0.4i$. The iteration was started with $S_0(t) = t$ and performed on a grid with $N = 1024$ points. The accuracy is about 0.02. The rate of convergence is $q = 0.855$. The rate of convergence for the (nonsmoothed) AP method was $q = 0.935$. This example demonstrates that the AP method works well even for regions with reentrant corners.

There are several variants of the projection method. All have in common the first projection step (135) which extracts the analytic part of $\eta \circ S_k$. But they differ in the way the nonanalytic part is used to change the boundary correspondence function.

The original version of Bergström [16] (see also Gaier [65, p. 109]) is applicable for star-shaped regions with parameterization $\eta(s) = \rho(s)e^{is}$. The second projection is along lines of equal argument, i.e.,

$$S_{k+1} := \arg f_k(t). \quad (150)$$

Bisshopp [21] determines a least-square approximation to the solution of the boundary correspondence equation. In an alternating iterative procedure he determines first an approximation to $\eta(S_k(t))$. This is the f_k of (134). The vanishing of the first variation of S_k gives the condition

$$\operatorname{Re}(\overline{\dot{\eta}(S_k(t))}(f_k(t) - \eta(S_k(t)))) = 0. \quad (151)$$

Newton's method for this equation gives an update for S_k which differs only slightly from (136).

Klonowska and Prosnak [136] arrange the new parameter points $S_{k+1}(t_j)$ in such a way that the distance between adjacent points $\eta(S_{k+1}(t_j))$ and $\eta(S_{k+1}(t_{j+1}))$ on Γ and the distance between two adjacent points $f_k(t_j)$ and $f_k(t_{j+1})$ on the curve parameterized by $f_k(t)$ (both measured by arclength) are in the same ratio for all intervals. This method is also applicable for nonsmooth curves. A normalization $\Phi(1) = \eta(s_0)$ must be used instead of $\Phi'(0) > 0$.

Li and Syngellakis [165] determine $S_{k+1}(t_j) = s^*$ as the parameter value s^* of that point on Γ which is closest to the point $f_k(t_j)$, i.e., gives the minimum Euclidean distance $|f_k(t_j) - \eta(s)|$ among all s . This method is applicable also for nonsmooth curves.

Li and Syngellakis [165] conjecture that their method converges globally. This is empirically supported by several examples. These authors point out that the numerical effort in each iteration of their algorithm is less than in the Newton methods. This is true. They claim even that their "algorithm should be more efficient than Wegmann's method" [165, p. 637]. This is not true in general (see Figure 16).

For a star-shaped region with boundary parameterization

$$\eta(s) = \rho(s)e^{is} \quad (152)$$

one can write the boundary correspondence equation in the form

$$\Psi(e^{it}) = \log \rho(S(t)) + i(S(t) - t) \quad (153)$$

with the auxiliary analytic function

$$\Psi(z) := \log(\Phi(z)/z), \quad (154)$$

which satisfies

$$\operatorname{Im} \Psi(0) = 0 \quad (155)$$

since $\Psi(0)$ is the logarithm of the positive number $\Phi'(0)$. Upon eliminating Ψ from equation (153) with Theorem 8 using (155), the *integral equation of Theodorsen* [258]

$$S(t) = t + \mathbf{K}[\log \rho(S(t))] \quad (156)$$

is obtained. It involves the operator $\mathbf{K}$ of conjugation. Equation (156) has a unique continuous solution S whenever the derivative ρ' is continuous. For unicity it is even sufficient that ρ' exists and is bounded (von Wolfersdorf [294]). Each solution S of (156) gives a function Φ which automatically satisfies the normalization $\Phi'(0) > 0$.

The usual way to solve the nonlinear integral equation (156) is by iteration (Theodorsen [258], Theodorsen and Garrick [259]). This approach has become even more attractive since now FFT can be applied to evaluate the conjugation operator numerically very efficiently (Henrici [103,104]).

Starting from a function S_0 such that $S_0(t) - t$ is 2π -periodic, the next iterates are calculated by *successive conjugation*

$$S_{k+1}(t) = t + \mathbf{K}[\log \rho(S_k(t))]. \quad (157)$$

There are several interpretations of this process. One can think of it as a projection method which in the first step calculates the boundary values of an analytic function

$$f_k(t) := \rho(S_k(t)) \exp(i(S_{k+1}(t))). \quad (158)$$

These values are in general not on the boundary curve. Therefore, a second projection must be applied which gives values $\rho(S_{k+1}(t)) \exp(i(S_{k+1}(t)))$ on the curve. The projection is here along lines of equal argument.

The iterates S_k defined by (157) converge uniformly whenever the function ρ is absolutely continuous and satisfies a so-called *epsilon condition*

$$\left| \frac{\dot{\rho}(s)}{\rho(s)} \right| \leq \varepsilon \quad \text{for almost all } s, \quad (159)$$

with a number $0 < \varepsilon < 1$. Convergence in L^2 is linear with a rate $\leq \varepsilon$. The rate q of uniform convergence is $\leq \sqrt{\varepsilon}$ (see Gaier [71,65]). This rate, however, can be improved if the boundary is sufficiently smooth (Gaier [65, p. 71]).

One can check the epsilon condition also in a general parameterization $\eta(t) = x(t) + iy(t)$ of the curve using

$$\frac{\dot{\rho}(s)}{\rho(s)} = \frac{xx' + yy'}{xy' - yx'}, \quad (160)$$

where on the right-hand side differentiation is with respect to the curve parameter t . For an ellipse with axes a, b with $b < a$ one obtains $\varepsilon = (a^2 - b^2)/2ab$ which is less than 1 for $b/a > \sqrt{2} - 1 = 0.414 \dots$

The Theodorsen equation (156) is discretized in an obvious way. The functions S_k are evaluated at $N = 2n$ grid points t_j , and the conjugation operator $\mathbf{K}$ is replaced by $\mathbf{K}_N$ as defined in (32). A system of N nonlinear equations is generated

$$S(t_j) = t_j + \mathbf{K}_N[\log \rho(S(t))], \quad j = 1, \dots, N. \quad (161)$$

Note that $\mathbf{K}_N$ requires only the values of $\log \rho(S(t))$ on the grid. These equations are very well investigated. We quote Gaier [71]: "...probably no nonlinear system of equations has received more attention than (T_d) [= the discrete Theodorsen equations (161)]".

When the boundary curve satisfies an epsilon condition (159) with $\varepsilon < 1$ then the discrete Theodorsen equation can be solved by the method of successive conjugation. The iterates converge linearly with a rate $\leq \varepsilon$ (Opitz [203]).

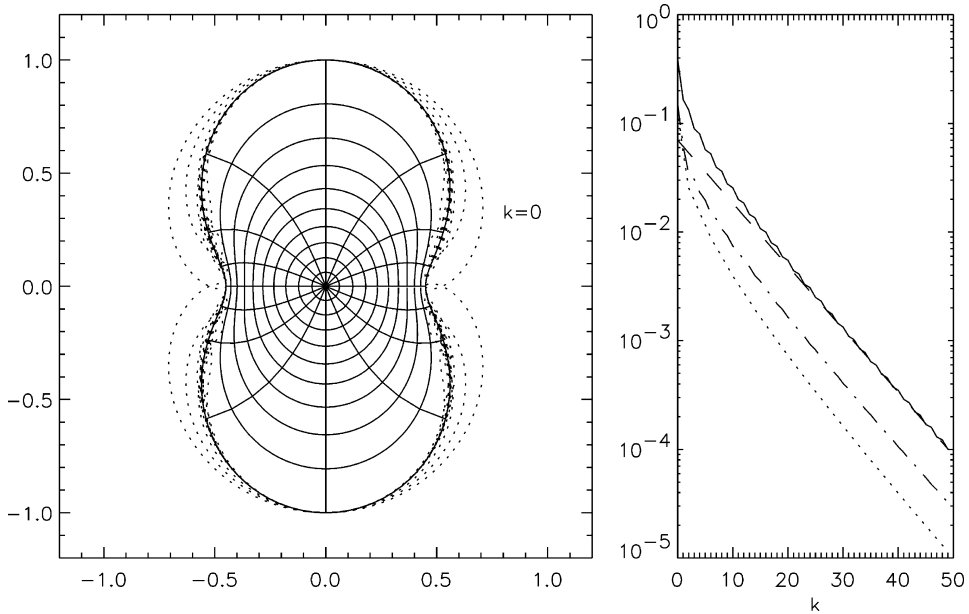


Fig. 9. Conformal mapping from the unit disk to an inverted ellipse with parameterization (162) by the Theodorsen method. The dotted lines are the curves parameterized by the functions f_k defined in (158). The right panel shows δ_k (solid), α_k (dotted), ϵ_k (dash-dotted) and Cq^k (dashed) for each iterative step k .

EXAMPLE 4. Inverted ellipse with parameterization (152) where ρ is the function

$$\rho(s) := \sqrt{1 - (1 - p^2) \cos^2 s}. \quad (162)$$

Figure 9 shows the result of a calculation with Theodorsen's method for the mapping of the disk to an inverted ellipse (Example 4) with parameter $p = 0.45$. This curve satisfies the epsilon condition for $\varepsilon = 0.886$. The calculation is performed with 256 grid points. The analytic functions f_k defined in (158) approach the boundary curve. The right panel of the figure shows the maximal change in each step and the measures of accuracy α_k and ϵ_k defined in (145) and (146). The iteration converges linearly with a rate $q = 0.883$. The images of the grid points crowd near the waist of the wasplike figure.

One can exploit the checkerboard structure of the Wittich matrix (see Gaier [65, p. 76] for definition and details) to separate the values with odd and even indices. This leads to two coupled systems of equations, which can be treated by an "Einzel-schrittverfahren". This converges twice as fast as the "Gesamtschrittverfahren", i.e., the method of successive conjugation (Niethammer [191], Hübner [119]).

For nearly circular regions with a boundary parameterization (125) the iteration starting from $S_0(t) = t$ gives already in the first step the approximation $S_1(t) = t + \tau \mathbf{K} \rho + o(\tau)$ which is accurate in first order of τ in view of Lavrentev's principle (128). Therefore, nothing can be gained by over- or underrelaxation – at least for nearly circular regions.

This is different for general regions. The iterative solution of the discrete Theodorsen equation can be interpreted as a Jacobi iterative method. It was found in several investigations (Niethammer [191], Gutknecht [92,94,95], Kaiser [131]) that the convergence can be accelerated by underrelaxation, i.e., instead of (157) the equation

$$S_{k+1}(t) = (1 - \omega)S_k(t) + \omega[t + \mathbf{K}(\log \rho(S_k))] \quad (163)$$

is used for iteration with an ω in the range $0 < \omega < 1$. For some cases, the choice of the relaxation parameter

$$\omega = \frac{2}{1 + \sqrt{1 + L^2}} \quad (164)$$

with $L := \|\dot{\rho}/\rho\|_\infty$ is optimal (Niethammer [191], Gutknecht [92,95]). Gutknecht [94] reports experiences (convergence and accuracy) from a number of test calculations with several variants of stationary iteration methods and optimal and nearly optimal relaxation parameters.

Gutknecht [94, p. 4] wrote in 1983: “It is still an open question which method is best in which situation. But numerical experiments presented here show that our methods [Theodorsen with underrelaxation] are definitely among the fastest”.

The iteration with underrelaxation converges sometimes even when the boundary does not satisfy the epsilon condition (159) with $\varepsilon < 1$. On the other hand, Hübner [120] has shown that there are regions satisfying an epsilon condition with $\varepsilon > 1$ such the iteration (163) diverges for any choice of ω .

EXAMPLE 5. Regular pentagon with parameterization (152), where ρ is the function

$$\rho(s) := 1/\cos(s - (2j - 1)\pi/5) \quad \text{for } 2j\pi/5 \leq s \leq 2(j + 1)\pi/5, \quad (165)$$

for the five sides $j = 0, 1, \dots, 4$.

Theodorsen’s method can easily be applied for regions with corners. This is demonstrated by Example 5. The result of a calculation with $N = 1024$ grid points is shown in Figure 10. The rate of convergence is $q = 0.5$. The pentagon satisfies the epsilon condition with $\varepsilon = 0.72$. In this example the rate of convergence is less than ε . The accuracy achieved is about 0.01.

The square satisfies (159) only with $\varepsilon = 1$. Theodorsen’s method does not converge.

Theodorsen’s method becomes inaccurate if the boundary is not smooth. Often the numerically calculated approximations to $S(t)$ are not monotone. Gutknecht [94] recommended to remove the corners by preliminary maps.

Nitsche [193] considers the solution of the Theodorsen equation (153) by a finite element method. The solution of Theodorsen’s equation by a Newton method will be discussed in Section 4.3.

Theodorsen’s method is very simple to program. It is remarkably stable. The iteration converges globally if only the epsilon condition is satisfied with $\varepsilon < 1$. This is a very

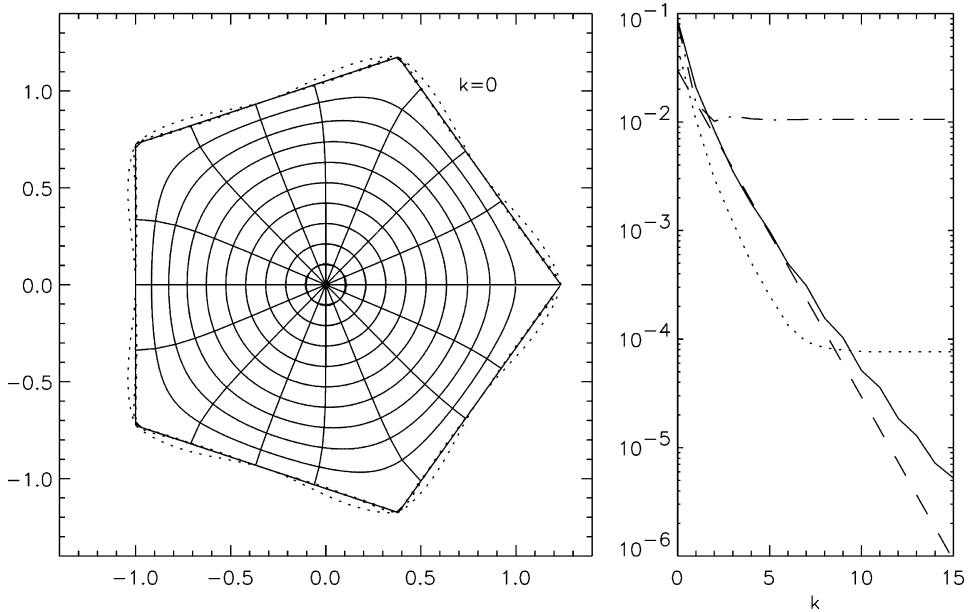


Fig. 10. Conformal mapping from the unit disk to the regular pentagon of Example 5 by the Theodorsen method. For the interpretation see Figure 9.

remarkable property. Most of the other iterative methods converge only locally, i.e., only if a sufficiently good initial approximation is known in advance. On the other hand, the requirement that the boundary curve must be parameterized in polar coordinates (152) is a real obstacle in practical applications.

There are other projection methods. Melentev (see [132, p. 415]) and Kulisch [151] propose for star-shaped regions the following iterative method which is based on the observation that the function $\mathcal{E}(z) := \Phi(z)/z$ has boundary values

$$\mathcal{E}(e^{it}) = \rho(S(t)) \exp[i(S(t) - t)]. \quad (166)$$

Starting from $\mathcal{E}_0(z) = z$, a sequence of analytic functions $\mathcal{E}_k$ and boundary correspondence functions S_k is constructed for $k \geq 1$ iteratively by

$$S_k(t) := t + \arctan \frac{\operatorname{Im} \mathcal{E}_{k-1}(e^{it})}{\operatorname{Re} \mathcal{E}_{k-1}(e^{it})}. \quad (167)$$

This is a projection of $\mathcal{E}(e^{it})$ onto the curve, since $\rho(S_k(t)) \exp[i(S_k(t) - t)]$ is on Γ . If S_k is inserted on the right-hand side of (166), it will not give the boundary values of an analytic function. The new $\mathcal{E}$ is constructed in such a way that it has the same real part as the function obtained by inserting S_k in the right-hand side of (166):

$$\mathcal{E}_k(e^{it}) := (\mathbf{I} + i\mathbf{K})[\rho(S_k(t)) \cos(S_k(t) - t)]. \quad (168)$$

This method is suitable for a graphical solution procedure (see [132]). In the first step (168) the points are pushed parallel to the imaginary axis and in the second step (167) in the radial direction. For nearly circular regions the Melentev–Kulisch method coincides with the Theodorsen method up to terms of order $O(\tau^2)$. In particular, it recovers in the first step the full first-order correction $\tau \mathbf{K} \rho$ to the initial guess $S_0(t) = t$. Kulisch [151] presents a wiring diagram for an analog computer which performs this iteration.

A test calculation with the Melentev–Kulisch method for an inverted ellipse with parameter $p = 0.35$ (Example 4) converged linearly with a rate $q = 0.86$. There is an xy -asymmetry in the method. It converges no longer when the boundary curve is rotated by 90° . It does not converge for the pentagon of Example 5.

Lotfullin [167] uses a projection method for the calculation of the inverse mapping $F: G \rightarrow D$. First the region is preprocessed by an osculation method to make it nearly circular. This gives an approximation $F_0(z)$ for F .

Starting with this analytic function the iteration proceeds as follows for $k \geq 1$. The function

$$g_k(s) := \exp[i \arg(F_{k-1}(\eta(s)))] \quad (169)$$

has values on the unit circle. Equation (169) means that the boundary values of F_{k-1} are projected onto the unit circle along rays through the origin. From the function g_k the analytic part, extracted by means of Plemelj's formula

$$F_k(\eta(t)) := \frac{1}{2} g_k(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{g_k(s)}{\eta(s) - \eta(t)} d\eta(s), \quad (170)$$

gives the next approximation F_k for F .

The numerical evaluation of the principal value integral in (170) is not so simple for a general region. Therefore, projection methods are mainly used to calculate the mapping Φ from the disk to the region, but rarely for the inverse function F .

4.3. Newton methods

One can solve the boundary correspondence equation (122) also by the Newton method. Linearization yields the equation

$$\Phi(e^{it}) + D\Phi(e^{it}) = \eta(S(t)) + \dot{\eta}(S(t)) DS(t) \quad (171)$$

which connects the change $D\Phi$ of the analytic function Φ with the change $DS(t)$ of the boundary correspondence function $S(t)$. One can use this relation to solve (122) at least in first order. Since $DS(t)$ must be a real function it can be eliminated from (171) and a boundary problem

$$\operatorname{Im} \frac{\Psi(e^{it})}{\dot{\eta}(S(t))} = \operatorname{Im} \frac{\eta(S(t))}{\dot{\eta}(S(t))} \quad (172)$$

for the analytic function $\Psi := \Phi + D\Phi$ remains. It is a Riemann–Hilbert problem as discussed in Section 2.5. The necessary change of S to satisfy (122) in first order is given by

$$DS(t) = \operatorname{Re} \frac{\Psi(e^{it})}{\dot{\eta}(S(t))} - \operatorname{Re} \frac{\eta(S(t))}{\dot{\eta}(S(t))} \quad (173)$$

with a solution Ψ of (172). This is the basis of the following iterative method first proposed by Wegmann [282, 283].

When the parameterization is differentiable with Lipschitz continuous derivative $\dot{\eta} \neq 0$, the derivative can be written in the form

$$\dot{\eta}(s) = r(s)e^{i\theta(s)} \quad (174)$$

with Lipschitz continuous functions θ and $r > 0$.

The iteration starts with a function $S_0(t)$ such that $S_0(t) - t$ is a 2π -periodic function in the Sobolev space W . (The natural choice is $S_0(t) = t$.)

When S_k is determined for some $k \geq 0$ then the following functions and numbers must be calculated ($\mathbf{K}$ and $\mathbf{J}$ are the conjugation and averaging operators defined in Section 2.5):

$$v(t) := \theta(S_k(t)) - t = \arg[e^{-it}\dot{\eta}(S_k(t))], \quad (175)$$

$$w := \mathbf{K}v, \quad \alpha := \mathbf{J}v, \quad (176)$$

$$g(t) := r(S_k(t)) \exp(w(t)) \operatorname{Im} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (177)$$

$$h := \mathbf{K}g, \quad \gamma := \mathbf{J}g, \quad (178)$$

$$DS(t) = -\frac{\gamma \cot \alpha + h(t)}{r(S_k(t)) \exp(w(t))} - \operatorname{Re} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (179)$$

$$S_{k+1}(t) := S_k(t) + DS(t). \quad (180)$$

The functions v, w, g, h, DS and the numbers α, γ depend on the iteration number k . We have omitted the index k for notational convenience.

When η is differentiable with Lipschitz continuous derivative $\dot{\eta}$, the method converges in W provided the iteration starts with a function S_0 which is sufficiently close to the correct boundary correspondence function $\tilde{S}$. When η is twice differentiable and $\dot{\eta}$ satisfies a Hölder condition with exponent μ then the order of convergence is at least $1 + \mu$. In particular, for Lipschitz continuous $\dot{\eta}$ convergence is quadratic (Wegmann [282, Theorem 1]).

The method can be applied numerically in a straightforward way on a grid of $N = 2n$ equidistant points $t_j = (j - 1)2\pi/N$ in the interval $[0, 2\pi]$. In the formulas above everywhere the operator $\mathbf{K}$ has to be replaced by its discrete approximation $\mathbf{K}_N$ on this grid. In each iterative step two conjugations must be calculated, namely in (176) and (178). This requires four FFTs. Using a grid with N points, where N is a power of 2, the computational cost is of the order $O(N \log_2 N)$.

This method works very well and follows the quadratic convergence closely as long as the discretization error is small compared with the changes DS . It has been noted by several people (Song [245], Wegert [279] and several personal communications) that the method approaches the solution very fast in the first iterations but then has a tendency to turn away. It may even diverge finally. Convergence cannot be enforced by increasing the number of grid points, or by starting with a better S_0 , or by calculating with higher precision. This was nicely demonstrated by Song [245]. She calculated with quadruple precision, got very precise results – but the iteration diverged finally.

Wegmann [288] investigated this convergence–divergence phenomenon. He showed that the method, when discretized in this straightforward way, does not converge. The nuisance is generated by a $\cos nt$ term. Although this term is negligibly small for smooth curves and large N , it is inevitably generated by roundoff errors. Wegmann [288] showed that the discretized method converges at least for nearly circular regions, when the annoying $\cos nt$ term is removed. This can be done by using instead of (179) the modified formula

$$DS(t) = -\frac{\gamma \cot \alpha + h(t) + \beta \cos nt}{r(S_k(t)) \exp(w(t))} - \operatorname{Re} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (181)$$

where β is the coefficient of the $\cos nt$ term in the trigonometric interpolation polynomial of order n for the function v on a grid with $N = 2n$ equidistant points.

Strictly speaking the discretized method converges only linearly. But this linear convergence pertains only to the final stage where small changes occur. In the first stage where the gross features of the parameter functions are modeled, convergence follows the quadratic behavior of the continuous method. Baty and Morris [8] report some practical experiences with this method.

This method has been carried over by Wegert [279] to the solution of nonlinear RH problems. Wegert noted that the convergence can be improved by smoothing the function DS . Song et al. [246,247] apply a low-frequency filter. It has been shown by Wegmann [291] that the problem of divergence is connected with the fact that the approximations $\mathbf{K}_N$ for the conjugation operator as Banach space operators converge strongly to $\mathbf{K}$ as $N \rightarrow \infty$ but not uniformly. But when the operators are restricted to a compact subset, convergence is uniform. This can be enforced by the restriction to a suitable finite subspace. This is the main reason why convergence of the discrete method can be reestablished by the following smoothing procedure.

Let $M := 2m < N$ be a fixed natural number. The function DS is calculated by (179). The Fourier coefficients are determined

$$DS(t) := \sum_{l=0}^n a_l \cos lt + \sum_{l=1}^{n-1} b_l \sin lt \quad (182)$$

and then DS is replaced by the truncated trigonometric polynomial

$$DS^*(t) := \sum_{l=0}^m a_l \cos lt + \sum_{l=1}^m b_l \sin lt \quad (183)$$

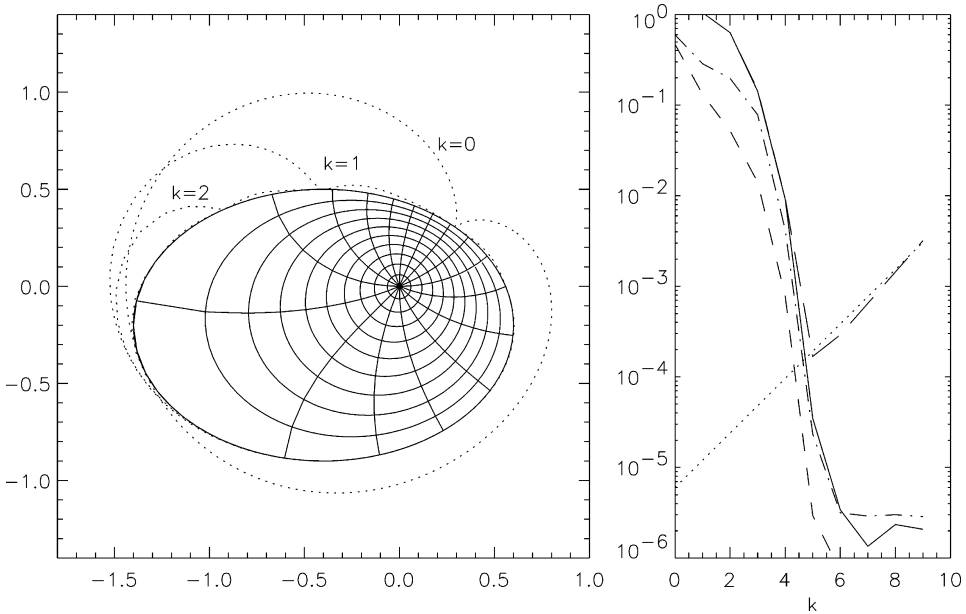


Fig. 11. Conformal mapping to the ellipse of Example 1 with the (damped) Wegmann method. The dotted lines show the boundary values of the analytic functions Φ_k of (184). The right panel shows the $\delta_k, \alpha_k, \varepsilon_k$ during the iteration. For the long-dashed and dotted lines see the text.

which then is used to calculate the new iterate $S_{k+1} = S_k + DS^*$.

The effect of this smoothing is demonstrated by a calculation for Example 1 (parameters: $a = 1, b = 0.7, z_0 = -0.4 - 0.2i, \alpha = 0$) with the Wegmann method with $N = 256$ grid points. A damping according to (183) is performed with $m = 108$. The result is shown in Figure 11. The curves

$$\Phi_k(e^{it}) = \eta(S_k(t)) + \dot{\eta}(S_k(t))(S_{k+1}(t) - S_k(t)) \quad (184)$$

which by construction are the boundary values of analytic function Φ_k differ noticeably from Γ only for $k = 0, 1, 2$. This can be compared with Figure 6.

After eight steps the iteration becomes stationary in single precision. Convergence is quadratic initially. For comparison, the values of δ_k are inserted for the method without damping (long-dashed). These values increase after the 6th iteration in geometric progression. The long-dashed curve points back to a point around 5×10^{-6} (dotted line) on the ordinate, indicating that the noise is generated by rounding errors, which are amplified in each step.

One might be afraid that the damping according to (182) and (183) requires two other Fourier transforms and therefore causes much additional cost. But in many cases it is quite sufficient to damp a few of the highest terms, with the number $L = n - m$ of damped coefficients independent of the number N of grid points. This requires only computing time of the order $O(N)$. We demonstrate this at the same example as before. Figure 12

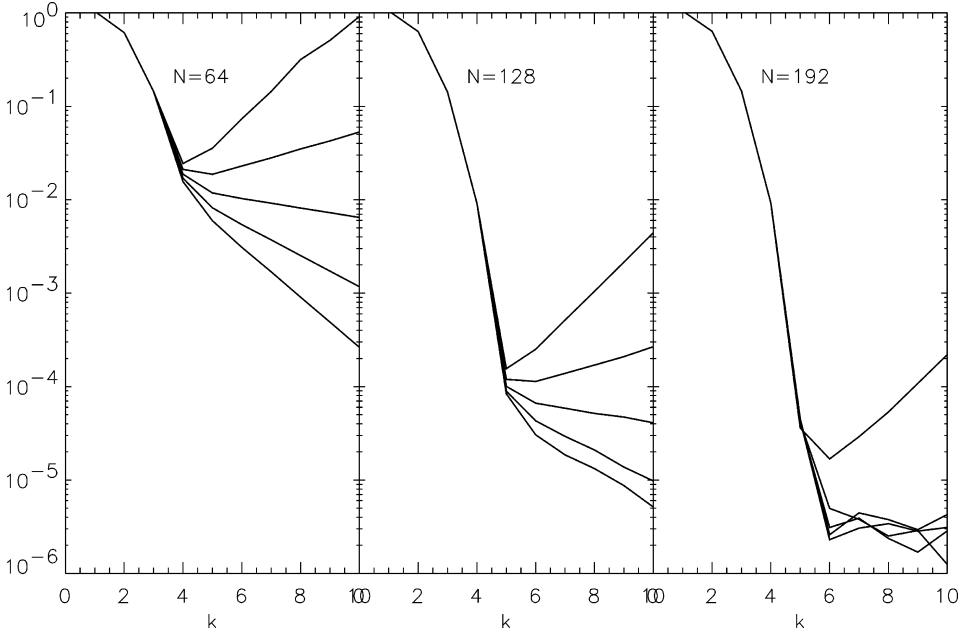


Fig. 12. The shift δ_k during the calculation with Wegmann's method for the same ellipse as in Figure 11 for three values of N and $L = 0, 1, 2, 3, 4$ damped Fourier coefficients.

shows the convergence, measured by the shift δ_k in each step, for different numbers of grid points and $L = 0, 1, 2, 3, 4$ damped Fourier terms. Apparently for each N the damping of 2 terms suffices to make the iteration convergent. The rate of the final convergence (or divergence) depends mainly on L . Convergence is faster, when more Fourier terms are damped.

Henrici [107, p. 422] wrote in 1986: "In view of its quadratic convergence [...] and its $O(n \log n)$ operations count per iteration step, Wegmann's method may well be the best mapping method in existence, but sufficient experimental documentation is as yet lacking".

Vertgeim [270] considered in 1958 boundary curves with the representation

$$\eta(s) = \exp(f_1(s) + i f_2(s)) \quad (185)$$

with real functions f_1, f_2 such that $f_1(s)$ and $f_2(s) - s$ are 2π -periodic. It is assumed that f_1, f_2 have Lipschitz continuous second derivatives and the first derivative $\dot{f}_2 - i\dot{f}_1$ does not vanish. The 2π -periodic functions θ and $r > 0$ are defined by

$$\dot{f}_2(s) - i\dot{f}_1(s) = r(s) \exp(i\theta(s)). \quad (186)$$

A boundary correspondence equation analogous to (122) can be written in the form

$$\Psi(e^{it}) = f_1(S(t)) + i[f_2(S(t)) - t] \quad (187)$$

with the analytic function $\Psi(z)$ defined in (154). This is equivalent to the functional equation for S ,

$$P[S] := f_2(S(t)) - t - \mathbf{K}f_1(S(t)) = 0. \quad (188)$$

The Newton method for this equation starts with an initial guess S_0 and calculates the change $DS = S_{k+1} - S_k$ in the k th iteration from the linearized equation (188)

$$\dot{f}_2(S_k(t))DS - \mathbf{K}[\dot{f}_1(S_k(t))DS] = -P[S_k]. \quad (189)$$

After multiplication of (189) by $\dot{f}_1$ the equation attains the form of an RH problem

$$\operatorname{Re}[(\dot{f}_2 + i\dot{f}_1)\mathcal{E}(e^{it})] = -\dot{f}_1 P \quad (190)$$

for the analytic function $\mathcal{E}$ in D with boundary values

$$\mathcal{E}(e^{it}) = (\mathbf{I} + i\mathbf{K})(\dot{f}_1 DS). \quad (191)$$

The function DS is obtained from $\mathcal{E}$ using $\dot{f}_1 DS = \operatorname{Re} \mathcal{E}(e^{it})$.

The RH problem (190) for $\mathcal{E}$ can be solved by the standard method using the operators $\mathbf{K}, \mathbf{J}$ of conjugation and averaging. When the functions and numbers

$$v(t) := \theta(S_k(t)), \quad w := \mathbf{K}v, \quad \alpha := \mathbf{J}v, \quad (192)$$

$$h := -\frac{e^w \dot{f}_1 P}{r} \quad (193)$$

are calculated, the change $DS = S_{k+1} - S_k$ is obtained by

$$DS = -\frac{\dot{f}_2 P}{r^2} + \frac{1}{r e^w}(\mathbf{K} - \tan \alpha \mathbf{J})h. \quad (194)$$

The $(\tan \alpha)$ -term in (194) comes from the condition $\operatorname{Im} \mathcal{E}(0) = 0$ which follows from the representation (191).

Vertgeim [270] did not take full advantage of his approach. He applied a quasi-Newton method, insofar as he inserted everywhere in the formulas (192), (193) and (194) the initial guess S_0 . He proved that the iterates converge in a Hölder-norm, provided the initial guess S_0 is sufficiently close to the correct parameter mapping S and $\cos \alpha \neq 0$. The same proof, however, can be applied to show the convergence of the full Newton method, where in the formulas (192)–(194) everywhere the last iterate S_k is inserted. Then convergence is quadratic.

The Vertgeim method needs in each iterative step three conjugations: one for the calculation of the right-hand side $P[S_k]$ of (189), one in (192) for the calculation of w and one for conjugation of h in (194). This must be compared with the Wegmann method which requires only two conjugations in each step.

One can parameterize any curve in the form (185). Even for star-shaped regions the representation (185) is flexible, since f_2 can be any monotone function. The usual representation (152) for star-shaped regions is obtained with

$$f_1(s) = \log(\rho(s)), \quad f_2(s) = s. \quad (195)$$

Then (188) gives Theodorsen's equation (156). When Vertgeim's method is specialized to this case, it gives an efficient solution procedure for Theodorsen's equation by a Newton method. This approach was (re)discovered and exploited by Hübner [123] in 1986, who proved that the iteration converges whenever the second derivative of ρ is Lipschitz continuous.

When discretized in the straightforward manner on a grid with N equidistant points, replacing $\mathbf{K}$ by $\mathbf{K}_N$, the Hübner–Vertgeim method converges for regions close to the unit disk (Wegmann [288]). Although in the initial phase the discretized method follows the quadratic convergence of the continuous version, convergence of the discretized method is finally only linear [288]. Similarly as for the Wegmann method, the turnover from quadratic to linear convergence depends on the number N of grid points, but the rate of the final linear convergence is independent of N . The Hübner–Vertgeim method converges not only in cases where the epsilon condition (159) is satisfied with $\varepsilon < 1$. It converges, e.g., for the inverted ellipse (162) with parameter $p = 0.23$, where (159) is satisfied only with $\varepsilon = 2$.

The iterative methods discussed in this section converge locally and quadratically in suitable function spaces. The corresponding discretized methods converge in the initial phase quadratically, but as soon as the changes are of the order of the discretization error, convergence becomes only linear (Wegmann [288]). The rate q of the final linear convergence is independent of the number of grid points, but the turnover from quadratic to linear convergence is rather sensitive to the number of grid points. This is shown in Figure 13 where the maximum changes in each step of the iteration are shown for a calculation of the mapping from the unit disk to a shifted ellipse (Example 1 with $a = 1$, $b = 0.7$, $z_0 = -0.3$, $\alpha = 0$) with the Wegmann method, using $N = 32, 64, 96, 128, 160$ grid points, respectively. The linear convergence occurs for all N with a rate $q = 0.38$. Also the rate $q = 1.77$ of divergence is the same for all N .

Chakravarty and Anderson [27] try to satisfy the condition

$$\operatorname{Re} \eta(S(t)) + \mathbf{K}[\operatorname{Im} \eta(S(t))] - \gamma = 0 \quad (196)$$

which must be satisfied by the boundary correspondence function S with $\gamma = \operatorname{Re} \Phi(0)$. They start with a guess S_0 and use a conjugate gradient method and a Newton method to change iteratively the function $S_k(t)$ and the real number γ_k in such a way that the left-hand side in (196) is minimized.

Sallinen [236] uses a continuation method to determine the conformal mappings to a family of regions $G(t)$ depending on a parameter $t \in [0, 1]$. Starting from the known mapping from the disk to $G(0)$, the mappings for the other t are calculated by increasing t in small steps and changing the mapping with Newton's method. The Newton step leads to an RH problem which can be solved explicitly.

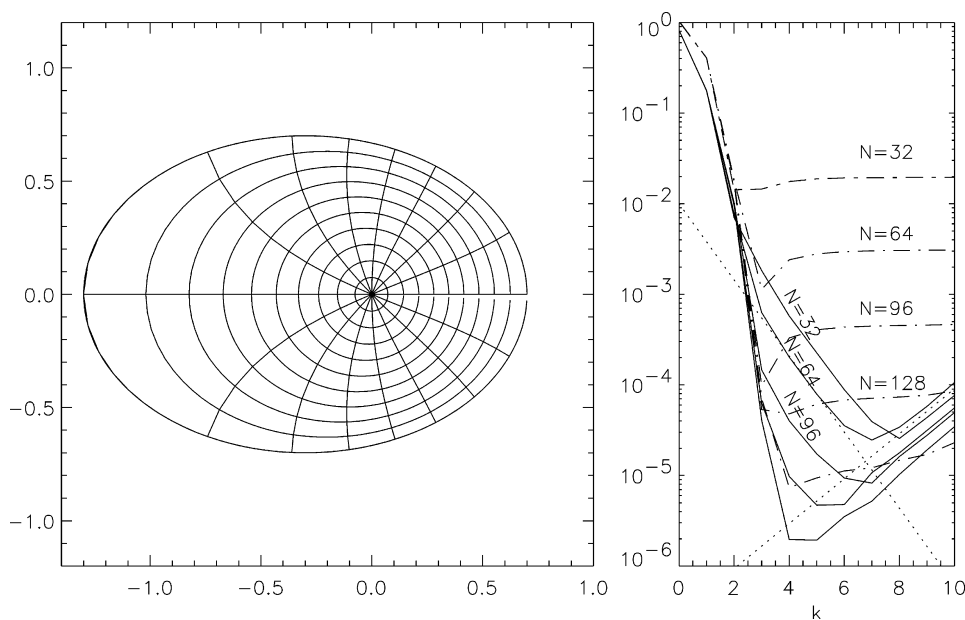


Fig. 13. Mapping to a shifted ellipse with the (undamped) Wegmann method. The right panel shows δ_k and ε_k during the iteration for calculations with $N = 32, 64, 96, 128$ and 160 grid points. The dotted lines are proportional to 0.38^k and 1.77^k .

4.4. Interpolation

The Newton methods discussed in Section 4.3 converge locally and quadratically in suitable function spaces. The corresponding discretized methods converge in the initial phase quadratically, but eventually only linearly. The discretized Newton methods are not Newton methods in the strict sense.

The methods of Section 4.3 first linearize the boundary correspondence equation (122) and then discretize. One can reverse the order. One can first discretize (122), and then solve the ensuing system of nonlinear equations by a Newton method. This then really gives a locally and quadratically convergent iteration. There are two major problems connected with this approach.

The first problem is a theoretical one: Does there exist a solution of the discretized boundary correspondence equation? Recall, that the proof of the convergence of the Newton method requires apart from mild differentiability conditions, the existence of a solution! While the existence of a solution of (122) follows from Riemann's mapping theorem, the solvability of the discretized equation is a nontrivial problem, as we will see below.

The second problem is a numerical one: How to perform the Newton method for the discretized equation numerically in an efficient way? This problem will also be discussed below.

We discuss first the question, of whether solutions to the discrete equations exist. We follow the reasoning of Wegmann [286]. Let $t_j = (j-1)2\pi/N$ be N equidistant grid points in the interval $[0, 2\pi]$ with an even number $N = 2n$. A *discrete boundary correspondence equation* can be written in the following way:

$$P_n(e^{it_j}) = \eta(s_j), \quad j = 1, \dots, N, \quad (197)$$

which must be satisfied by a polynomial P_n ,

$$P_n(z) = \sum_{l=1}^{n+1} p_l z^l, \quad (198)$$

of degree $n+1$ with complex coefficients $p_l, l = 2, \dots, n$, but real lowest- and highest-order coefficients, i.e.,

$$\operatorname{Im} p_1 = 0, \quad \operatorname{Im} p_{n+1} = 0. \quad (199)$$

The polynomial P_n is an approximation for the mapping function $\Phi: D \rightarrow G$. The real numbers s_j are approximations for the values $S(t_j)$ of the boundary correspondence function S . The condition that P_n must be a polynomial of form (198) can be written as a condition on the Fourier coefficients of the right-hand side of (197), namely

$$\sum_{j=1}^N e^{ilt_j} \eta(s_j) = 0 \quad (200)$$

for $l = 0, 1, \dots, n-2$, and

$$\operatorname{Im} \sum_{j=1}^N e^{ilt_j} \eta(s_j) = 0 \quad (201)$$

for $l = -1$ and $l = n-1$. Thus, the polynomial P_n is eliminated from (197), and a system of N real nonlinear equations for the N real unknowns s_j remains. The solvability of this system of equations has been discussed by Wegmann [286].

One can interpret (197) as an interpolation problem of the following kind: Determine a polynomial P_n of degree $n+1$ of the form (198) satisfying the constraints (199) such that the values of the polynomial at the grid points e^{it_j} lie on the boundary curve Γ of the region G . In this sense the polynomial P_n interpolates the curve Γ . The values at the grid points are not given explicitly, but only in the implicit form, that they are required to lie on Γ . In interpolation problems usually the number of grid points equals the degree of the polynomial. Here only half of the information about the function values is prescribed, and the number of grid points is about twice the degree.

Wegmann [286] proved the following result:

THEOREM 17. *When η is twice differentiable with Lipschitz continuous second derivative then there exists for all sufficiently large $N = 2n$ a polynomial (198) normalized by (199) such that the interpolation condition $P_n(e^{ij}) \in \Gamma$ is satisfied for all $j = 1, \dots, N$. In a neighborhood of the conformal mapping Φ there is exactly one such P_n . The sequence of these P_n converges to Φ in the sense that the boundary values $P_n(e^{it})$ converge to $\Phi(e^{it})$ in the Sobolev norm $\|\cdot\|_W$.*

In addition, Wegmann [286] has proved for curves η with a Lipschitz continuous third derivative that for sufficiently large N the derivatives P'_n of the polynomials do not vanish in D , and the P_n are conformal mappings of D to some regions G_n which are close to G .

The second of the conditions (199) is somewhat artificial. It has no counterpart in the continuous conformal mapping theory. It has been pointed out by Wegert [280] that this condition can be replaced by the more general one,

$$\operatorname{Re}(e^{i\alpha} p_{n+1}) = 0 \quad (202)$$

for any prescribed angle α .

Fornberg [62] made an ansatz by a polynomial P_n of degree n ,

$$P_n(z) = \sum_{l=1}^n p_l z^l, \quad (203)$$

with complex coefficients $p_1, \dots, p_n$. The interpolation problem (197) with this type of polynomials seems to be well posed at first glance, since the number N of unknowns s_j equals the number N of equations (200) which must be satisfied for $l = 0, 1, \dots, n-1$. The polynomials (203), however, do not satisfy the second normalization condition that $P'_n(0)$ is real. Therefore, one must not expect uniqueness. In fact, it has been proved by Wegmann [286] that for sufficiently large N there exist at least N solutions of (197) with polynomials of type (203). More precisely: For every $k = 1, \dots, N$, there exists such a solution $P_{n,k}$ with the property

$$(k-1) \frac{2\pi}{N} < \arg P'_{n,k}(0) \leq k \frac{2\pi}{N}. \quad (204)$$

Hence, one can find a solution of (197) with an n th degree polynomial (203) which satisfies the second normalization condition, $\arg P'_n(0) = 0$, with an error of at most $2\pi/N$.

When using polynomials (203) one should, instead of the condition $\operatorname{Im} P'_n(0) = 0$, better use the normalization

$$P_n(1) = \eta(s_1) \quad (205)$$

with a prescribed value of s_1 . If the interpolation condition (197) for $j = 1$ is omitted, a system of $N - 1$ equations for the remaining unknowns $s_2, \dots, s_N$ is obtained, which is solvable for sufficiently large N (see Fornberg [62], Wegmann [286]).

This theoretical discussion is in a sense academic, since for sufficiently large N the highest-order coefficient p_{n+1} is below roundoff error anyway. Then, numerically, one cannot distinguish between the conditions $p_{n+1} = 0$ or $\text{Im } p_{n+1} = 0$ and will find a solution numerically, despite the fact that an exact solution of the underlying system of equations does not exist. Theory guarantees that this numerical solution is close to the conformal mapping.

Fornberg [62] proposed in 1980 an efficient method for solving the problem (197). We follow here Wegmann [284] who adapted Fornberg's method to the calculation of polynomials of form (198).

The Newton method for equations (197) starts from the linearized equations

$$\eta(s_j) + \dot{\eta}(s_j)Ds_j = P_n(e^{it_j}) \quad (206)$$

which must be satisfied with a polynomial P_n of form (198) and real numbers Ds_j .

The conditions (206) are satisfied by a polynomial P_n if and only if the Fourier coefficients c_l of the left-hand side vanish for $l = n + 2, \dots, N$ and are real for $l = 1$ and $l = n + 1$. This is equivalent to the property that the Fourier polynomial formed with these coefficients,

$$f_0(t) = i(\text{Im } c_1)e^{it} + i(\text{Im } c_{n+1})e^{i(n+1)t} + \sum_{l=n+2}^N c_l e^{ilt}, \quad (207)$$

vanishes identically. In what follows, the necessary condition, $f_0(t_j)/\dot{\eta}(s_j) = 0$ for all j , is used to build up a system (208) of linear equations.

Let $\mathbf{x}$ and $\mathbf{u}$ be the vectors with components $x_j := \dot{\eta}(s_j)/|\dot{\eta}(s_j)|$ and $u_j := Ds_j |\dot{\eta}(s_j)|$, respectively. Then equations (206) can be written as a linear system of equations

$$\mathbf{A}\mathbf{u} = -\mathbf{r} \quad (208)$$

with a real $N \times N$ matrix $\mathbf{A}$ and a real vector $\mathbf{r}$.

It is not necessary to calculate the matrix $\mathbf{A}$. One needs only a recipe of how to calculate the product $\mathbf{A}\mathbf{u}$ for a given real vector $\mathbf{u}$. The product $\mathbf{A}\mathbf{u}$ is calculated by the following steps.

First calculate the Fourier coefficients

$$a_l := \frac{1}{N} \sum_{j=1}^N x_j u_j e^{-ilt_j}, \quad l = 1, \dots, N, \quad (209)$$

and the values of the Fourier polynomial

$$e_j = i(\text{Im } a_1)e^{it_j} + i(\text{Im } a_{n+1})e^{i(n+1)t_j} + \sum_{l=n+2}^N a_l e^{ilt_j}, \quad j = 1, \dots, N. \quad (210)$$

Then the components of $\mathbf{A}\mathbf{u}$ are equal to

$$(\mathbf{A}\mathbf{u})_j = \text{Re}(\overline{x_j} e_j), \quad j = 1, \dots, N. \quad (211)$$

The right-hand side of (208) is calculated in a similar way. The function η is Fourier analyzed

$$b_l := \frac{1}{N} \sum_{j=1}^N \eta(s_j) e^{-ilt_j}, \quad l = 1, \dots, N, \quad (212)$$

and the values

$$h_j = i(\text{Im } b_1)e^{it_j} + i(\text{Im } b_{n+1})e^{i(n+1)t_j} + \sum_{l=n+2}^N b_l e^{ilt_j}, \quad j = 1, \dots, N, \quad (213)$$

are calculated. The components r_j of $\mathbf{r}$ are then equal to $r_j := \text{Re}(\overline{x_j} h_j)$.

It has been shown by Wegmann [286] that the matrix $\mathbf{A}$ so defined is symmetric, positive semidefinite and has norm ≤ 1 in the Euclidean $\mathbb{R}^N$. If equation (206) has a unique solution (which is the generic case) then $\mathbf{A}$ is even positive definite.

Since the matrix $\mathbf{A}$ has these favorable properties, the system (208) can be solved by the conjugate gradient method (CGM). The calculation of the right-hand side $\mathbf{r}$ requires two Fourier transforms. In each iteration of the CGM $\mathbf{A}\mathbf{u}$ has to be evaluated for a vector $\mathbf{u}$. This again requires two Fourier transforms. This can be done efficiently with FFT. The computational cost is of the order $O(N \log N)$ with a coefficient $K_{\text{CGM}} + 2$ which depends on the number K_{CGM} of iterations needed in the CGM. It has been shown by Wegmann [286] that the matrix $\mathbf{A}$ is up to a matrix of rank less than or equal to 2 a discretized version of the operator $\frac{1}{2}(\mathbf{I} + \mathbf{R}_\beta)$ with the operator $\mathbf{R}_\beta$ introduced in (54) formed with the tangent angle β defined by $\exp(i\beta(s)) := \dot{\eta}(s)/|\dot{\eta}(s)|$. It follows from Theorem 11 that for sufficiently smooth boundary functions η only a few eigenvalues of $\mathbf{R}_\beta$ differ significantly from zero. Therefore, the eigenvalues of $\mathbf{A}$ cluster around $1/2$. This has the consequence that the CGM converges fast. The number K_{CGM} of iterations needed to achieve a desired accuracy depends only on the eigenvalue distribution of $\mathbf{A}$. It is largely independent of N . Therefore, the computational cost of this method is $O(N \log N)$.

One can exploit the checkerboard structure of the matrix $\mathbf{A}$ to reduce the system (208) to two-coupled systems of equations of order n . From these an n -order equation is obtained for the components of $\mathbf{u}$ with even index j with a matrix which is a discretization of the operator $\mathbf{I} - \mathbf{R}_\beta^2$. This matrix is positive definite. For sufficiently smooth curves

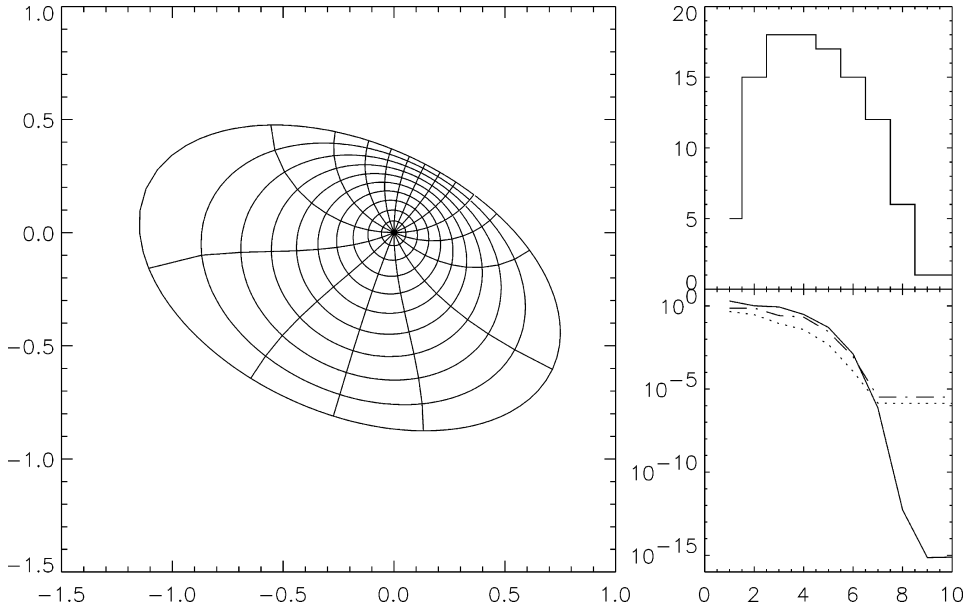


Fig. 14. The mapping from the disk to a shifted and rotated ellipse calculated by the interpolation method with 256 grid points. The lower right panel shows the maximum change δ_k in each step (solid) and the measures α_k and ε_k of accuracy, the upper right panel shows the number of iterations needed in the CGM.

the eigenvalues cluster at 1. This property has been detected in numerical experiments by Fornberg [62]. It is very favorable for the use of CGM (for details see Fornberg [62], Wegmann [286]). For an analysis of Fornberg's method, and its variant for the mapping of exterior regions see DeLillo and Pfaltzgraff [44].

Hübner [122] observed, that the Newton method for the discrete Theodorsen equation can be performed very efficiently since the ensuing linear equations can be transformed to a system with a Toeplitz matrix. For such systems fast solution methods are available which require work of the order $O(N \log^2 N)$. Unfortunately, these fast Toeplitz solvers can become unstable in some cases. Wegmann [286] carried this idea over to the solution of the linear system (206) which can also be transformed into a Toeplitz system.

Figure 14 shows the result of a calculation for an ellipse (Example 1 with $a = 1$, $b = 0.6$, $z_0 = -0.2 - 0.2i$, $\alpha = -0.4$) with the interpolation method with 256 grid points. The lower right panel shows that the method converges quadratically. The upper right panel shows that in the initial stages at most 18 iterations are needed in the CGM. This is reduced to 1 when the outer iteration has converged.

Figure 15 shows the result of a calculation for an ellipse (Example 1 with $a = 1$, $b = 1.4$, $z_0 = 0.3 - 0.4i$, $\alpha = 0.5$) with the Fornberg method with 256 grid points. The lower right panel demonstrates that the method converges quadratically. The upper right panel shows that in the initial stages at most 10 iterations are needed in the CGM. This is reduced to 1 when the outer iteration has converged.

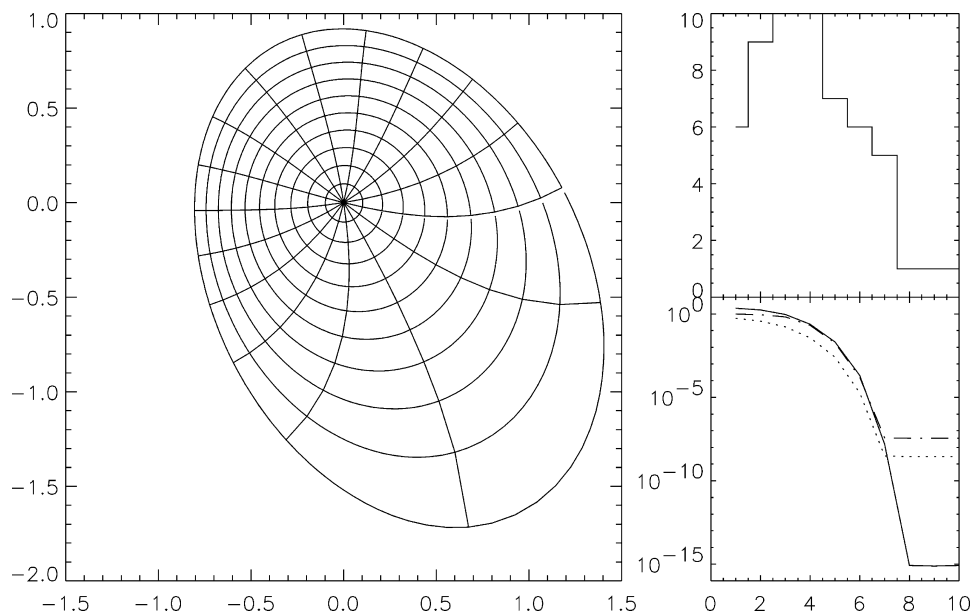


Fig. 15. The mapping from the disk to a shifted and rotated ellipse calculated by the Fornberg method with 256 grid points. The lower right panel shows the maximum change δ_k in each step (solid) and the measures α_k and ε_k of accuracy, the upper right panel shows the number of iterations needed in the CGM.

The discretized version of the method of alternating projections described in Section 4.2 converges to an interpolating polynomial (if it converges at all).

Porter [225] constructs the interpolating polynomial of degree n by alternatingly interpolating an n th degree polynomial at the points z_j with even/odd index j . The values w_j which the polynomial has to take are calculated from the values of the last polynomial at the points z_j with even/odd index j projected back to the curve. The resulting projection method converges linearly. This method is not new. It is already described in the book of Fil'čakov [57, p. 404]. A similar iterative method was used by Ugodčikov [267], who started from initial data obtained by an electrical model.

Opitz [203] interpreted the discrete Theodorsen equation (161) as an interpolation problem. The equation asks for a polynomial P_n of degree less than or equal to n with real lowest- and highest-order coefficients p_0 and p_n , which satisfies

$$z_j \exp(P_n(z_j)) \in \Gamma \quad (214)$$

for all $z_j := e^{ij2\pi/N}$, $j = 1, \dots, N = 2n$. The condition (214) resembles closely the condition (197) with the only difference that there is an exponential in (214).

In view of this close similarity, it is surprising that the discrete Theodorsen equation has a solution for all N provided only that ρ is a continuous function (Gutknecht [90]). If the boundary curve satisfies an epsilon condition (159) with $\varepsilon < 1$ then the solution of the discrete Theodorsen equation is unique. Hübner [120] has shown that for any $\varepsilon \geq 1$

and for any $n \geq 2$ there is a region G whose boundary curve satisfies (159) with this value of ε so that the Theodorsen equation discretized with $2n$ points has infinitely many solutions. Gutknecht [94] notes that there is strong experimental evidence, that even for quite simple boundaries the discrete Theodorsen equation may have more than one solution. He proposes to use a continuation method to reach the only “reasonable solution”.

Hübner [121] showed that for analytic boundary curves there is for all sufficiently large N a solution of the discrete Theodorsen equation which is close to the boundary correspondence function. This solution can be computed by the Newton method described in Section 4.3.

4.5. Accuracy

The result of a numerical calculation is an approximation $\hat{\Phi}$ to the mapping function $\Phi : D \rightarrow G$ which in most cases is a polynomial. When the calculation is done on a grid with $N = 2n$ points, then the approximating polynomial is of degree n or $n + 1$. This is most obvious in the projection and interpolation methods. It is well known from approximation theory that the degree of approximation by polynomials to Φ depends on the smoothness of Φ . We have seen in Section 2.2 that the smoothness of Φ is about the same as the smoothness of the boundary curve Γ of the region.

Methods based on function conjugation can be only as accurate as the approximation $\mathbf{K}_N$ of the conjugation operator on the grid. The error of $\mathbf{K}_N$ is discussed in Section 2.5.

When $\eta(s)$ is analytic in a strip $A_\tau := \{s + i\sigma : |\sigma| < \tau\}$ then Φ can be extended to an analytic function in the disk $\{z : |z| < R\}$ with $R := e^\tau$. Uniform approximation on the unit disk D by polynomials of degree less than or equal to n is possible with an error of order $O(R^{-n})$. This is the error for mappings of D to regions G with analytic boundaries. For Theodorsen’s method this is shown in Gaier’s book [65, p. 95]. Wegmann [282] obtained an error estimate with the same order for his method in a Sobolev norm under the additional hypothesis that $\dot{\eta} \neq 0$ in A_τ . DeLillo [37] reports a series of test calculations which confirm these estimates.

When η is in a Hölder space $C^{l,\alpha}$ for $l \geq 2$ and $0 < \alpha < 1$, the Sobolev norm of the error of Wegmann’s method is of order $O(N^{2-l-\alpha})$. The numerical results reported by Wegmann [282] indicate that the supremum norm of the error behaves like $O(N^{-(l+\alpha)})$.

To get a feeling for the efficiency of several of the methods discussed in this section we have calculated the mapping from the disk to an inverted ellipse (Example 4) with parameter $p = 0.45$ by the AP, the OAP, the Theodorsen and the Wegmann methods. After each iteration we calculated the achieved accuracy (measured by ε) and the number N_F of FFTs. Since the computational cost is mainly determined by the Fourier transforms, we measure it by the quantity $P := N_F N \log_2 N$ where N is the number of grid points. The result is shown in Figure 16. For low accuracy the OAP method is the most efficient. Its rate of convergence for this example is $q = 0.45$. The AP and the Theodorsen methods converge with rates $q = 0.77$ and $q = 0.81$, respectively. Newton methods have a slow start but they accelerate. They are therefore best suited for high accuracy calculations. This is most clearly seen in the right panel of Figure 16.

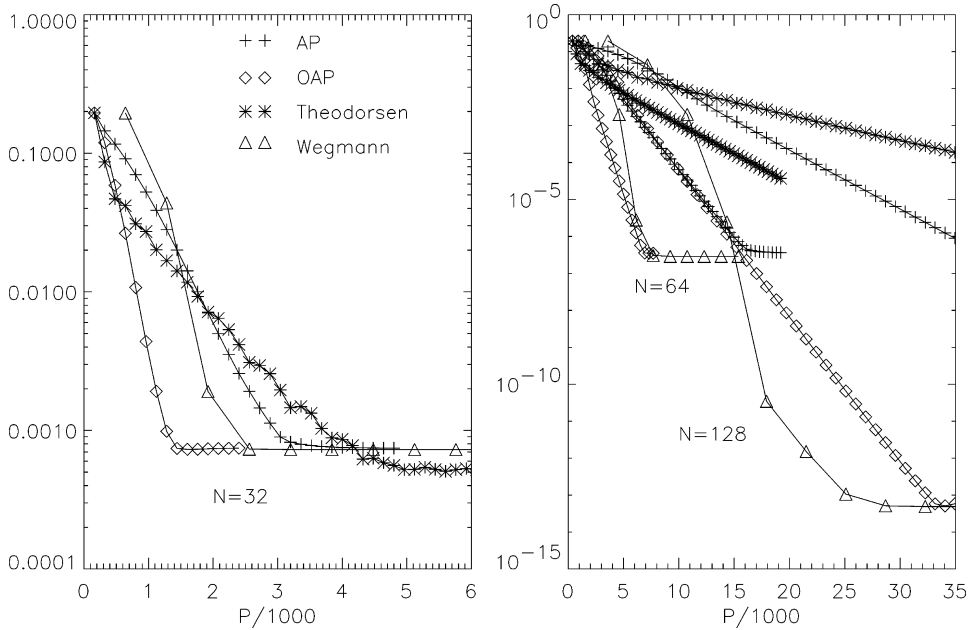


Fig. 16. The accuracy ε as a function of the computational cost P for the calculation of the mapping to an inverted ellipse with the AP (+ sign), OAP (diamond), Theodorsen (asterisk) and Wegmann (triangle) method with $N = 32$ (left panel) and $N = 64$ and $N = 128$ (right panel).

5. Mapping from an ellipse to the region

The fast Newton methods of Section 4.3 for calculating the conformal mapping from the unit disk to a region rely mainly on the following two properties:

- (a) The conjugate harmonic function can be calculated very efficiently using FFT.
- (b) Functions f analytic in the disk can be developed in Taylor series converging in the disk.

The fast methods for the disk can be carried over to other canonical regions where these properties hold in a suitable modified form.

For general regions the Taylor series has to be replaced by the Faber series in order to get a development in a series of polynomials convergent in the whole region (see Henrici [107, p. 507 ff]).

The Faber polynomials of ellipses are multiples of the Chebyshev polynomials. For $q > 1$ let E_q be the ellipse whose boundary is parameterized by

$$\zeta(t) := \frac{1}{2}(qe^{it} + q^{-1}e^{-it}). \quad (215)$$

The Chebyshev polynomial T_n of degree n has the boundary values

$$T_n(\zeta(t)) = \frac{1}{2}(q^n e^{int} + q^{-n} e^{-int}). \quad (216)$$

This has been exploited by DeLillo and Elcrat [39] to carry over Fornberg's method (see Section 4.4) from the disk to an ellipse as canonical region. DeLillo, Elcrat and Pfaltzgraff [41] carried this idea further and applied it to cross-shaped regions.

The conjugate harmonic function on an ellipse can be calculated very easily. When a real function on the boundary of E_q as a function of the parameter t in the representation (215) is developed in a Fourier series

$$f(\zeta(t)) = a_0 + \sum_{l=1}^{\infty} (a_l \cos lt + b_l \sin lt), \quad (217)$$

the conjugate harmonic function $g := \mathbf{K}_E f$ has Fourier series

$$g(\zeta(t)) = \sum_{l=1}^{\infty} \left(-D_l b_l \cos lt + \frac{a_l}{D_l} \sin lt \right) \quad (218)$$

with the factors

$$D_l := \frac{1 + q^{-2l}}{1 - q^{-2l}} \quad (219)$$

(Wegmann [291]). The l th Fourier coefficients of the conjugate function depend only on the l th coefficients of the original function. The factors D_l are close to 1 for large l . For high-order coefficients conjugation on the ellipse is almost the same as conjugation on the disk (compare (218) with (27)).

The formula (218) resembles the formula (299) for conjugation on an annulus (see Section 11.1). In fact, there is a close relationship between these two types of regions, since the *Joukowski map* $\Psi(z) := \frac{1}{2}(z + z^{-1})$ maps the annulus $\{z: 1/q < |z| < q\}$ onto a Riemann surface, which covers the ellipse E_q twice. In particular, Ψ maps the unit circle to the interval $[-1, 1]$.

The operator $\mathbf{K}_E$ plays the same role for the ellipse E_q as the operator $\mathbf{K}$ for the disk. In particular, Theorem 8 holds with D and $\mathbf{K}$ replaced by E_q and $\mathbf{K}_E$, respectively. This has the consequence that methods which are based on function conjugation can easily be carried over to ellipses. For the Wegmann method this has been done in [291]. The behavior of this modified method is quite similar to that for the disk. It is more prone to divergence. Convergence can be enforced by damping of the higher Fourier terms as described in (183).

Ellipses as canonical regions have the advantage that crowding can be avoided to some extent. For this purpose the q should be chosen so that the ellipse E_q has nearly the same aspect ratio as the target region (see DeLillo and Elcrat [39]).

EXAMPLE 6. Sportground of length 2π is bounded by two semicircles of radii R_S and two straight line segments of length $(1 - R_S)\pi$.

Figure 17 shows the mapping of an ellipse with $q = 1.25$ (aspect ratio 0.22) to the sportground of Example 6 with $R_S = 0.25$ which has aspect ratio 0.175. The example is

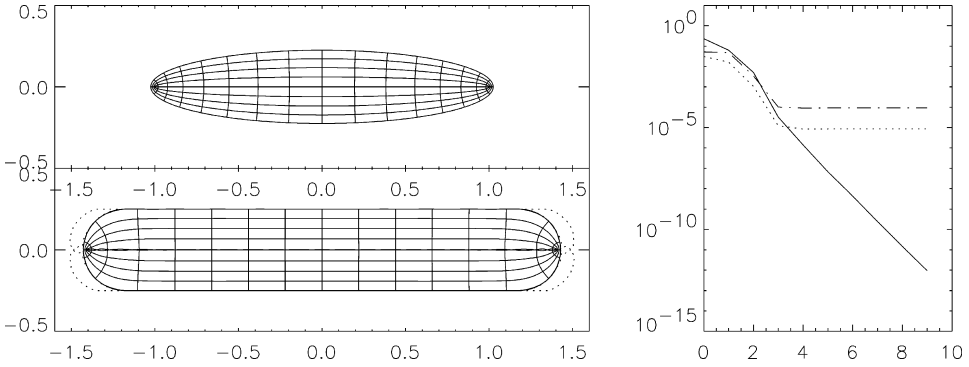


Fig. 17. The mapping of the ellipse with $q = 1.25$ (upper panel) to the sportground of Example 6 (lower panel). The right panel shows the maximum change δ_k (solid) in each iteration and the error measures α_k and ε_k .

calculated on a grid with $N = 512$ points by the method described in [291] with damping as described in (183). Convergence is quadratic in the beginning but then turns to linear. There is little distortion and no crowding.

6. Waves

An important application for conformal mapping is the study of waves (see, e.g., Lamb [153, p. 363]). There are a few explicitly known mapping functions. In general the wave form must be calculated from the equilibrium conditions of gravity and capillary forces. This requires special techniques.

We consider here only the case where the wave form is known, and the velocity field (u, v) must be determined. The region G is bounded from below by the wave with wavelength λ . We assume that $\lambda = 2\pi$. The region G is mapped by $z \rightarrow w = \exp(iz)$ to a simply-connected bounded region G' . Since the flow field is 2π -periodic, it can be expressed as a function of the variable $w \in G'$. Therefore, it is only necessary to find a conformal mapping Φ from the unit disk to the region G' satisfying $\Phi(0) = 0$. This function Φ can be calculated with the methods described before. When the wave is described by $s + i\hat{y}(s)$ with a 2π -periodic function $\hat{y}(s)$, the auxiliary region G' is star-shaped with boundary parameterization $\eta(s) = \exp(-\hat{y}(s))e^{is}$.

EXAMPLE 7. Cosine wave is bounded by $\hat{y}(s) = D \cos s$.

Figure 18 shows the flow in a cosine wave (Example 7 with $D = 0.7$) over two wavelengths. The lines are contours of the streamfunction (= the streamlines) and of the velocity potential. The figure shows also the auxiliary region G' and the conformal mapping from the disk to G' . This mapping is calculated with the Wegmann method on a grid with $N = 256$ points.

Menikoff and Zemach [176] consider the mapping Φ from the upper half-plane $U := \{w: \text{Im } w > 0\}$ to the region $G := \{z = x + iy: y > \hat{y}(x)\}$ above a line described by an even

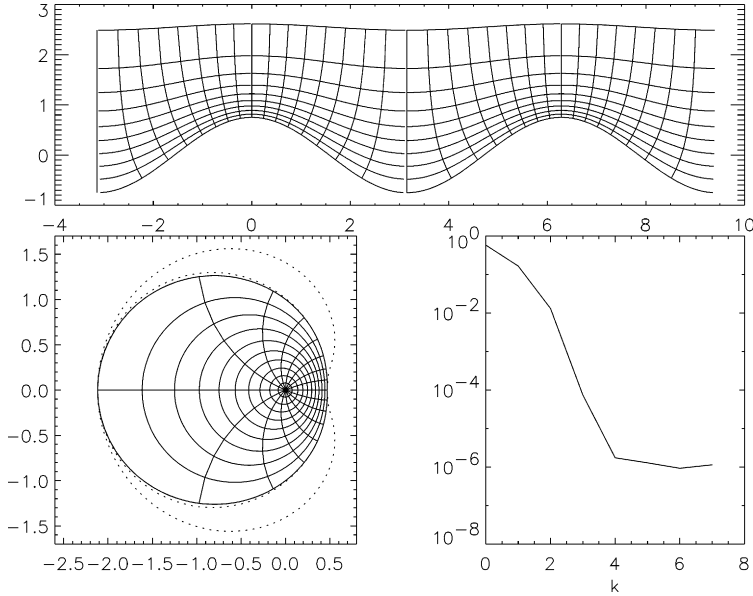


Fig. 18. Upper panel: streamlines and velocity potential for the flow in the cosine wave of Example 7 for $D = 0.7$. Lower left: the conformal mapping from the disk to the auxiliary region G' . Lower right: the δ_k during the iteration.

2π -periodic function $\hat{y}(x)$. Such regions occur in a natural way in the study of water waves and other free surface flows (see Menikoff and Zemach [176] and Meiron et al. [175]). The mapping is standardized by the condition that the interval $[-\pi, \pi]$ corresponds to the boundary part $x + i\hat{y}(x)$, $-\pi \leq x \leq \pi$, of G and $i\infty$ is mapped to $i\infty$. The mapping can be described by a boundary correspondence function $u(x)$ defined by the property $\Phi(u(x)) = x + i\hat{y}(x)$. A nonlinear singular integral equation (which is related to Theodorsen's) for the function $u(x)$ is derived, discretized and solved by a Newton method. The singular integral equation is transformed by integration by parts to an equation with a logarithmic kernel. This form of the equation gives the best results. Gauss quadrature is applied. There occurs severe crowding near boundary points where the function $\hat{y}(x)$ attains a minimum. Zemach [297] gives an approximate formula which allows calculation of $u(x)$ near points of severe crowding, which are characterized by the condition $du/dx \ll 1$. He calculates cosine waves (Example 7) with $D = 100$, a "very hard case", where crowding is of the order $du/dx \approx 10^{-124}$.

7. Mapping from a quadrilateral to a rectangle

A Jordan region Q where on the boundary Γ four distinct points A, B, C, D are prescribed in counterclockwise order, is called a *quadrilateral*. There is a uniquely defined number q such that there is a conformal mapping F of Q to the rectangle $R_q := \{z = x + iy: 0 < x < 1, 0 < y < q\}$ in such a way that the points A, B, C, D are mapped to the corners

$0, 1, 1 + qi, qi$, in this order. The height q of the rectangle is unknown. It must be determined together with F . The quantity $m(Q) := q$, called the *conformal module* of the quadrilateral, is of interest in problems of electric conductivity. When Q is a sheet of metal of specific resistivity 1 and the segments AB and CD are kept at the potentials 0 and 1, respectively, while the remaining segments are insulated, then the sheet Q between the electrodes AB and CD has resistance $m(Q)$ (see, e.g., Henrici [107, p. 431]). Papamichael [207] gave a review of possible applications of the conformal mapping $F: Q \rightarrow R_q$, and of available numerical methods to compute approximations to F and $m(Q)$. Gaier [75] gave a survey of old and new methods to determine the module by direct methods (without computation of the conformal map). Gaier [66] calculated approximations for $m(Q)$ by finite element methods.

Since the mapping from a rectangle to the disk can be expressed explicitly in terms of elliptic integrals of the first kind, the problem of approximating F is solved (at least in theory) when a mapping $\tilde{F}: Q \rightarrow D$ to the disk is known.

A method for the mapping of a rectangle R_q onto a region of the form $G = \{z = x + iy: 0 < x < 1, 0 < y < f(x)\}$ is described by Challis and Burley [26]. The method works for functions f which can be developed into cosine series (the so-called even periodic geometry). The boundary values of the mapping are constructed by a projection method. Underrelaxation with a factor of 0.5 is recommended. Gaier and Papamichael [79] point out that this method is closely related to Garrick's method for mapping an annulus to a doubly-connected region (see Section 11.2).

For some quadrilaterals of a special form one can transform the problem to a mapping of a doubly connected domain onto an annulus (see Papamichael [207, p. 69]). Papamichael et al. [211] compare calculations via an annulus with calculations via the disk. It turns out that the former are preferable in the case of "long" quadrilaterals where crowding prevents a sufficiently exact location of the images of the specified points A, B, C, D .

We have already mentioned in Section 2.4 that crowding may cause problems. Let ϕ_1 be the smallest arc between the images of A, B, C, D on the unit circle. Then it follows from (19) that $\phi_1 \sim 8 \exp(-\pi m(Q)/2)$ for large $m(Q)$ (see, e.g., Papamichael and Stylianopoulos [216, p. 34]). Therefore, for quadrilaterals with large module, the images of the points A, B, C, D can hardly be distinguished on the circle unless the calculation is done with very high precision. For this reason it is sometimes advisable to calculate the mapping F to the rectangle directly.

For the numerical calculation of F one can use the fact that $\operatorname{Re} F$ and $\operatorname{Im} F$ are harmonic functions in Q , which are coupled via the boundary conditions. This is used by Vabishchevich and Pulatov [268] to construct an iterative scheme which requires in each step the solution of a Dirichlet problem for the Laplace equation on Q . A similar approach is used by Seidl and Klose [241] to construct iteratively the inverse mapping $\Phi: R_q \rightarrow Q$. This method takes advantage of the fact that for a rectangle fast Laplace solvers are available.

As shown before, crowding occurs for quadrilaterals with large module $m(Q)$. A way out comes from the observation that the mapping of elongated regions is often "localized" in the sense that a change of the region at one end has very little effect on the mapping function at the other end. This can be exploited by *domain decomposition methods* which approximate the mapping of a rectangle to an elongated region by the mapping of rectan-

gles to suitable subregions. For some geometries explicit exponentially small error bounds are given by Papamichael and Stylianopoulos [214], Laugesen [156] and Falcão et al. [56].

When a “long” quadrilateral Q is decomposed by suitable crosscuts into two or more component quadrilaterals Q_j , the module $m(Q)$ can be well approximated by the sum $\sum m(Q_j)$ (Gaier [77,78], Papamichael and Stylianopoulos [215,216] and Falcão et al. [55]).

8. Mapping of exterior regions

One of the main problems of aerodynamics is the calculation of the flow around an airfoil. This requires in a natural way the conformal mapping from the exterior of the unit circle to the exterior of a curve. Therefore, it is not surprising that many methods have been developed for the calculation of the mapping to such exterior regions.

Let G now be the region exterior to a Jordan curve Γ . Then there exists a unique conformal mapping function Φ from the exterior D^- of the unit circle to G subject to the normalization

$$\Phi(z) = \gamma z + a_0 + O\left(\frac{1}{z}\right) \quad (220)$$

with a constant $\gamma > 0$, called the *capacity* of the boundary curve Γ or the *transfinite diameter* of the region G .

We distinguish sometimes exterior mappings and interior mappings by subscripts “e” and “i”. By applying the mapping R , defined by $R(z) := 1/z$, to both regions G and D^- , the function Φ_e is given by $\Phi_e = R \circ \Phi_i \circ R$, where Φ_i maps D conformally to the bounded region $R(G)$ and satisfies the usual conditions $\Phi_i(0) = 0$ and $\Phi_i'(0) > 0$. Therefore, the mapping of exterior regions can be calculated with the methods for interior regions.

For many of the methods presented so far, there are variants which can be used to calculate the mapping of exterior regions directly. And there are a few methods developed especially for exterior regions.

8.1. Mapping from the exterior region to the exterior of the disk

The mapping $F: G \rightarrow D^-$ is normalized by the condition at infinity

$$F(z) = \frac{1}{\gamma} z + b_0 + O\left(\frac{1}{z}\right) \quad (221)$$

with the capacity $\gamma > 0$. When Γ is a Jordan curve parameterized by a 2π -periodic complex function η , F can be extended to a continuous function on $\overline{G}$ and the boundary values can be described by the (exterior) *boundary correspondence function* $T(s)$ which satisfies

$$F(\eta(s)) = \exp(iT(s)). \quad (222)$$

The counterpart to Gershgorin's equation (76) is the integral equation of Kantorovich–Krylow [132, p. 464]; see also Gaier [65, p. 13], Henrici [107, p. 397],

$$(\mathbf{I} + \mathbf{K}_1)T = \beta(s) := 2 \arg(\eta(s) - \eta(0)). \quad (223)$$

The ansatz (80) as a single layer potential is modified for the exterior problem by (Symm [254], Henrici [107, p. 379])

$$u(z) = -\log \gamma + \frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - \frac{\eta(s)}{z} \right| \sigma(s) ds. \quad (224)$$

The density σ satisfies Symm's exterior equation

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| 1 - \frac{\eta(s)}{\eta(t)} \right| \sigma(s) ds = \log \gamma - \log |\eta(t)|. \quad (225)$$

The density σ is normalized by

$$\frac{1}{2\pi} \int_0^{2\pi} \sigma(s) ds = 1. \quad (226)$$

In view of this, condition (225) can be reduced to the simpler form

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\eta(t) - \eta(s)| \sigma(s) ds = \log \gamma. \quad (227)$$

If the capacity $\gamma \neq 1$, the pair of equations (227) and (226) has the unique solution $\sigma(s) = T'_e(s)$, the derivative of the exterior boundary correspondence function T_e . Equation (227) contains besides the unknown function σ also the unknown parameter γ . This makes the use of Symm's method for exterior regions more complicated than for interior regions. From (227) it becomes clear why the case $\gamma = 1$ is exceptional. The operator with logarithmic kernel then has an eigenvalue 0 with eigenfunction $T'_e(s)$. This affects the uniqueness of the interior equation (81) (see Theorem 12).

Murid et al. [185] carry the approach of Kerzman and Stein [133] over to exterior regions. DeLillo and Elcrat [40] propose an “inverse Timann method” which can be considered as a generalization of the Schwarz–Christoffel method and is also applicable for curved boundaries with corners. Papamichael and Kokkinos [209] study the Ritz method and the Bergman kernel method for the calculation of the conformal mapping F of a region G exterior to a Jordan curve to the unit disk satisfying the constraint $F(\infty) = 0$.

With any n points $z_1, \dots, z_n \in \Gamma$ one can form the *Vandermonde determinant*

$$V(z_1, \dots, z_n) := \prod_{1 \leq j < k \leq n} (z_j - z_k). \quad (228)$$

Points $z_j^{(n)}$, $j = 1, \dots, n$, for which $|V(z_1, \dots, z_n)|$ attains its maximum value V_n are called *Fekete points*. The power $V_n^{2/n(n-1)}$ converges to the transfinite diameter γ from above for $n \rightarrow \infty$. For the polynomials

$$P_n(z) := \prod_{j=1}^n (z - z_j^{(n)}), \quad (229)$$

the n th root $F_n := P_n^{1/n}$ converges to $\gamma F(z)$ for $n \rightarrow \infty$ when the branch of the root is taken so that $F_n(z) = z + \dots$ near ∞ (see Gaier [65, p. 177]). Convergence cannot be uniform in G since all P_n have zeros on Γ . Kleiner [135] gave a pointwise estimate $|\log F_n(z) - \log F(z)| = O(\log n / \sqrt{n})$ for each $z \in G$ (see also Pommerenke [221]). When Fekete points are represented as $z_j^{(n)} = \Phi(\exp(i\theta_{jn}))$ with the conformal mapping $\Phi: D^- \rightarrow G$, then the θ_{jn} are approximately equidistributed in the interval $[0, 2\pi]$ (see Pommerenke [222, 223] for more precise estimates). In [222] there are numerically calculated Fekete points for ellipses and for the square up to $n = 320$.

Fekete points are difficult to determine numerically. Leja [161] defined another system of points which can be recursively calculated.

Menke [177, 178] introduces *stationary point systems* which are determined by certain extremum properties. These points approximate directly the *Fejér points*, i.e., the images $z_j := \Phi(\exp(i(j-1)2\pi/n))$ of equidistant points on the circle. When the mapping $\Phi: D^- \rightarrow G$ is known, the Fejér points can easily be calculated.

Reichel [233] has shown that Fejér points can also be used to calculate the mapping of the interior. Let F_i be the mapping of the interior of Γ to the unit disk with $F_i(0) = 0$, and let P_n be the polynomial of degree $n-1$ which interpolates $F_i(z)/z$ at the n Fejér points z_j then $zP_n(z)$ is an approximation to $F_i(z)$.

Kühnau [150] describes a sort of osculation method which constructs an approximation for the mapping of the exterior of a curved slit S to the exterior of a disk. (Notice that pages 632 and 634 in [150] have to be interchanged.) The slit is subdivided into K adjacent pieces S_i . In the first step the first piece S_1 , which is assumed to be an endpiece, is approximated by a straight line and the exterior of this straight line is mapped by a function f_1 to the exterior of a disk. The image $f_1(S_1 \cup S_2)$ is approximately a circle with an attached circular arc. The exterior is mapped with f_2 to the exterior of a disk, and so on. All mappings are normalized by $f_i(z) = z + O(1/z)$ at infinity. Then the radii of the disks increase and the slit S is finally mapped approximately to a circle. The images $f_i(S)$ of the slits change like a growing tadpole. By an additional step the method can be applied also for the mapping of the exterior or the interior of a Jordan curve. The “zipper method” of Marshall (see Section 3.3) is based on similar ideas.

Homentcovschi [111] calculates an asymptotic expansion for the conformal mapping $F(z, \tau)$ of the exterior of a slender region bounded by the curves $\tau y_{\pm}(x)$ with $|x| \leq 1$ to the plane with a slit $[x_1, x_2]$ on the real axis.

8.2. Mapping from the exterior of the disk to the exterior region

The mapping $\Phi : D^- \rightarrow G$ has the form (220). It is quite sufficient to know the boundary values of Φ , since Φ can be reconstructed by Cauchy's formula

$$\Phi(z) = \gamma z + a_0 - \frac{1}{2\pi} \int_0^{2\pi} \frac{\Phi(e^{it})e^{it}}{e^{it} - z} dt, \quad (230)$$

with the coefficients

$$\gamma = \frac{1}{2\pi} \int_0^{2\pi} \Phi(e^{it})e^{-it} dt, \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(e^{it}) dt. \quad (231)$$

The boundary values can be expressed by an (exterior inverse) boundary correspondence function $S(t)$ which is determined by the same equation (122) as for the interior mapping with the only difference that now Φ is required to be analytic in D^- and subject to the normalization (220) at infinity.

When the boundary is close to a circle, i.e., it has a parameterization (127), then the boundary values of Φ are in first order of τ given by

$$\Phi(e^{it}) = e^{it} (1 + \tau [\rho(t) - i\mathbf{K}\rho(t)]). \quad (232)$$

The function S is in first order of τ :

$$S(t) = t - \tau \mathbf{K}\rho(t). \quad (233)$$

The method of alternating projection (AP) described in Section 4.2 can easily be adapted for the exterior problem. The formulas must be modified in the following way: The iteration starts with a function S_0 such that $S_0(t) - t$ is 2π -periodic. The obvious choice is $S_0(t) \equiv t$.

When S_k is determined for some $k \geq 0$ then the boundary function is Fourier analyzed

$$\eta(S_k(t)) = \sum_{l=-\infty}^{\infty} B_l e^{ilt}. \quad (234)$$

The function

$$f_k(t) = (\operatorname{Re} B_1) e^{it} + \sum_{l=-\infty}^0 B_l e^{ilt} \quad (235)$$

represents the boundary values of an analytic function

$$\Phi_k(z) = (\operatorname{Re} B_1) z + \sum_{l=-\infty}^0 B_l z^l \quad (236)$$

which obviously is of form (220) with the only exception that the coefficient of z is real but not necessarily positive. This is the first projection step. The second step calculates a new parameter mapping by

$$S_{k+1}(t) := S_k(t) - \operatorname{Re} \frac{g_k(t)}{\dot{\eta}(S_k(t))}, \quad (237)$$

with the nonanalytic part of $\eta \circ S_k$

$$g_k(t) = \sum_{l=2}^{\infty} B_l e^{ilt} + i(\operatorname{Im} B_1) e^{it}. \quad (238)$$

This definition is applicable for differentiable curves. The nonanalytic part of $\eta \circ S_k$ is projected onto the tangent at the point $\eta(S_k(t))$ and the shift along the tangent is then replaced by a shift along the curve induced by a shift in the parameter function S . This gives a new boundary correspondence $t \rightarrow \eta(S_{k+1}(t))$. This construction of a boundary mapping is the second step of the alternating projection.

Figure 19 shows the mapping of the exterior of the disk to the exterior of an inverted ellipse (Example 4) with parameter $p = 0.4$ calculated by the AP method with $N = 256$ grid points. We use the measure ε_k of accuracy defined in (146). The measure of analyticity is changed to

$$\alpha_k := \|g_k\|_2 = \left(\sum_{l=2}^n |B_l|^2 + (\operatorname{Im} B_1)^2 \right)^{1/2}. \quad (239)$$

Fil'čakova [58] describes a method of trigonometric interpolation which determines iteratively a mapping function of form (236).

Now let G be a star-shaped region with a boundary curve parameterized by (152). It follows from (220) that the function Ψ defined as in (154) is analytic in D^- and assumes at infinity the real value $\Psi(\infty) = \log \gamma$. The boundary values of Ψ are as in (153). Real and imaginary parts are connected by the operator $\mathbf{K}$ of conjugation which leads to Theodorsen's integral equation for exterior domains

$$S(t) = t - \mathbf{K}[\log \rho(S(t))] \quad (240)$$

which differs from (156) only in the minus sign in front of $\mathbf{K}$. The numerical treatment of (240) is quite analogous to the interior case, described in Section 4.2. In particular, the method of successive conjugation can be applied. It converges when the epsilon condition (159) is satisfied with $\varepsilon < 1$.

The application of Theodorsen's method is limited by the condition that the contour Γ must be close to the circle (the epsilon condition!). Moretti [184] proposed a way to overcome this difficulty by a sort of embedding which is based on the simple observation that, for a suitable α in the range $0 < \alpha < 1$, the contour Γ_1 parameterized by

$$\eta_1(s) := \alpha e^{is} + (1 - \alpha)\eta(s) \quad (241)$$

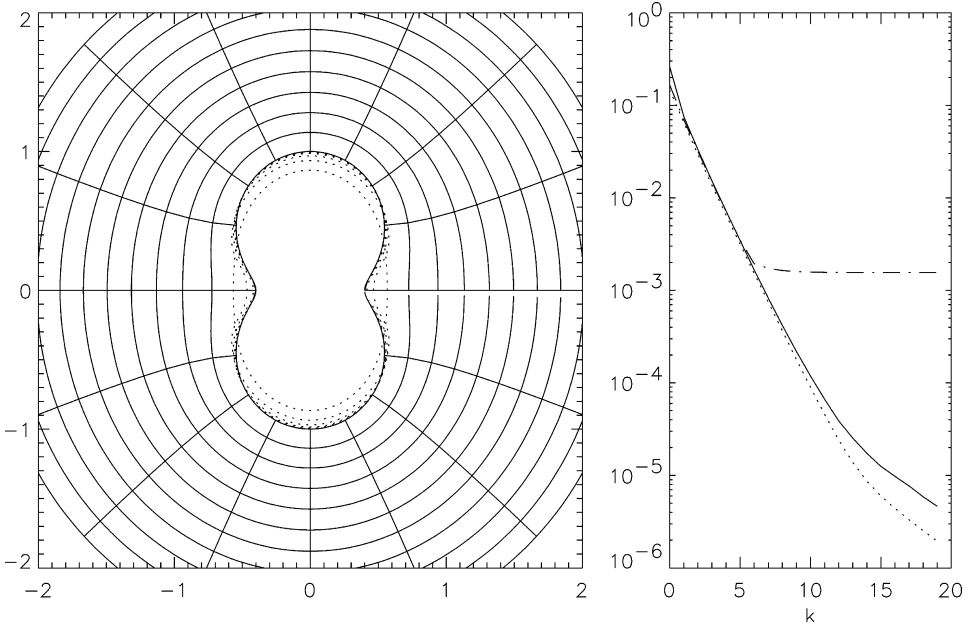


Fig. 19. Mapping from the exterior of the unit disk to the outside of an inverted ellipse with parameter $p = 0.4$ calculated with the AP method. The solid lines show images of 11 concentric circles with radii 1, 1.2, ..., 3. All other quantities as in Figure 6.

satisfies the epsilon condition.

Differentiation of (122) with respect to t yields the equation

$$ie^{it}\Phi'(e^{it}) = \dot{\eta}(S(t))S'(t) \quad (242)$$

which connects the boundary values of the analytic function Φ' with the derivatives of η and S . The function $\log \Phi'$ is analytic in D^- with boundary values

$$\log \Phi'(e^{it}) = \log r(S(t)) + \log S'(t) + i \left[\theta(S(t)) - t - \frac{\pi}{2} \right] \quad (243)$$

with the functions r and θ as defined in (174). The derivative Φ' as well as its logarithm have no $(1/z)$ -term in their Laurent expansion. Therefore, the integral

$$\int_0^{2\pi} \log \Phi'(e^{it}) e^{it} dt = 0 \quad (244)$$

vanishes. It has been noticed by Timman [263] that equation (243) can be used to devise a projection method (see also James [130], Henrici [107] and Gutknecht [95]).

Start from a function S_0 so that $S_0(t) - t$ is 2π -periodic.

If S_k is determined for some $k \geq 0$, calculate v as in (175). In order to take the condition (244) into account, the function w is not calculated by $w = \mathbf{K}v$ as in (176) but by $w = \mathbf{K}^{(1)}v$ with the conjugation operator modified so that it omits the first-order terms. It is defined in terms of the Fourier series (26) by

$$\begin{aligned}\mathbf{K}^{(1)}\phi(s) &= \sum_{l=2}^{\infty} (-iA_l e^{ils} + iA_{-l} e^{-ils}) \\ &= \sum_{l=2}^{\infty} (-b_l \cos ls + a_l \sin ls).\end{aligned}\quad (245)$$

The boundary correspondence function is determined from

$$S_{k+1}(t) := \frac{2\pi}{\int_0^{2\pi} \exp(w(\tau))/r(S_k(\tau)) d\tau} \int_0^t \frac{\exp(w(\tau))}{r(S_k(\tau))} d\tau \quad (246)$$

which is derived from the condition that in view of (243) the function $\exp(w)/r$ should be the derivative of the boundary correspondence function S . Henrici [107, p. 411] recommends the evaluation of the integrals in (246) by means of Fourier series.

If θ is twice continuously differentiable and the curvature θ' is nearly constant and does not change rapidly (i.e., $|\theta''|$ is small), then the iterates S_k converge uniformly to the exact boundary correspondence function S as $k \rightarrow \infty$ (Nasyrov and Fokin [188]). Convergence in L^∞ was proved for nearly circular curves by Kaiser [131] under more restrictive smoothness requirements.

A simple convergence proof can be based on the theorem of Ostrowski (see Ortega and Rheinboldt [204, p. 300]). To this aim one has to show that the iteration operator $S_{k+1} = \mathbf{M}S_k$ defined in (246) is Fréchet differentiable and the derivative $\mathbf{M}'$ has spectral radius < 1 . For the circle $\eta(s) = e^{is}$ one calculates

$$\begin{aligned}\mathbf{M}'(1) &= \mathbf{M}'(\cos s) = \mathbf{M}'(\sin s) = 0, \\ \mathbf{M}'(\cos ls) &= -\frac{1}{l} \cos ls, \quad \mathbf{M}'(\sin ls) = -\frac{1}{l} \sin ls \quad \text{for } l \geq 2.\end{aligned}\quad (247)$$

If the method would be applied with the full conjugation operator $\mathbf{K}$ instead of $\mathbf{K}^{(1)}$, one would get $\mathbf{M}'(\cos s) = -\cos s$ and $\mathbf{M}'(\sin s) = -\sin s$ instead of the first of the equations (247). The spectral radius of $\mathbf{M}'$ would be 1 and the method would not even converge for the unit circle.

It follows from (247) that for nearly circular regions the iteration converges linearly with a rate $q \approx 1/2$ and that it oscillates around the limit function. This behavior suggests a relaxation method with the aim to damp the oscillations and to speed up convergence. Instead of the iteration defined by (246) one iterates functions $\tilde{S}_k$ defined by

$$\tilde{S}_{k+1} := \alpha S_{k+1} + (1 - \alpha) \tilde{S}_k \quad (248)$$

with S_{k+1} defined by (246) and a relaxation parameter α . In view of the convergence rate $1/2$ for the disk, a value of $\alpha = 2/3$ seems to be a good choice. When the conformal mapping to the exterior of the peanut (Example 10) is calculated by Timman's method without relaxation, a convergence rate $q = 0.86$ is observed. Relaxation (248) with $\alpha = 2/3$ decreases the rate to $q = 0.40$.

EXAMPLE 8. Tear is parameterized by

$$\eta(s) = \cos s + 0.2 \cos(2s) + i[0.5 \sin s - 0.2 \sin(2s)]. \quad (249)$$

Figure 20 shows the result of a calculation for the mapping from D^- to the exterior of the tear (Example 8) with Timman's method on a grid with $N = 256$ points. The convergence rate with relaxation ($\alpha = 2/3$) is $q = 0.25$. Without relaxation the rate would be $q = 0.65$.

It must be noted that the only normalization achieved by Timman's method is $S(0) = 0$. The mapping function can have the general form near infinity $\Phi(z) = az + a_0 + O(1/z)$ with complex numbers a and a_0 .

The real part of (242) is up to a constant the conjugate of the imaginary part. When one uses the representation (28) of the operator of conjugation, a relation

$$\log[r(S(t))S'(t)] = \frac{1}{\pi} \int_0^{2\pi} \log \left| \sin \frac{s-t}{2} \right| d\theta(S(t)) + c \quad (250)$$

follows which expresses the ratio $d\sigma/dt = r(S(t))S'(t)$ of arclength $d\sigma$ on the curve to the arclength dt on the unit circle under conformal mapping by the changes $d\theta(S(t))$ in the tangent angle. This formula is applicable also for regions with corners, where the function $\theta(s)$ has jumps. When G is a polygon, (250) expresses $\log \Phi'(e^{it})$ in terms of the angles of the polygon. Thus the Schwarz–Christoffel formula can be easily derived. Equation (250) in its general form is the basis of the generalized Schwarz–Christoffel methods of Floryan and Zemach [59–61], the inverse Timman method of DeLillo and Elcrat [40] and the method of Nieto et al. [192]. With suitable modifications it can also be used for the mapping of interior regions.

One can combine Timman's and Theodorsen's methods in a way which was first studied by Friberg [64]. One can use a parameterization

$$\eta(s) = \rho(s)e^{i\beta(s)} \quad (251)$$

of the boundary curve. It follows from (242) and (122) that

$$\begin{aligned} \log[ie^{it}\Phi'(e^{it})/\Phi(e^{it})] \\ = \log[r(S(t))S'(t)/\rho(S(t))] + i[\theta(S(t)) - \beta(S(t))] \end{aligned} \quad (252)$$

holds with r and θ as defined in (174). Therefore, S' can be reconstructed from the function $\theta - \beta$ (this is the angle of the tangent with the radius vector) by means of conjugation.

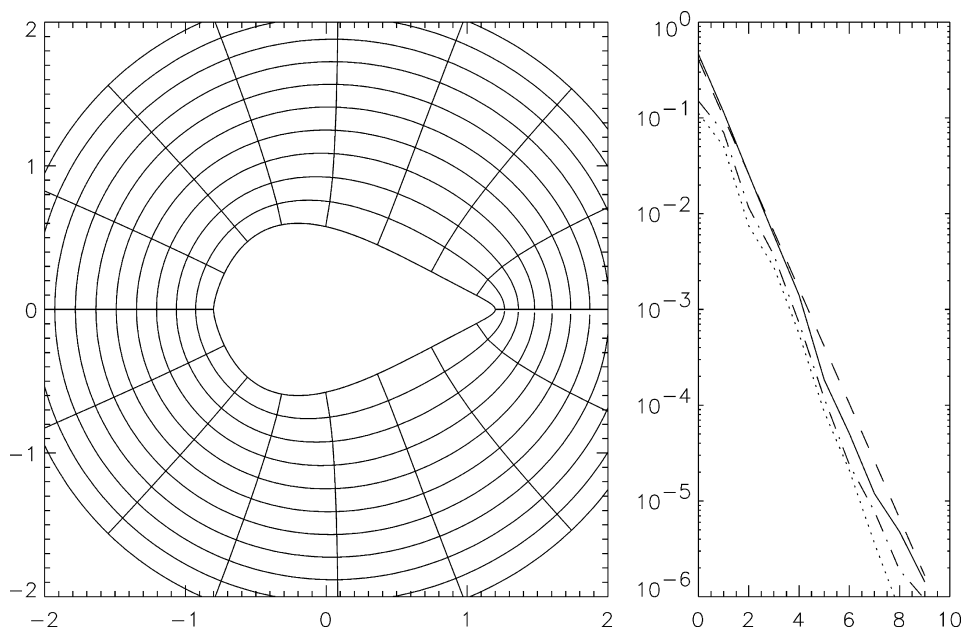


Fig. 20. Mapping from the exterior of the unit disk to the outside of the tear (Example 8) by Timman's method with relaxation parameter $\alpha = 2/3$.

This can be the basis of an iterative method. Start from a function S_0 such that $S_0(t) - t$ is 2π -periodic. For $k \geq 0$ put

$$v(t) := \theta(S_k(t)) - \beta(S_k(t)) \quad \text{and} \quad w = \mathbf{K}v. \quad (253)$$

The next approximation for the boundary correspondence function is then determined from

$$S_{k+1}(t) := c \int_0^t \frac{\rho(S_k(\tau)) \exp(w(\tau))}{r(S_k(\tau))} d\tau \quad (254)$$

with the norming factor

$$c := \frac{2\pi}{\int_0^{2\pi} \rho(S_k(\tau)) \exp(w(\tau)) / r(S_k(\tau)) d\tau}. \quad (255)$$

Since $iz\Phi'(z)/\Phi(z)$ tends to i as $z \rightarrow \infty$, the constant c is equal to 1 for the exact mapping. But during the iteration values of c different from 1 can occur. This must be corrected in order to guarantee that $S_k(t) - t$ is a 2π -periodic function.

The iteration operator is described by equations (253)–(255). Its derivative, evaluated for the circle, is the null-operator. Therefore, the Friberg method converges for regions whose boundary is close to the circle in the same sense as discussed before for the Timman method. Convergence can be arbitrarily fast for curves very close to the circle. The method

does not converge for the peanut of Example 10 when started with $S_0(t) = t$. The iterates oscillate around the true function $S(t)$. Also, in this case relaxation helps. When the method is modified by relaxation with parameter $\alpha = 2/3$ then it converges at a rate $q = 0.59$ for the peanut example.

EXAMPLE 9. Curve is parameterized by (251) with

$$\rho(s) = 2 + 0.3 \cos s + 0.8 \cos(2s), \quad \beta(s) = s - 0.1 \cos s + 0.2 \sin s. \quad (256)$$

The conformal mapping from the disk to the exterior of the curve of Example 9 is calculated by Friberg's method on a grid of 256 points with a relaxation factor of $\alpha = 0.666$. The result is shown in Figure 21. Convergence is linear with a rate of $q = 0.92$. Without relaxation the rate would be $q = 0.94$.

Gutknecht [95] gives a very general framework which covers several of these projection methods. From Φ an *auxiliary function* H is derived. The condition that H must be analytic gives a nonlinear integral equation or integrodifferential equation for the boundary correspondence function which involves the conjugation operator $\mathbf{K}$. It is known a priori that the two functions $\Phi(z)/z$ and $\Phi'(z)$ do not vanish. Therefore, their logarithm is analytic in D^- . Linear combinations of these logarithms give suitable auxiliary functions from which "successive conjugation" methods are derived. The methods of Theodorsen, Timman and Friberg provide illustrative examples.

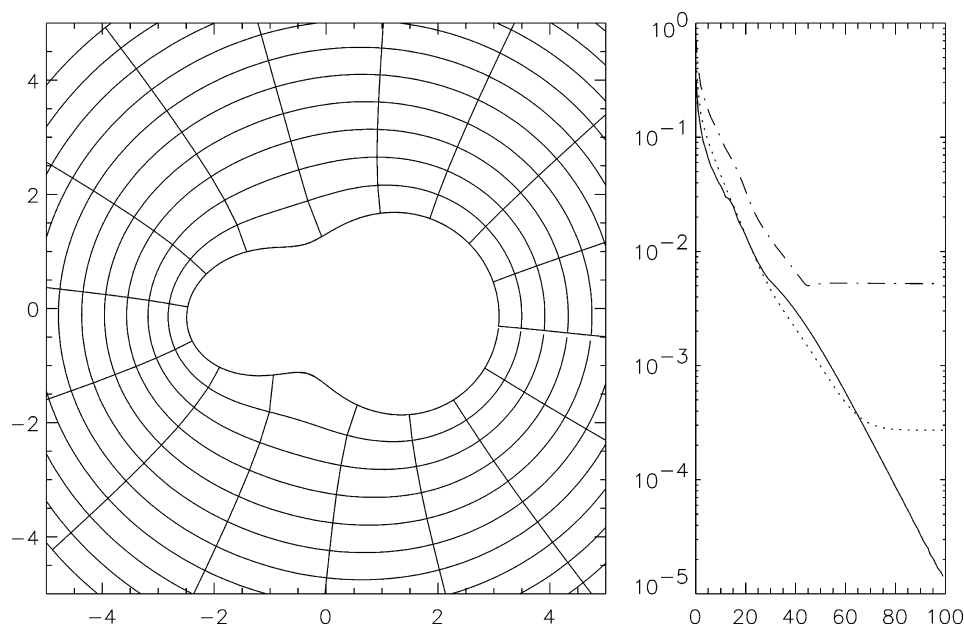


Fig. 21. The mapping from the exterior of a disk to the exterior of the curve of Example 9 calculated with Friberg's method.

Halsey [97] found in numerical experiments that Timman's method converges better than Theodorsen's. He attributes this to the possibility that the numerically calculated approximations to S can become nonmonotone in Theodorsen's method but not in Timman's. It must be noted, however, that all examples, where Halsey finds failure of Theodorsen's method, do not satisfy the epsilon condition with $\varepsilon < 1$. Therefore, a theoretician's explanation of divergence would be that the examples are outside Theodorsen's reach anyway.

Theodorsen's method converges for regions with nearly circular boundary linearly with a rate approximately $\|\rho'\|_\infty$, when the boundary curve is parameterized by (126). Therefore, convergence can be very fast or very slow, depending on ρ . Timman's method, on the other hand converges with a rate around $1/2$. Therefore, each of these methods has its range, where it converges better than the other.

The Newton methods described in Section 4.3 can be readily adapted to the exterior mapping problem. For the Wegmann method described by equations (175) to (179) only the signs in front of the functions w and h have to be changed.

Start with a function $S_0(t)$ such that $S_0(t) - t$ is a 2π -periodic function in W . (The natural choice is $S_0(t) = t$.)

When S_k is determined for some $k \geq 0$ then calculate the functions and numbers

$$v(t) := \theta(S_k(t)) - t = \arg[e^{-it} \dot{\eta}(S_k(t))], \quad (257)$$

$$w := \mathbf{K}v, \quad \alpha := \mathbf{J}v, \quad (258)$$

$$g := \frac{r(S_k(t))}{\exp(w(t))} \operatorname{Im} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (259)$$

$$h := \mathbf{K}g, \quad \gamma := \mathbf{J}g, \quad (260)$$

$$DS(t) = \frac{\exp(w(t))(h(t) - \gamma \cot \alpha)}{r(S_k(t))} - \operatorname{Re} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (261)$$

$$S_{k+1}(t) := S_k(t) + DS(t). \quad (262)$$

EXAMPLE 10. A peanut with boundary parameterization

$$\eta(s) = [1 + 0.5 \cos(2s) + 0.1 \cos(4s)]e^{is}. \quad (263)$$

Figure 22 shows the mapping from D^- to the exterior of a peanut described in Example 10 calculated with the Wegmann method on a grid with 256 points starting from $S_0(t) = t$.

DeLillo and Elcrat [38] performed a series of test calculations for mappings from the exterior of the unit disk to the exterior of ellipses, sportgrounds, inverted ellipses, cosine airfoils, perturbed circles and general spline curves. They compare the methods of Timman, Friberg, Wegmann and Theodorsen with regard to accuracy, computing time and convergence behavior. They come to the conclusion that "Wegmann is the most efficient and most robust. Friberg is superior to Timman for nearly circular domains [...] for extreme regions Timman is preferable if arclength is used as the parameter". They find also that the methods of Timman, Friberg and Wegmann after initial convergence may diverge finally. This is attributed to the term $r(S)$. An indication for this is the observation that this did not occur for Timman's method when parameterization by arclength is used [38, p. 415].

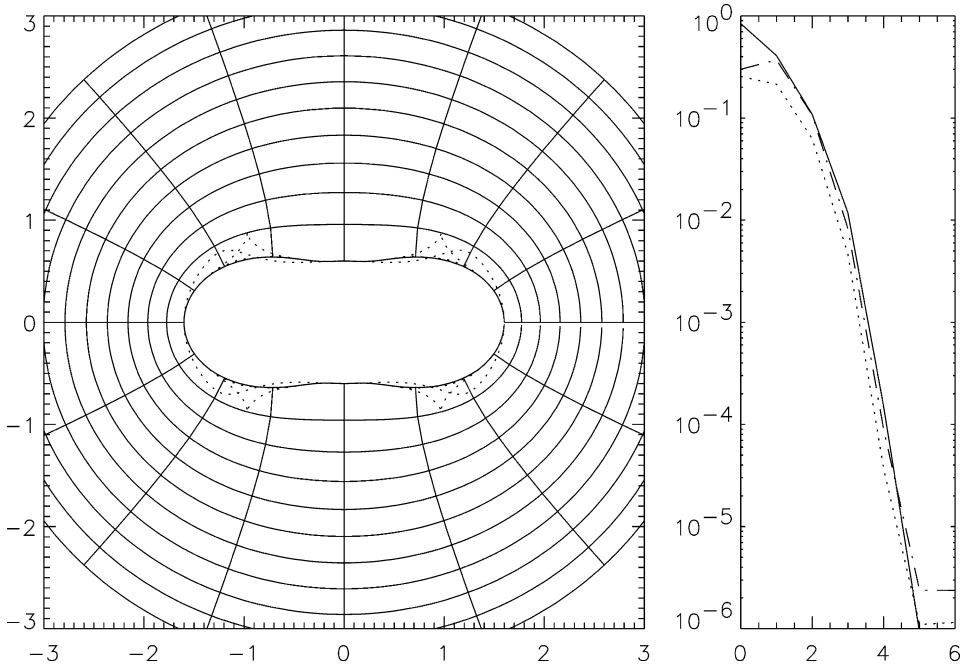


Fig. 22. The mapping from the exterior of a disk to the exterior of the peanut of Example 10 calculated with Wegmann's method.

9. Mapping to Riemann surfaces

The derivative Φ' of a conformal mapping Φ of the disk to a simply connected region does not vanish in D . The differentiated boundary correspondence equation (242) shows that the winding number of $\Phi'(e^{it})$ is connected to the winding number of $\dot{\eta}$ by

$$\text{wind}(\Phi'(e^{it})) = \text{wind}(\dot{\eta}) - 1. \quad (264)$$

Only in the case $\text{wind}(\dot{\eta}) = 1$, the winding number of $\Phi'(e^{it})$ is zero, which ensures by the argument principle that Φ' does not vanish in D . The same holds for exterior mappings: if $\text{wind}(\dot{\eta}) = 1$ then Φ' does not vanish in the exterior D^- of the disk.

When the curve Γ has self intersections, it can still be the boundary of a simply connected region G on a suitable Riemann surface. If $\text{wind}(\dot{\eta}) = 1$, the branch points of the Riemann surface must be outside G . In this case the methods for mapping D (or D^-) to the region, as described in Sections 4 and 8.2, can be applied without modification.

EXAMPLE 11. Figure 8 is parameterized by

$$\eta(s) = \cos s + i[0.5 \sin s + 0.8 \sin(3s)]. \quad (265)$$

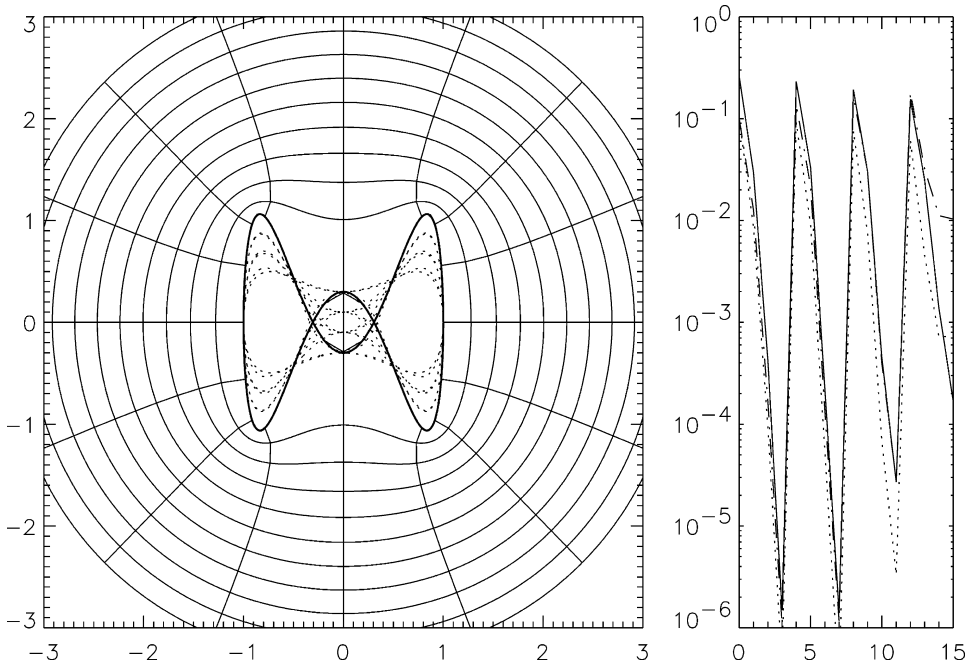


Fig. 23. The mapping from the exterior of a disk to the Riemann surface with boundary curve (265).

The region G outside of the curve of Example 11 covers a neighborhood of the origin twice. Figure 23 shows the conformal mapping from D^- to this region calculated by the Wegmann method on a grid with $N = 2048$ points. The computation is done in combination with a continuation method. The factor d in front of the $\sin(3s)$ term is gradually enlarged in four steps from 0.2, 0.4, 0.6, to 0.8. For each value of d four iterations are performed. Then the value of d is enlarged and the method is started with the result of the last iteration as S_0 . This explains the sawtooth behavior of the changes δ on the right panel of Figure 23. The achievable accuracy deteriorates as the region becomes more and more difficult. Crowding occurs at the flanks of the Figure 8. The wasp tail is only poorly resolved.

Figure 24 shows the image of 16 concentric circles of radii 1 (0.01) 1.15. For greater clarity only the image of the sector re^{it} for $|t + \pi/2| \geq \pi/128$ is shown.

The situation changes when $\text{wind}(\dot{\eta}) = 1 + l$, with $l > 0$ for the interior and $l < 0$ for the exterior mapping problem. It follows from the argument principle that, for each mapping Φ , the derivative Φ' must have l zeros in D or $-l$ zeros in D^- . These zeros correspond to branch points of the Riemann surface where G resides. These branch points lie in the target region G .

If the curve Γ is parameterized by a 2π -periodic function η such that $m := \text{wind}(\dot{\eta}) = 1 + l > 0$, one can still try to satisfy the boundary correspondence equation (122) with a function Φ analytic in D . The derivative Φ' of this function has zeros at l (not necessarily distinct) points $\zeta_1, \dots, \zeta_l \in D$. The curve Γ is then the boundary of a region G which

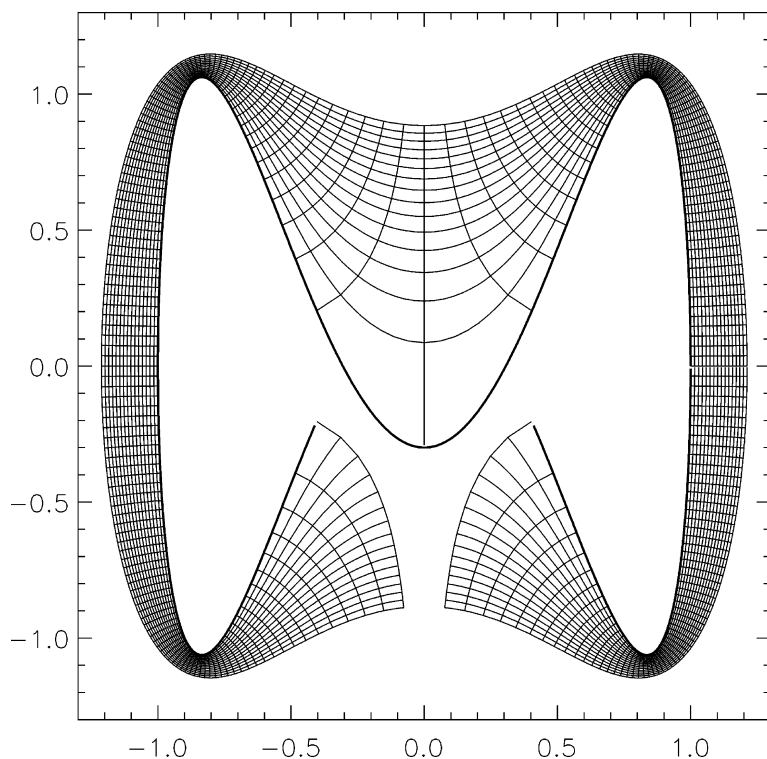


Fig. 24. Detail of Figure 23.

is part of the Riemann surface of the function Φ . The mapping $z \rightarrow \Phi(z)$ from D to G is conformal everywhere with the exception of the zeros ζ_j which are mapped to branch points of G .

The mapping is unique when the values

$$\Phi(z_j) = w_j \quad (266)$$

at m different points $z_j \in D$, $j = 1, \dots, m$, and at a boundary point $z_0 \in \partial D$ are prescribed. The values w_j are in G for $j = 1, \dots, m$ and in Γ for $j = 0$. There are other possibilities. One can prescribe instead the position of the preimages of the branch points, i.e., the zeros $\zeta_1, \dots, \zeta_l$ of Φ' in D .

Wegmann [283] showed that his method (see equations (175) to (180)) can be adapted in the following way.

Let $m := \text{wind } \dot{\eta}$ be a positive number.

Start with $S_0(t)$ such that $S_0(t) - t$ is a 2π -periodic function in W . When S_k is determined for some $k \geq 0$, the change DS is calculated from the condition (171), which is equivalent to the RH problem (172). The RH problem is solved with Theorem 9. One has

to calculate the functions

$$v(t) := \theta(S_k(t)) - mt = \arg[e^{-imt} \dot{\eta}(S_k(t))], \quad (267)$$

$$w := \mathbf{K}v, \quad (268)$$

$$g := r(S_k(t)) \exp(w(t)) \operatorname{Im} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}, \quad (269)$$

$$h := \mathbf{K}g. \quad (270)$$

There are analytic functions Y and $\mathcal{E}$ in D with boundary values

$$Y(e^{it}) = -w + iv, \quad \mathcal{E}(e^{it}) = -h + ig. \quad (271)$$

According to Theorem 9, the general solution of the RH problem (172) is

$$\Psi(z) = z^m \exp(Y(z)) (\mathcal{E}(z) + P_m(z)) \quad (272)$$

with a Laurent polynomial P_m of form

$$P_m(z) = \sum_{j=-m}^m p_j z^j \quad (273)$$

with complex coefficients p_j satisfying

$$p_{-j} = \overline{p_j} \quad \text{for } j = 0, \dots, m. \quad (274)$$

When the calculation is done on a grid of $N = 2n$ points, then the functions Y and $\mathcal{E}$ are obtained as polynomials of degree n . The coefficients of these polynomials are obtained as a by-product of the process of conjugation. Therefore, the values of Y and $\mathcal{E}$ at the points z_j , $j = 0, \dots, m$, can be easily calculated. Using the condition that Ψ must satisfy the interpolation conditions (266), the values of P_m at these points are evaluated as

$$P_m(z_j) = \frac{w_j}{z_j^m \exp(Y(z_j))} - \mathcal{E}(z_j). \quad (275)$$

The condition (274) is equivalent to

$$P_m(1/\overline{z_j}) = \overline{P_m(z_j)}. \quad (276)$$

Equations (275) and (276) yield interpolation conditions at the $2m + 1$ points $z_0, z_1, \dots, z_m, 1/\overline{z_1}, \dots, 1/\overline{z_m}$ for the polynomial $z^m P_m(z)$ of degree $\leq 2m$. Therefore, P_m is uniquely defined by (275) and (276). Inserting (272) into (171) gives the change

$$DS(t) = \frac{-h(t) + P_m(e^{it})}{r(S_k(t)) \exp(w(t))} - \operatorname{Re} \frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}. \quad (277)$$

Note that in view of (274) the function $P_m(e^{it})$ is real.

EXAMPLE 12. Trefoil with boundary parameterization

$$\eta(s) = \cos(2s) + 0.2 \sin s + i[\sin(2s) + 0.4 \cos s]. \tag{278}$$

Figure 25 shows the result of a calculation with this method for the mapping Φ of the unit disk to the trefoil of Example 12 satisfying the interpolation conditions (266) with

$$\begin{aligned} z_0 = 1, \quad z_1 = 0.8i, \quad z_2 = -0.8i, \\ w_0 = 1 + 0.4i, \quad w_1 = w_2 = -0.64. \end{aligned} \tag{279}$$

The calculation is done with 256 grid points.

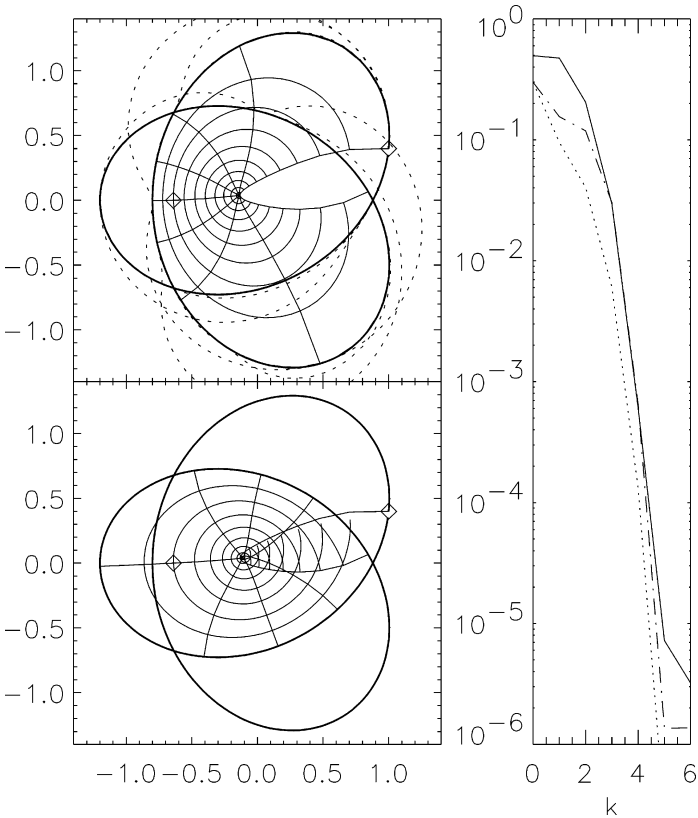


Fig. 25. Mapping from the unit disk to the trefoil of Example 12. The upper (lower) panel shows the image of the upper (lower) semidisc. The interpolation points (279) are indicated by diamonds. The graph on the right shows δ, α, ϵ during the iteration.

10. Mapping of a doubly-connected region to an annulus

For $0 < q < 1$ let A_q be the *circular annulus* $A_q := \{z: q < |z| < 1\}$. It is well known that any doubly-connected region G such that each boundary component consists of more than one point is conformally equivalent to an annulus A_q . The number q is uniquely determined by G . The inverse $M := 1/q$ is called the *module* of G .

When a boundary component of G consists of a single point z_0 , then each conformal mapping of G has a removable singularity at z_0 and can therefore be extended to a conformal mapping of the simply connected region $G \cup \{z_0\}$.

Let G be a doubly-connected region bounded by Jordan curves Γ_1 from outside and Γ_2 inside (see Figure 26). The curves Γ_j are parameterized by 2π -periodic complex functions $\eta_j(s)$. Both curves are orientated in the counterclockwise direction. Let $F: G \rightarrow A_q$ be the conformal mapping which maps Γ_1 to the unit circle and Γ_2 to the circle with radius q . The mapping F is unique up to a rotation $e^{i\alpha}$ with an arbitrary real number α .

10.1. Potential theoretic methods

Symm [255] adapted his method for the mapping of doubly-connected regions in the following way. Assume that 0 is inside the inner boundary Γ_2 . Then the function $H(z) := \log(F(z)/z)$ defined as in (64) is analytic in G . Its real part $u := \operatorname{Re} H$ is a harmonic function in G with boundary values

$$u(\eta_1(s)) = -\log|\eta_1(s)|, \quad u(\eta_2(s)) = \log q - \log|\eta_2(s)| \quad (280)$$

on $\Gamma := \partial G = \Gamma_1 \cup \Gamma_2$. These conditions ensure that $|F| = 1$ on Γ_1 and $|F| = q$ on Γ_2 . The harmonic function u is represented as a single-layer potential with densities σ_1 and σ_2 on the boundary components as in (80):

$$u(z) = -\frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \log|z - \eta_j(s)| \sigma_j(s) ds. \quad (281)$$

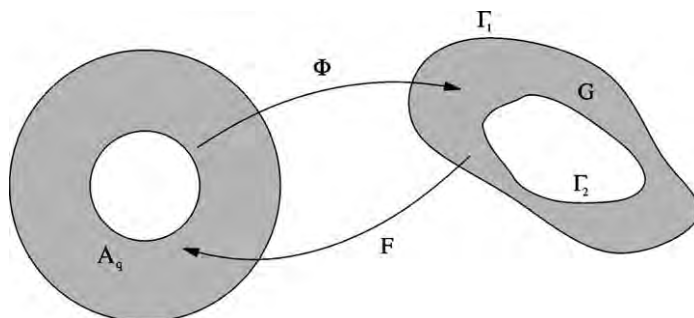


Fig. 26. The conformal mapping F from a doubly-connected region G to an annulus A_q and its inverse Φ .

This gives a pair of integral equations of the first kind:

$$\frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \log|\eta_1(t) - \eta_j(s)| \sigma_j(s) ds = \log|\eta_1(t)| \quad (282)$$

and

$$\frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \log|\eta_2(t) - \eta_j(s)| \sigma_j(s) ds + \log q = \log|\eta_2(t)|, \quad (283)$$

for the densities σ_j which still contain the yet unknown parameter q .

The single-layer potential u is single valued. The conjugate function v , however, which is needed in the representation $H = u + iv$ will in general not be single valued. Let C be a simple closed curve in G which surrounds Γ_2 once in the counterclockwise direction and let τ be arclength on C . Different branches of the (possibly) multivalued function v differ by integer multiples of

$$A := \int_C \frac{\partial v}{\partial \tau} d\tau = \int_C \frac{\partial u}{\partial n} d\tau. \quad (284)$$

The Cauchy–Riemann equations have been used to transform the first integral in (284) to an integral involving the derivative of u with respect to the outer normal n . The function v is single-valued if and only if $A = 0$. The last integral in (284) is proportional to the “charges” inside C . In view of the representation (281) the condition $A = 0$ is equivalent to the condition

$$\int_0^{2\pi} \sigma_2(s) ds = 0. \quad (285)$$

Equations (282), (283) and (285) are coupled integral equations for σ_1, σ_2 and q , which have a unique solution. The conformal mapping is $F(z) = z \exp(H(z))$ with

$$H(z) = -\frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \log(z - \eta_j(s)) \sigma_j(s) ds + i\alpha. \quad (286)$$

Inoue [125,126] proposed a charge simulation method. He investigated theoretically the distribution of charge points and charges and discussed convergence of the approximations to the mapping function.

One can also transfer the integral equations with Neumann kernel to doubly-connected regions (see, e.g., Gaier [65, p. 190]).

10.2. Extremum principles

Let G be a finite doubly connected Jordan region. We assume that 0 is in the hole inside the inner curve Γ_2 . The norm $\|\cdot\|_B$ and the scalar product $(\cdot, \cdot)_B$ for functions f, g on G are defined by the integrals over G as in the simply connected case (see (92) and (93)). The Bergman space $B(G)$ consists of all analytic functions in G for which $\|f\|_B$ is finite and which possess a single-valued indefinite integral in G .

Let F be the conformal mapping of G to an annulus A_q . The derivative

$$H'(z) := \frac{F'(z)}{F(z)} - \frac{1}{z} \quad (287)$$

of the auxiliary function $H(z) = \log(F(z)/z)$ is in $B(G)$. The following theorem characterizes H' by an extremal property (see Gaier [65, p. 249] and Papamichael and Kokkinos [210]).

THEOREM 18. *The unique function $f_0 \in B(G)$, which has minimal norm $\|f\|_B$ among all functions $f \in B(G)$ with the side condition*

$$(f, H')_B = 1, \quad (288)$$

is related to H' by the equation

$$H'(z) = \frac{f_0(z)}{\|f_0\|_B^2}. \quad (289)$$

This characterization seems to be useless at first glance, since the constraint (288) involves the unknown function H' . However, the integral in (288) can be evaluated using Green's formula (100) and the boundary values of H simply by

$$(f, H')_B = i \int_{\partial G} f(z) \log |z| dz. \quad (290)$$

Papamichael and Kokkinos [210] treat the minimization problem of Theorem 18 numerically by a Ritz procedure.

When, on the other hand, an orthonormal set f_j of basis functions in $B(G)$ is given, one can represent H' directly by the formula

$$H'(z) = \sum_{j=1}^n \beta_j f_j(z) \quad \text{with } \beta_j := \overline{(f_j, H')}. \quad (291)$$

The coefficients β_j can be calculated via (290) without knowing H' . This method is called *orthonormalization method* (ONM) (see [210]).

The conformal module M of G can be calculated from H' by

$$2\pi \log M = \int_G \frac{dx dy}{|z|^2} - \|H'\|_B^2 = -i \int_{\partial G} \frac{\log |z|}{z} dz - \|H'\|_B^2 \quad (292)$$

(see Papamichael et al. [212], Gaier [65, p. 250]).

The natural set of basis functions consists of the monomials z^j , $j \neq -1$. It is important to augment this set of basis functions by singular functions which are more suitable to describe the conformal mapping near corners of G . In [210] some results and examples are reported. The method works very well when the region has certain symmetries. The performance for nonsymmetric regions is disappointing. This may be due to the presence of pole type singularities in the complement of G , which cannot be dealt with. Papamichael and Warby [217] investigate pole type singularities of H' for some special geometries. It turns out that the poles occur in pairs at points ζ_1, ζ_2 , where ζ_1 and ζ_2 are symmetric with respect to both curves Γ_1 and Γ_2 .

When the monomials are taken as basis functions the approximation of H' is by rational functions of the form

$$R_{m,n}(z) := \sum_{j=-m, j \neq -1}^n a_j z^j \quad (293)$$

with the term $j = -1$ omitted. When the boundary curves Γ_1, Γ_2 are analytic, the mapping function F and the function H' can be extended analytically into a region containing $\bar{G}$. Papamichael et al. [212] investigate the question of how the portion of positive and negative powers in (293) should be chosen in an optimal way depending on the location of the singularities of the analytic extension of H' nearest to Γ_1 and Γ_2 . They show furthermore, that the approximations $R_{m,n}^*$ and the corresponding approximations to F obtained by the ONM converge uniformly on $\bar{G}$ as $m, n \rightarrow \infty$ with an error proportional to r^{m+n} , for some $r \in (0, 1)$. There are also convergence results for piecewise analytic boundaries similar to the results for Bieberbach polynomials for simply connected regions given in (115). The ONM approximation to the conformal module M is more accurate than the corresponding approximation to the conformal mapping F (Papamichael et al. [212, p. 494]).

Burbea [24] determines the module of a doubly-connected region using a relation of M with the Bergman kernel function.

11. Mapping from an annulus to a doubly-connected region

The conformal mapping $\Phi : A_q \rightarrow G$, the inverse of the mapping F , is uniquely determined up to a rotation of A_q . To fix this ambiguity one can impose the condition

$$\Phi(1) = \eta_1(0). \quad (294)$$

Instead of (294) one can also impose the condition that the coefficient of z in the Laurent expansion of Φ is real, i.e.,

$$\operatorname{Im} \int_0^{2\pi} e^{-it} \Phi(e^{it}) dt = 0. \quad (295)$$

The function Φ is completely determined by its boundary values. These are described by boundary correspondence functions S_1, S_2 , which satisfy the boundary correspondence equations

$$\Phi(e^{it}) = \eta_1(S_1(t)), \quad \Phi(qe^{it}) = \eta_2(S_2(t)). \quad (296)$$

These two equations, together with the norming (294), determine the functions S_1, S_2 and the parameter q . The condition (294) is equivalent to

$$S_1(0) = 0. \quad (297)$$

11.1. Boundary value problems

Conjugation on the annulus A_q is effected by a (real) linear operator $\mathbf{K}_q(\phi_1, \phi_2)$ which is most easily defined in terms of the complex or real Fourier series

$$\phi_j(t) = \sum_{l=-\infty}^{\infty} A_{l,j} e^{ilt} = a_{0,j} + \sum_{l=1}^{\infty} (a_{l,j} \cos lt + b_{l,j} \sin lt) \quad (298)$$

of the functions $\phi_j, j = 1, 2$. Then

$$\mathbf{K}_q(\phi_1, \phi_2)(t) = \sum_{l=-\infty}^{\infty} B_l e^{ilt} = \sum_{l=1}^{\infty} (\alpha_l \cos lt + \beta_l \sin lt) \quad (299)$$

with the coefficients

$$B_0 = 0, \quad B_l := \frac{2iA_{l,2} - (q^{-l} + q^l)iA_{l,1}}{q^{-l} - q^l} \quad \text{for } l \neq 0 \quad (300)$$

and

$$\alpha_l := \frac{2b_{l,2} - (q^{-l} + q^l)b_{l,1}}{q^{-l} - q^l}, \quad \beta_l := -\frac{2a_{l,2} - (q^{-l} + q^l)a_{l,1}}{q^{-l} - q^l} \quad (301)$$

for $l = 1, 2, \dots$. Since $q < 1$, it follows from (300) that for $l \rightarrow \infty$ the coefficients satisfy

$$B_l \approx -iA_{l,1}, \quad B_{-l} \approx iA_{-l,1}. \quad (302)$$

Comparison with (27) shows that the conjugation $\mathbf{K}_q(\phi_1, \phi_2)$ acts on the high-order Fourier-terms like ordinary conjugation $\mathbf{K}\phi_1$ on the unit disk.

The following theorem, analogous to Theorem 8, shows how to construct analytic functions in an annulus with prescribed real part on the two boundary circles.

THEOREM 19. *Let ϕ_1, ϕ_2 be real functions in W . There exists an analytic function Φ in A_q with boundary values*

$$\operatorname{Re} \Phi(e^{it}) = \phi_1(t), \quad \operatorname{Re} \Phi(qe^{it}) = \phi_2(t), \quad (303)$$

if and only if the right-hand sides satisfy

$$\int_0^{2\pi} \phi_1 dt = \int_0^{2\pi} \phi_2 dt. \quad (304)$$

The general solution of (303) can be constructed in terms of the conjugation operator

$$\begin{aligned} \Phi(e^{it}) &= \phi_1(t) + i\mathbf{K}_q(\phi_1, \phi_2)(t) + i\gamma, \\ \Phi(qe^{it}) &= \phi_2(t) - i\mathbf{K}_q(\phi_2, \phi_1)(t) + i\gamma \end{aligned} \quad (305)$$

with an arbitrary real constant γ .

Condition (304) is in contrast to the situation for the disk where a solution of the boundary problem (24) exists for every right-hand side (Theorem 8).

Let λ be a real number. One can consider instead of (303) the more general RH problem

$$\operatorname{Re} \Phi(e^{it}) = \phi_1(t), \quad \operatorname{Re}(e^{i\lambda} \Phi(qe^{it})) = \phi_2(t). \quad (306)$$

When λ is a multiple of π then this problem reduces to (303). In this case it is solvable only if (304) is satisfied. But when λ is not a multiple of π , the problem (306) has a unique solution for every right-hand side. The solution can be expressed by an operator $\mathbf{K}_{q,\lambda}(\phi_1, \phi_2)$ which is quite analogous to $\mathbf{K}_q$ (see Wegmann [285] for details). This means that the RH problem (306) becomes degenerate when $\lambda = 0 \bmod \pi$.

The general RH problem on the annulus

$$\operatorname{Re}(\overline{A_1(t)} \Phi(e^{it})) = \psi_1(t), \quad \operatorname{Re}(\overline{A_2(t)} \Phi(qe^{it})) = \psi_2(t), \quad (307)$$

is formulated with two complex Hölder continuous functions A_1, A_2 . This RH problem can be solved explicitly in terms of the operator $\mathbf{K}_{q,\lambda}$ (Banzuri [7], Wegmann [285]). For regions with connectivity greater than or equal to 2, there are cases where the solvability properties of the RH problem are not only determined by the *index* of the problem. These are the *special classes* discussed by Vekua [269, p. 257]. Unfortunately, just these degenerate problems are needed in the conformal mapping methods discussed below. On the other hand, these degeneracies offer opportunities to determine the accessory parameters such as the modulus of a doubly-connected region.

11.2. Projection

Fornberg [63] proposed a projection method. We adapt it slightly so that the similarity with the method for the simply connected regions (see (133)–(136)) becomes clear.

The iteration starts with functions $S_{0,1}$ and $S_{0,2}$ such that $S_{0,j}(t) - t$ are 2π -periodic for $j = 1, 2$. When the functions $S_{k,j}$ are determined for some $k \geq 0$, then Fourier analysis of the boundary functions

$$\eta_j(S_{k,j}(t)) = \sum_{l=-\infty}^{\infty} B_l^{(j)} e^{ilt} \quad (308)$$

gives the coefficients $B_l^{(j)}$.

When the mapping function Φ has Laurent expansion

$$\Phi(z) = \sum_{l=-\infty}^{\infty} B_l z^l \quad (309)$$

then the solution

$$B_l = B_l^{(1)} = q^{-l} B_l^{(2)} \quad (310)$$

holds for all l . The strategy is to change $S_{k,1}$ in such a way that the relation (310) is approximately satisfied for $l \leq 0$, and to change $S_{k,2}$ so that (310) is improved for $l \geq 1$. The normalization (295) is implemented into the change of $S_{k,1}$. This leads to a prescription as follows. We assume that the functions η_j are differentiable with derivative $\dot{\eta}_j \neq 0$.

First form, with the coefficients $B_l^{(j)}$ from (308), the functions

$$g_1(t) = \sum_{l=-\infty}^0 (B_l^{(1)} - q_k^{-l} B_l^{(2)}) e^{ilt} + i \operatorname{Im} B_1^{(1)} e^{it}, \quad (311)$$

$$g_2(t) = i \operatorname{Im} B_1^{(2)} e^{it} + \sum_{l=2}^{\infty} (B_l^{(2)} - q_k^l B_l^{(1)}) e^{ilt}. \quad (312)$$

Then new approximations for the boundary correspondence functions are calculated by

$$S_{k+1,j}(t) := S_{k,j}(t) - \operatorname{Re} \frac{g_j(t)}{\dot{\eta}_j(S_{k,j}(t))}. \quad (313)$$

A new estimate for q is calculated by

$$q_{k+1} = \frac{|B_1^{(2)}| + |B_{-1}^{(1)}|}{|B_1^{(1)}| + |B_{-1}^{(2)}|}. \quad (314)$$

The boundary values of the conformal mapping can be approximated in the k th iteration by the functions f_1 and f_2 on the outer and inner contour as follows:

$$f_1(t) = \sum_{l=-\infty}^{-1} q_k^{-l} B_l^{(2)} e^{ilt} + B_0^{(1)} + \operatorname{Re} B_1^{(1)} e^{it} + \sum_{l=2}^{\infty} B_l^{(1)} e^{ilt}, \quad (315)$$

$$f_2(t) = \sum_{l=-\infty}^0 B_l^{(2)} e^{ilt} + \operatorname{Re} B_1^{(2)} e^{it} + \sum_{l=2}^{\infty} q_k^l B_l^{(1)} e^{ilt}. \quad (316)$$

Fornberg [63] gives more details about the most convenient organization of the computation. He gives also a graphical illustration of how the change in the S_j is determined from the error functions g_j . Fornberg changes S_1 and S_2 at the same time, as in formulas (313). But one can use instead a “Einzelschritt-iteration”. Then one determines the new $S_{k+1,1}$ first and then uses $S_{k+1,1}$ to calculate new $B_l^{(1)}$ via (308), before g_2 in (312) and $S_{k+1,2}$ in (313) are evaluated.

EXAMPLE 13. Region is bounded by two ellipses (see Figure 27)

$$\begin{aligned} \eta_1(s) &= \cos s + 0.2 \sin s + 0.7i \sin s, \\ \eta_2(s) &= -0.1 + 0.3 \cos s + i(-0.05 + 0.4 \sin s). \end{aligned} \quad (317)$$

The calculation for Example 13 is started with $S_{0,j}(t) = t$ and $q_0 = 0.5$. The calculated inverse module is $q = 0.46300$. The iteration converges linearly with a rate of 0.90. On each boundary component 128 grid points have been used. The approximation for q converges much faster than the iteration for the S_j . In general, the calculated q is more accurate than the boundary correspondence functions.

Garrick [81] carried over the method of Theodorsen to doubly-connected regions. Let G be bounded by two star-shaped curves

$$\eta_1(s) = \rho_1(s)e^{is}, \quad \eta_2(s) = \rho_2(s)e^{is} \quad (318)$$

with $0 < \rho_2(s) < \rho_1(s)$. The boundary correspondence equation for the auxiliary function $\Psi(z) := \log(\Phi(z)/z)$ can be written in the form

$$\Psi(e^{it}) = \log \rho_1(S_1(t)) + i(S_1(t) - t), \quad (319)$$

$$\Psi(qe^{it}) = \log \rho_2(S_2(t)) - \log q + i(S_2(t) - t). \quad (320)$$

Since Ψ is analytic in the annulus A_q (with unknown q), the real and imaginary parts of Ψ on the boundary circles are connected by the operator of conjugation. This leads to the pair of integral equations of Theodorsen–Garrick

$$S_1(t) = t + \mathbf{K}_q(\psi_1, \psi_2), \quad S_2(t) = t - \mathbf{K}_q(\psi_2, \psi_1), \quad (321)$$

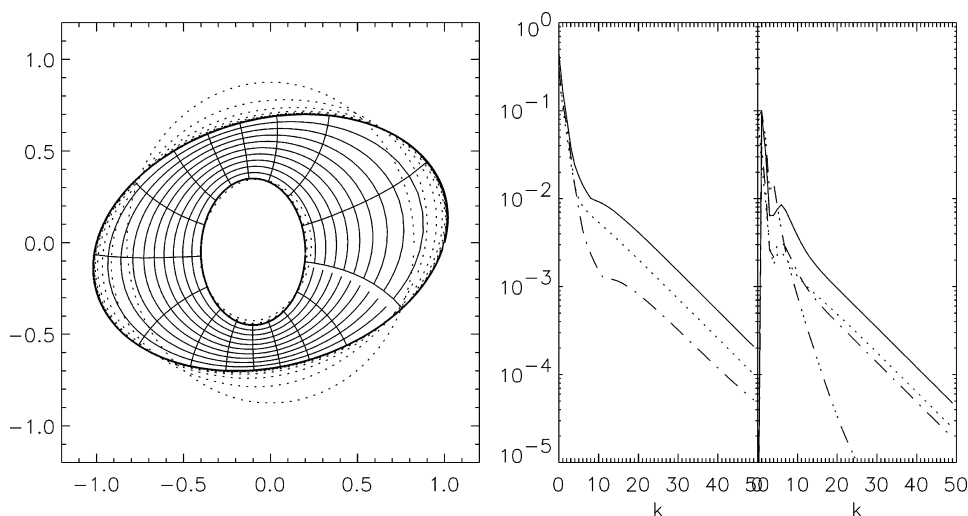


Fig. 27. Mapping from an annulus to the region of Example 13 calculated with the projection method. The left panel shows the images of 11 concentric circles and 16 spokes in A_q . The approximations f_1, f_2 according to (315) and (316) are dotted. The right panel shows the maximal changes δ_k of S_j in each iterative step and the L_2 norm of g_j (dotted) and the error ε_k (dash-dotted) on the outer (left) and inner contour (right). The long dash-dotted curve in the right panel shows the change in q in each step.

with

$$\psi_1(t) = \log \rho_1(S_1(t)), \quad \psi_2(t) = \log \rho_2(S_2(t)). \quad (322)$$

Equations (321) imply that

$$\int_0^{2\pi} (S_1(t) - t) dt = \int_0^{2\pi} (S_2(t) - t) dt = 0. \quad (323)$$

Hence the normalization (295) is satisfied. From $\int \operatorname{Re} \Psi(e^{it}) dt = \int \operatorname{Re} \Psi(qe^{it}) dt$ and equations (319) and (320) the representation

$$\log q = \frac{1}{2\pi} \int_0^{2\pi} (\log \rho_2(S_2(t)) - \log \rho_1(S_1(t))) dt \quad (324)$$

for q follows. This equation means that the module $1/q$ is the ratio of the geometric means of $\rho_1(S_1)$ and $\rho_2(S_2)$.

The system (321) of nonlinear equations can be solved by iteration (see Gaier [65, p. 202]). Start with a guess $q_0 \in (0, 1)$ for q and with functions $S_{0,j}(t)$ such that $S_{0,j}(t) - t$ are 2π -periodic and in W . When functions $S_{k,j}$ and a number $q_k \in (0, 1)$ are determined for some $k \geq 0$, then insert these into the right-hand side of (321) to calculate the new iterates $S_{k+1,j}$ and then use these to calculate a new value q_{k+1} by means of (324). The

iteration converges under conditions which roughly mean that G is already close to an annulus (see Gaier [65, p. 200] and Ostrowski [206] for more detail).

The discretized form of the equations (321) has been studied by Hammerschick [98]. He showed that under certain conditions on the curve there is a unique solution and the iterative method of solution converges.

EXAMPLE 14. A region bounded by two star-shaped curves is parameterized as in (318) with

$$\rho_1(s) = 1 + 0.5 \sin s, \quad \rho_2(s) = 0.5 + 0.2 \cos(2s). \quad (325)$$

Example 14 calculated on a grid with 256 points with the method of Theodorsen–Garrick is shown in Figure 28. The iteration converged with a rate of 0.82. The calculated inverse module is $q = 0.640174$.

Garrick expressed conjugation by integral operators (see Gaier [65, p. 194]). Calculations became much more efficient when the representation in terms of Fourier series (299) was used in combination with FFT (Ives [129]). The method of Garrick is described in detail by Gaier and Papamichael [79].

11.3. The Newton method

The Newton method for the equations (296) starts from the linearized equations

$$\Psi(e^{it}) = \eta_1(S_1(t)) + \dot{\eta}_1(S_1(t))DS_1(t), \quad (326)$$

$$\Psi(qe^{it}) + e^{it}\Phi'(qe^{it})Dq = \eta_2(S_2(t)) + \dot{\eta}_2(S_2(t))DS_2(t). \quad (327)$$

Since the functions DS_j are real, these equations can be transformed to an RH problem for the analytic function $\Psi := \Phi + D\Phi$ in A_q . The complication arises that this RH problem changes its character according to whether the number λ defined in (332) below is zero or not. One can incorporate the side condition (294) only when the RH problem becomes degenerate, i.e., in the case $\lambda = 0$. Therefore, in order to avoid this difficulty, the equation (327) is replaced by

$$\Psi(qe^{it}) + e^{it}\Phi'(qe^{it})Dq = \eta_2(S_2(t)) + e^{-i\lambda}\dot{\eta}_2(S_2(t))DS_2(t). \quad (328)$$

This leads to the following procedure. We follow here the approach of Lucchini and Manzo [168] which is simpler than that of Wegmann [285]. The method is analogous to the method for simply-connected regions as described in Section 4.3.

Assume that the functions $\eta_j(s)$ are differentiable with Lipschitz continuous derivatives $\dot{\eta}_j \neq 0$. These can be represented by

$$\dot{\eta}_j(s) = r_j(s)e^{i\theta_j(s)} \quad (329)$$

with Lipschitz continuous functions θ_j and $r_j > 0$.

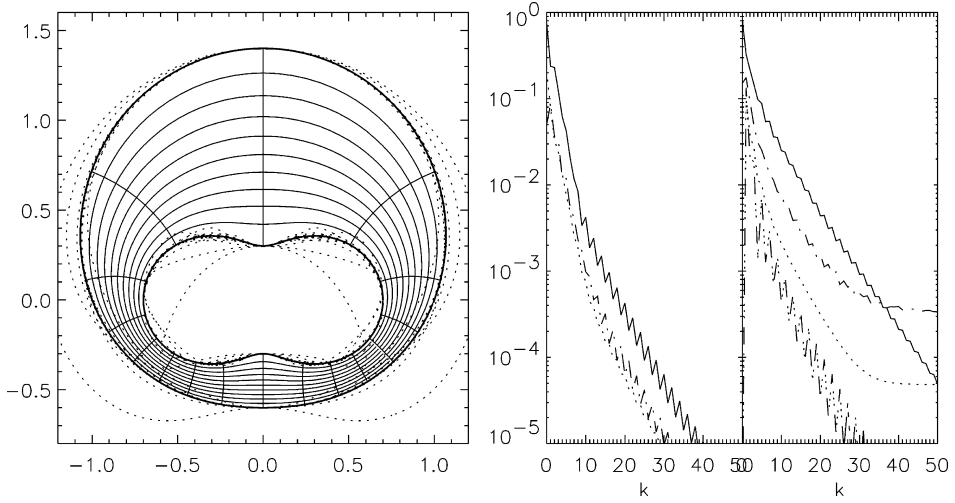


Fig. 28. Mapping from an annulus to the region of Example 14 calculated with the Theodorsen–Garrick iterative method. Meaning of the lines as in Figure 27.

Start with a guess $q_0 \in (0, 1)$ for q and with functions $S_{0,j}(t)$ such that $S_{0,j}(t) - t$ are 2π -periodic functions in W .

When functions $S_{k,j}$ and a number $q_k \in (0, 1)$ are determined for some $k \geq 0$, then calculate functions and numbers as follows. We put $p := q_k$ for brevity. The functions θ_j , r_j , and η_j are all evaluated at $S_{k,j}$.

The iteration from k to $k + 1$ is in several steps. First put

$$v_j(t) := \theta_j - t, \quad j = 1, 2, \quad (330)$$

$$w_1 := \mathbf{K}_p(v_1, v_2), \quad w_2 := -\mathbf{K}_p(v_2, v_1), \quad (331)$$

$$\lambda := \mathbf{J}(v_2) - \mathbf{J}(v_1). \quad (332)$$

Then calculate the functions

$$f_1 := \frac{r_1 e^{w_1} \eta_1}{\dot{\eta}_1}, \quad f_2 := \frac{r_2 e^{w_2} e^{i\lambda} \eta_2}{p \dot{\eta}_2}, \quad (333)$$

and put

$$g_j := \operatorname{Im} f_j, \quad j = 1, 2, \quad (334)$$

$$h_1 := \mathbf{K}_p(g_1, g_2), \quad h_2 := -\mathbf{K}_p(g_2, g_1), \quad (335)$$

$$\alpha := -h_1(0) - \operatorname{Re} f_1(0) + r_1(S_{k,1}(0)) \exp(-w_1(0)) S_{k,1}(0). \quad (336)$$

This gives finally the changes

$$DS_1(t) = -\frac{\exp(w_1(t))}{r_1}(h_1(t) + \operatorname{Re} f_1(t) + \alpha) \quad (337)$$

and

$$DS_2(t) = -\frac{p \exp(w_2(t))}{r_2}(h_2(t) + \operatorname{Re} f_2(t) + \alpha), \quad (338)$$

and the new functions

$$S_{k+1,j}(t) := S_{k,j}(t) + DS_j(t), \quad j = 1, 2. \quad (339)$$

The constant term α in (338) calculated by (336) enforces $S_{k+1,1}(0) = 0$. The next iteration satisfies the normalization (294).

The change of q is calculated from the term $e^{it}\Phi'(qe^{it})Dq$ in equation (328) which has not yet been taken care of. Differentiation of the boundary correspondence equations (296) yields for the conformal mapping

$$ie^{it}\Phi'(e^{it}) = \dot{\eta}_1(S_1(t))S_1'(t), \quad iqe^{it}\Phi'(qe^{it}) = \dot{\eta}_2(S_2(t))S_2'(t). \quad (340)$$

Comparison with (330) shows that

$$\arg(i\Phi'(e^{it})) = v_1(t), \quad \arg(iq\Phi'(qe^{it})) = v_2(t), \quad (341)$$

and with (331),

$$\begin{aligned} i\Phi'(e^{it}) &= C \exp(-w_1(t) + iv_1(t)), \\ i\Phi'(qe^{it}) &= C \exp(-w_2(t) + iv_2(t)) \end{aligned} \quad (342)$$

with a constant $C > 0$. The function $\Phi(e^{it})$ parameterizes the outer curve Γ_1 . Therefore, $L_1 := \int |\Phi'(e^{it})| dt$ is the length of Γ_1 . Integrating the modulus in (342) on both sides yields

$$C = \frac{L_1}{\int e^{-w_1} dt}. \quad (343)$$

The term $e^{it}\Phi'(qe^{it})Dq$ in (327) gives the following contribution to g_2 in (334):

$$-\operatorname{Im} \frac{\Phi'(qe^{it})Dq}{q \exp(-w_2 + iv_2)} = \frac{CDq}{q}. \quad (344)$$

The solvability condition (304) for the conjugation problem

$$\operatorname{Im} \mathcal{E}(e^{it}) = g_1(t), \quad \operatorname{Im} \mathcal{E}(qe^{it}) = g_2(t) + \frac{CDq}{q} \quad (345)$$

yields for the constant term the condition $CDq/q = \mathbf{J}(g_1) - \mathbf{J}(g_2)$. This gives the following prescription for the update of q

$$q_{k+1} := q_k \left(1 + (\mathbf{J}(g_1) - \mathbf{J}(g_2)) \frac{\int \exp(-w_1(t)) dt}{L_1} \right). \quad (346)$$

EXAMPLE 15. Region is bounded by two ellipses

$$\begin{aligned} \eta_1(s) &= \cos s + 0.7i \sin s, \\ \eta_2(s) &= -0.2 + 0.3 \cos s + i(-0.1 + 0.5 \sin s). \end{aligned} \quad (347)$$

Figure 29 shows the result of the calculation of the mapping from an annulus to the region of Example 15 by the Newton method as described before. The calculation is done on a grid of 256 points. The iteration converges quadratically and reaches in the fifth step a stationary state. The calculated inverse module is $q = 0.593051$.

DeLillo and Pfaltzgraff [44] describe a quadratically convergent Fornberg-type method. They discretize the linearized boundary correspondence equations (326) and (327) and solve the system of equations by a conjugate gradient method. This approach is very efficient for the same reasons as described in Section 4.4 for the Fornberg method.

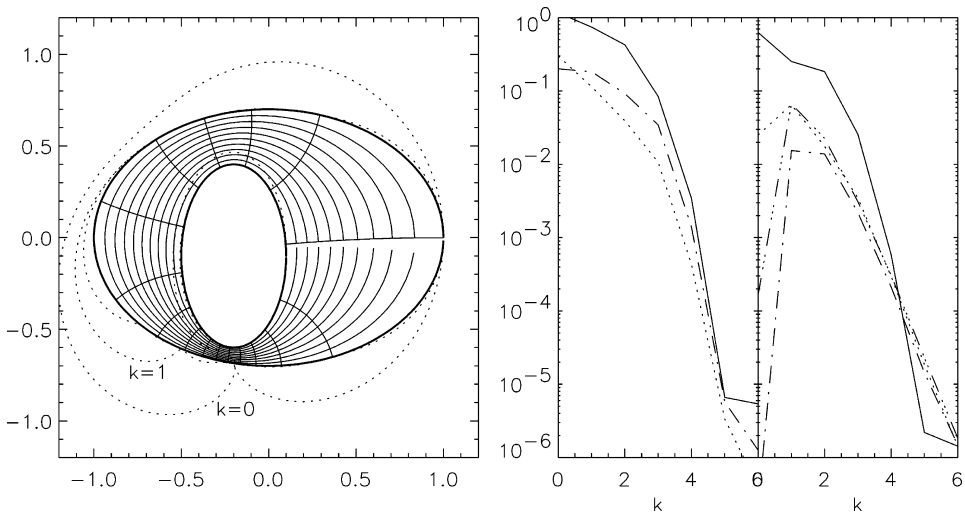


Fig. 29. Mapping from an annulus to the region of Example 15 calculated with the Newton iterative method. Meaning of the lines as in Figure 27.

11.4. Other methods

When the region G is already close to an annulus one can calculate first-order approximations similar to those explained in Section 4.1. When the boundary curves are parameterized by

$$\eta_1(s) = (1 + \tau\rho_1(s))e^{is}, \quad \eta_2(s) = p(1 + \tau\rho_2(s))e^{is}, \quad (348)$$

then the boundary correspondence functions are approximated by

$$S_1(t) \approx t + \tau \mathbf{K}_p(\rho_1, \rho_2), \quad S_2(t) \approx t - \tau \mathbf{K}_p(\rho_2, \rho_1). \quad (349)$$

An estimate for the inverse module is given by

$$q \approx p(1 + \tau \mathbf{J}(\rho_2 - \rho_1)). \quad (350)$$

Rabinovich et al. [228,230] use this principle in a simplified form (based on the simple operator $\mathbf{K}$ of conjugation only) to make the region G iteratively more and more annular.

Menke [179,180] considers regions G bounded by the unit circle from outside and by a Jordan curve Γ_2 from inside such that 0 is not in the closure $\bar{G}$. He defines for each natural number n “extremal point systems” z_j on Γ_2 and obtains from these point systems estimates for the inverse module q . An approximation to the conformal mapping $\Phi: A_q \rightarrow G$ is obtained by the Laurent polynomial which takes on equidistributed points on the circles $|z| = q$ and $|z| = 1/q$, the values z_j and their mirror images $1/\bar{z}_j$, respectively.

Hoidn [109] considered an osculation method for the construction of the mapping of G to an annulus. His experiences from a number of experiments are similar to those reported in Section 3.3 for the simply connected case. The method converges asymptotically slow but is fast at the beginning. It works especially well for regions that are known to be difficult for other methods, including regions with corners and slits.

Komatu [144] developed a method for the mapping of a doubly-connected region to an annulus which is closely related to the Koebe method which will be described in Section 12.2.

Opfer [197] computes approximations to the module M of a ring domain by calculating the harmonic measure with difference schemes. Mizumoto and Hara [182] determine M by finite element methods using characterizations of M by extremum properties of the Dirichlet integral.

12. Multiply-connected regions

We consider multiply-connected regions of the following kind: The boundary of an unbounded region G of connectivity $m \geq 1$ consists of m Jordan curves Γ_j , $j = 1, \dots, m$. The boundary curves are parameterized by complex 2π -periodic functions $\eta_j(s)$. The orientation is clockwise, so that the region is to the left of the curves. The point ∞ is an inner point of G when considered as a subregion of the closed plane $\bar{\mathbb{C}}$ (see Example 18).

The boundary of a bounded region G of connectivity $m + 1 \geq 1$ consists of $m + 1$ Jordan curves Γ_j , $j = 0, 1, \dots, m$. The outer boundary curve Γ_0 is oriented counterclockwise, the inner boundaries Γ_j , $j = 1, \dots, m$, are oriented clockwise, so that also in this case the region is to the left of the curves (see Example 19).

A region H whose boundary consists only of circles C_j is called a *circular region*. The boundary circles C_j of H with centers z_j and radii R_j are parameterized by

$$\zeta_j(t) = z_j + R_j e^{-it} \quad \text{for } j = 1, \dots, m. \quad (351)$$

For a bounded region the outer circle is given by

$$\zeta_0(t) = z_0 + R_0 e^{it}. \quad (352)$$

The orientation of the circles, determined by the sign in the exponential in (351) and (352) is so that H is always to the left of C_j . For an unbounded H all circles are negatively oriented. When H is bounded, then there is a positively oriented outer circle C_0 .

It was shown by Koebe [138] that a conformal mapping $\Psi: H_1 \rightarrow H_2$ of a circular region H_1 to another circular region H_2 must be a Moebius transformation $\Psi(z) = (a + bz)/(c + dz)$. In a later paper [139] Koebe proved that, for each m -connected region G , there exists a circular region H with m boundary circles and a conformal mapping Φ from H to G . The circular region H as well as the mapping Φ are uniquely determined by G when normalization conditions are applied according to the following theorem.

THEOREM 20. (a) *For an unbounded region G with m holes there exists an unbounded circular region H and a conformal mapping $\Phi: H \rightarrow G$. Both H and Φ are uniquely determined by the region G when the hydrodynamic normalization*

$$\Phi(z) = z + O\left(\frac{1}{z}\right) \quad (353)$$

near ∞ is imposed.

(b) *For a bounded region G with m holes there exists a bounded circular region H and a conformal mapping $\Phi: H \rightarrow G$. Both H and Φ are uniquely determined by the region G when the normalization conditions are imposed:*

1. *The outer boundary C_0 of H is the unit circle.*
2. *Φ interpolates η_0 at three given points of the outer boundary, i.e.,*

$$\Phi(\zeta_0(t_j)) = \eta_0(s_j) \quad (354)$$

for prescribed parameter values $0 \leq t_1 < t_2 < t_3 < 2\pi$ and $s_1 < s_2 < s_3 < s_1 + 2\pi$.

There are other canonical regions: Each m -connected region can be conformally mapped to an unbounded region with m parallel slits. Such regions are particularly convenient for flow problems. The slits can also be radial, on concentric circles or on logarithmic spirals (see Nehari [189, Chapter VII], Courant [32, Chapter II] or Grunsky [89, Chapter 3]). One

can also require that the slits form preassigned angles α_j with the real axis (see Koebe [141] and Figure 31). A comprehensive review of results about the conformal mapping of multiply-connected regions to suitable canonical regions was given by Gaier [69].

12.1. Potential theoretic methods

Let G be a bounded region of connectivity $m + 1$ with boundary curves $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ and $0 \in G$. Then there exists a conformal mapping F of G into the unit disk D in such a way that Γ_0 is mapped to the unit circle, and the $\Gamma_j, j = 1, \dots, m$, are mapped to circular slits of radii R_j . The mapping is unique when the conditions (61) are imposed.

The function $H = \log(F(z)/z)$ is analytic in G . The real part is a harmonic function in G with boundary values

$$u(\eta) = \log R_j - \log |\eta| \quad \text{for } \eta \in \Gamma_j. \quad (355)$$

Only $R_0 = 1$ is known in advance, all other $R_j, j = 1, \dots, m$, must be determined from the additional condition that u is the real part of a univalent analytic function in G .

The boundary value problem (355) is a special case of an RH problem, the “problem D” of Vekua [269]. It has a unique solution consisting of a harmonic function u in G and $R_1, \dots, R_m$ (see [269, p. 264]).

If the point 0 is not in G but inside one of the inner curves, say inside Γ_1 , then one can consider a conformal mapping F of G to an annulus A_q with $m - 1$ concentric circular slits, such that Γ_0 is mapped to the unit circle, and Γ_1 to the circle with radius q . The mapping is then uniquely defined up to a factor $e^{i\alpha}$ of unit modulus (see Ellacott [51]). When $H = \log(F(z)/z)$ is formed with this mapping function F , its real part u is described by the same Dirichlet problem (355) as before, with $R_1 = q$.

One can make an ansatz for the harmonic function u as a single-layer potential and arrive at a system of integral equations with logarithmic kernel for the density σ which generalizes Symm’s equations for simply and doubly-connected regions (see Sections 3.1 and 10.1). Gaier [70] studies these equations for the case of the mapping to an annulus with circular slits. He shows that the equations have a unique solution if and only if the capacity of the outer curve Γ_0 is different from 1. The density σ is closely related to the derivative of the boundary correspondence functions in analogy to Theorem 12. The integral equations contain the radii R_j as unknown parameters. One has to impose the additional equations

$$\int_0^{2\pi} \sigma_j(s) ds = 0 \quad (356)$$

for $j = 1, \dots, m$, where σ_j is the density σ restricted to Γ_j . These conditions are derived in the same way as for the doubly-connected case (see equation (285)). One can also impose condition (356) for $j = 0$ and consider R_0 as free parameter. Then R_0 is the capacity of Γ_0 (Reichel [234]).

For the efficient numerical treatment of Symm's equations for multiply-connected bounded regions see Reichel [234].

Mikhlin [181] calls a problem of the kind (355) a *modified Dirichlet problem*. He solves it by a double-layer ansatz (79). Let $a(z, \zeta)$ denote the function which is equal to unity if the points z and ζ belong to the same internal curve Γ_j and equal to zero otherwise, and let $\mathbf{A}$ be the integral operator with kernel a and $\mathbf{K}_1$ the operator with Neumann kernel (see Section 3.1). There is a unique solution μ of the integral equation

$$(\mathbf{I} + \mathbf{K}_1 - \mathbf{A})\mu = -2 \log |\eta| \quad \text{for } \eta \in \Gamma. \quad (357)$$

The double-layer potential u with this density μ satisfies (355) with

$$R_0 = 1 \quad \text{and} \quad \log R_j = \frac{1}{\pi} \int \mu(\eta_j(s)) ds \quad \text{for } j = 1, \dots, m \quad (358)$$

(see [181, p. 151]). The function H is given by the Cauchy integral

$$H(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\mu(\eta)}{\eta - z} d\eta + \alpha i \quad (359)$$

whence $F(z) = z \exp(H(z))$ can be computed. The imaginary constant αi corresponds to a rotation of the canonical region $F(G)$.

The method of Mayo [172, 173], described in Section 3.1, can be used to evaluate the mapping function in the interior of G in an efficient way.

Amano [3] solves the Dirichlet problem (355) by a charge simulation method. Inoue [127] extends his scheme (see [126]) from doubly- to multiply-connected regions. Ogata et al. [195] propose a charge simulation method for the mapping of periodic structure domains onto parallel slit domains.

Ellacott [51] considers the mapping to an annulus with circular slits. He calculates u by uniform approximation of $-\log |\eta|$ on the boundary Γ by the real part of Laurent polynomials, and of polynomials in $(z - a_j)^{-1}$ for $j = 2, \dots, m$, where the a_j are fixed points inside the curves Γ_j . The harmonic measures ω_j of Γ_j with respect to G are included for $j = 1, \dots, m$ in the set of basis functions. (Recall that the *harmonic measure* ω_j is a harmonic function in G with the boundary values $\omega_j = 1$ on Γ_j and $\omega_j = 0$ on Γ_v for $v \neq j$.) The solution of the approximation problem readily gives a rational approximation of $\log(F(z)/z)$. The coefficients of ω_j in the approximating function are estimates for $\log R_j$.

12.2. Osculation methods

Koebe's iterative method [139] determines the conformal mapping F from an unbounded m -connected region G with boundary components Γ_j to a circular region H . The mapping F is calculated as a composition of maps

$$F = \dots \circ F^{k,j} \circ \dots \circ F^{1,2} \circ F^{1,1}. \quad (360)$$

The factors are arranged from right to left in lexicographic order of the index pairs (k, j) , $k = 1, 2, \dots, j = 1, \dots, m$. If all functions $F^{l,v}$ for indices (l, v) which precede (k, j) in the lexicographic order are calculated, then the analytic function $F^{k,j}$ is determined as the conformal mapping of the exterior of the image of Γ_j under the composition of all previous mappings, to the exterior of a circle, normalized at ∞ by $F^{k,j}(z) = z + O(1/z)$. The method converges (see Gaier [65, p. 230], Henrici [107, p. 499]). The image regions become more and more circular. This is nicely illustrated by a figure in [107, p. 498], which shows the effect of the first three iterative steps on a region of connectivity 3.

Koebe's iterative method was rediscovered and applied by Halsey [96], who notes that the method converges rapidly. The region becomes nearly circular after a few iterations. The asymptotic rate of convergence is then the same as for the nearly circular region. It was shown by Wegmann [292] that this rate is the same as the rate of convergence for a "general conjugation problem" (see Section 12.4). The latter can be determined as the maximum eigenvalue of a certain matrix. The asymptotic rate of convergence depends only on the circular region H which is conformally equivalent to G .

For a doubly-connected region with module M the rate of convergence is M^{-2} (see Gaier [65, p. 216]).

A similar iterative procedure has been applied by Grötzsch [88] to construct the mapping from an unbounded m -connected region G to the plane with m parallel slits. More general iterative methods of this kind are investigated by Lind [166].

These methods occur in Gaier's book [65] under the heading "function theoretic iteration methods". But the underlying idea is apparently closely related to that of the osculation methods, described in Section 3.3. The mapping is composed of simpler mappings which make the region locally more circular. The osculation family consists, in the case of the Koebe iteration, of all normalized conformal mappings of the exterior of a Jordan curve to the exterior of a circle.

These methods construct the conformal mapping of a multiply-connected region as an infinite composition of mappings of simply-connected regions. It is interesting to note that there even exists a representation as a finite composition. The following result was proved by Hübner [118].

THEOREM 21. *Let G be an unbounded m -connected region with boundary components Γ_j . Then each univalent function F in G can be factorized as*

$$F = F_m \circ F_{m-1} \circ \cdots \circ F_1 \quad (361)$$

with functions F_j univalent in the simply-connected region exterior to Γ_j . If F and all F_j behave like $z + O(1/z)$ near ∞ then the factors F_j are uniquely determined.

This factorization holds true in particular for the conformal mapping $F: G \rightarrow H$. This special case, announced by Erohyn [54], generalizes previous results of Grunsky, Landau and Gaier. The factors F_j of the conformal map can be determined by an iterative method which is closely related to Koebe's method (see Hübner [117]).

12.3. Projection

Let H be a circular region of connectivity m . The boundary circles C_j are parameterized by ζ_j as defined in (351). Here and in what follows we denote for analytic functions F in H the values $F(\zeta_j(t))$ on the boundary circle C_j by $F|_j$. These are considered as functions of t .

An analytic function Ψ in an unbounded circular region H which vanishes at infinity can be written in a unique way as a sum

$$\Psi(z) = \sum_{j=1}^m h_j(z), \quad (362)$$

where each function h_j is analytic outside C_j and is represented by a Laurent series around the centers z_j of the circles

$$h_j(z) = \sum_{l=1}^{\infty} b_{l,j}(z - z_j)^{-l}. \quad (363)$$

Let G be an unbounded region of connectivity m as described at the beginning of Section 12. An unbounded circular region H and a conformal mapping Φ of H to G must be determined so that Φ satisfies the condition (353) near infinity. The function Φ is completely described by its boundary values. These can be represented by boundary correspondence functions $S_j(t)$. These functions have the property that $S_j(t) - t$ is 2π -periodic. The boundary correspondence equations

$$\Phi|_j = \eta_j(S_j(t)), \quad j = 1, \dots, m, \quad (364)$$

determine the functions S_j as well as the parameters z_j, R_j of the circles.

The simplest way to treat these equations is by alternating projection. Such a method was devised by Prosnak [227] and further elaborated by Klonowska and Prosnak [136]. We give here a simplified version which is closer to the method for simply-connected regions described in Section 4.2. We assume that the functions η_j are continuously differentiable with $\dot{\eta}_j \neq 0$.

In view of the normalization (353), Φ has the representation

$$\Phi(z) = z + \sum_{j=1}^m h_j(z) \quad (365)$$

with functions h_j analytic outside C_j and represented by Laurent series (363). In view of (365), (351) and (363), the boundary correspondence equation on the circle C_j takes the form

$$z_j + R_j e^{-it} + \sum_{l=1}^{\infty} \frac{b_{l,j}}{R_j^l} e^{ilt} = \eta_j(S_j(t)) - \sum_{v \neq j} h_v|_j. \quad (366)$$

This can be used as the basis of an iterative method.

The iteration starts with estimates $z_j^{(0)}$ and $R_j^{(0)}$ for the centers and for the radii of the boundary circles of H , and with guesses $S_j^{(0)}$ for the boundary correspondence functions, and $h_j^{(0)}$ for the functions h_j in (365). One can choose $S_j^{(0)}(t) = t$ and $h_j^{(0)} = 0$.

When $z_j^{(k)}, R_j^{(k)}, S_j^{(k)}, h_j^{(k)}$ are determined for some $k \geq 0$ then the Fourier coefficients $A_{l,j}$ of the functions

$$\eta_j(S_j^{(k)}(t)) - \sum_{v \neq j} h_{v|j}^{(k)} = \sum_{l=-\infty}^{\infty} A_{l,j} e^{ilt} \quad (367)$$

are calculated. From the form (366) of the boundary correspondence equations the conditions

$$z_j^{(k+1)} + R_j^{(k+1)} e^{-it} + \sum_{l=1}^{\infty} \frac{b_{l,j}^{(k+1)}}{R_j^{(k)l}} e^{ilt} = \eta_j(S_j^{(k)}(t)) - \sum_{v \neq j} h_{v|j}^{(k)} \quad (368)$$

are obtained. Upon comparing the coefficients in the Fourier series (367) and (368) the new circle parameters and the coefficients $b_{j,l}^{(k+1)}$ of the new functions $h_j^{(k+1)}$ can be determined

$$R_j^{(k+1)} := \operatorname{Re} A_{-1,j}, \quad z_j^{(k+1)} := A_{0,j}, \quad (369)$$

$$b_{l,j}^{(k+1)} := (R_j^{(k)})^l A_{l,j} \quad \text{for } l = 1, 2, \dots \quad (370)$$

With the remaining coefficients the function

$$g_j(t) = \sum_{l=-\infty}^{-2} A_{l,j} e^{ilt} + i(\operatorname{Im} A_{-1,j}) e^{-it} \quad (371)$$

is formed which gives the change in the boundary correspondence function

$$S_j^{(k+1)}(t) := S_j^{(k)}(t) - \operatorname{Re} \frac{g_j(t)}{\dot{\eta}_j(S_j^{(k)}(t))}. \quad (372)$$

The new approximation for Φ is

$$\Phi^{(k+1)}(z) := z + \sum_{j=1}^m h_j^{(k+1)}(z), \quad (373)$$

where the functions $h_j^{(k+1)}$ are given by the Laurent series (363) with the new coefficients (370).

EXAMPLE 16. The exterior of the inverted ellipse with parameter $p = 0.4$

$$\eta_1(s) = 2 + 0.5i + 2\sqrt{1 - 0.84 \cos^2 s} e^{-is} \quad (374)$$

and ellipses

$$\eta_j := w_j + e^{i\alpha_j} (a_j \cos s + ib_j \sin s) \quad (375)$$

with $w_2 = -2.5i$, $w_3 = -1 + 0.5i$, and axes $a_2 = 1$, $b_2 = 0.8$, $a_3 = 1$, $b_3 = 1.6$ rotated by angles $\alpha_2 = 0$, $\alpha_3 = 0.5$.

Example 16 is calculated by the projection method with 128 grid points on each curve. The rate of convergence is $q = 0.88$. Figure 30 shows the images of a grid of size 0.15 in the circular region. The parameters of the circles are $z_1 = 2.0613 + 0.5837i$, $R_1 = 1.5859$, $z_2 = -0.1660 - 2.2176i$, $R_2 = 0.9586$, $z_3 = -1.2685 + 0.5544i$, $R_3 = 1.1938$.

As an indicator for convergence we take

$$\delta_k := \max_{j,t} |S_j^{(k+1)}(t) - S_j^{(k)}(t)| \quad (376)$$

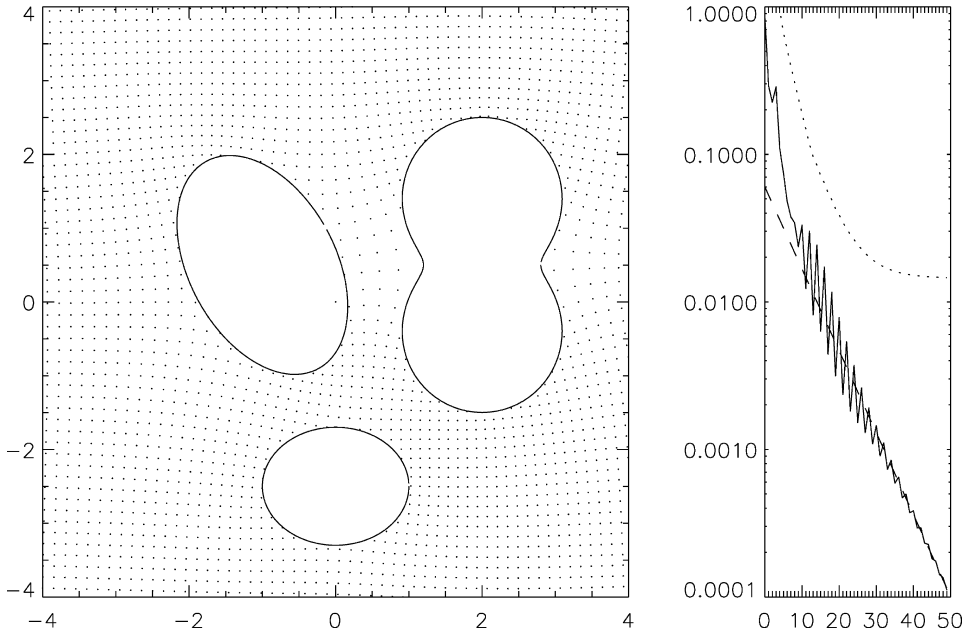


Fig. 30. Mapping to the region of Example 16 calculated by the projection method. The right panel shows the maximum change δ_k (solid) and the measure of accuracy α_k (dotted). The dashed line is proportional to 0.88^k .

and as a measure of the accuracy

$$\alpha_k := \max_j \|g_j\|_2 \quad (377)$$

with the functions g_j defined in (371) in the k th iteration.

There are several other ways of projecting $\Phi^{(k+1)}$ onto the Γ_j . Prosnak [227] studies airfoil profiles. The projection is perpendicular to a “chord”. Klonowska and Prosnak [136] associate points on the image curve of $\Phi^{(k+1)}$ with points on the curve Γ_j according to their arclength parameterization. Kosma [146] gives a version of the projection method which keeps the collocation points on the curves Γ_j fixed and varies in each iteration the preimages on the circles C_j . This method requires the solution of large systems of nonlinear equations by the Newton method. The application of this particular version of the projection method “requires a certain amount of experience” [146, p. 51].

Fil’čakova [58] extends her method of trigonometric interpolation to latticed domains whose boundaries consist of disjoint congruent continua.

12.4. Riemann–Hilbert problems

RH problems on multiply-connected regions have been studied by Vekua [269]. Krutitskii [149] investigated the relation to the directional derivative problem for harmonic functions. For the conformal mapping of multiply-connected regions, RH problems of a special kind on circular regions are needed.

Let H be an m -connected unbounded circular region with boundary circles parameterized by (351). Let l be a nonnegative integer, let λ_j be real numbers and let ψ_j be real functions. The problem of *general conjugation* asks for an analytic function Ψ in H with $\Psi(\infty) = 0$, real numbers a_{j0} and complex numbers $a_{j1}, \dots, a_{jl}$ such that

$$\operatorname{Re}(e^{i\lambda_j} e^{ilt} \psi_j + a_{jl} e^{ilt} + \dots + a_{j1} e^{it} + a_{j0}) = \psi_j \quad (378)$$

for $j = 1, \dots, m$. One can apply instead of the normalization $\Psi(\infty) = 0$ the more general condition that Ψ behaves near infinity like

$$\Psi(z) = p(z) + O(1/z) \quad (379)$$

with a given polynomial $p(z)$. When on the right-hand side of (378) the functions ψ_j are replaced by $\psi_j - \operatorname{Re}(e^{i\lambda_j} e^{ilt} p|_j)$ this situation is reduced to the general conjugation problem (378) for the function $\tilde{\Psi} := \Psi - p$ with normalization $\tilde{\Psi}(\infty) = 0$. This covers the usual normalization condition (353) for conformal mapping functions.

Wegmann [292] introduced a method of *successive conjugation* which reduces the problem (378) to a sequence of RH problems on the circles C_j . For $l = 0$ this method was first applied by Halsey [96].

Using the representation (362) for Ψ , the equation (378) is written in the form

$$\begin{aligned} & \operatorname{Re}(e^{i\lambda_j} e^{ilt} h_{j|j} + a_{jl} e^{ilt} + \cdots + a_{j1} e^{it} + a_{j0}) \\ &= \psi_j - \operatorname{Re}\left(e^{i\lambda_j} \sum_{v \neq j} e^{ilt} h_{v|j}\right) =: \psi_j^*. \end{aligned} \quad (380)$$

The left-hand side of (380) contains only quantities on the circle C_j . This suggests the following iterative procedure of successive conjugation.

The iterative procedure starts with a set of functions $h_j^{(0)}$. One can choose all $h_j^{(0)} = 0$. When after the k th iterative step, functions $h_j^{(k)}$ are available, these functions are inserted into the right-hand side of (380) and new functions $h_j^{(k+1)}$ are determined successively for $j = 1, \dots, m$, from the equations

$$\begin{aligned} & \operatorname{Re}(e^{i\lambda_j} e^{ilt} h_{j|j}^{(k+1)} + a_{jl} e^{ilt} + \cdots + a_{j1} e^{it} + a_{j0}) \\ &= \psi_j - \operatorname{Re}\left(e^{i\lambda_j} \sum_{v < j} e^{ilt} h_{v|j}^{(k+1)}\right) - \operatorname{Re}\left(e^{i\lambda_j} \sum_{v > j} e^{ilt} h_{v|j}^{(k)}\right) =: \psi_j^*. \end{aligned} \quad (381)$$

The parameters $a_{j0}, \dots, a_{jl}$ in (381) depend also on the iterative step $k+1$. But they need not be updated during the iteration, since only the functions $h_j^{(k+1)}$ are needed for the next step. Therefore, it is sufficient to determine the parameters $a_{j0}, \dots, a_{jl}$ at the end of the iteration.

The prescription by equation (381) may be considered as a Gauss–Seidel method or Einzelschrittverfahren for the system of equations (380). One can also apply a Jacobi method (Gesamtschrittverfahren) when on the right-hand side of (381) all h_v are taken from step k .

The conjugation on a single circle can be very efficiently done in Fourier space. To this aim the left-hand side of (381) must be compared with the Fourier series of the right-hand side,

$$\psi_j^*(t) = \sum_{n=-\infty}^{\infty} A_{n,j} e^{int}. \quad (382)$$

Since ψ_j^* is a real function, the coefficients satisfy $A_{-n,j} = \overline{A_{n,j}}$. In particular, $A_{0,j}$ is real. By comparing the coefficients of the left-hand side of (380) and the right-hand side of (382) we get the unique solution

$$a_{j0} = A_{0,j}, \quad a_{ji} = 2A_{i,j} \quad \text{for } i = 1, \dots, l, \quad (383)$$

for the parameters a_{ji} , and

$$b_{n,j} = 2R_j^n e^{-i\lambda_j} A_{n+l,j} \quad \text{for } n = 1, 2, \dots, \quad (384)$$

for the coefficients $b_{n,j}$ of the series (363). Wegmann [292] proved that this method converges for $l \geq 1$. Convergence is linear and the rate of convergence can be determined from an eigenvalue problem. This result gives a constructive proof for the existence of a solution of the RH problem. It follows then from the general theory of RH problems that the solution is unique. For $l = 0$ this result is well known (see, e.g., Vekua [269, p. 265]). We note this result since it will be important in Section 12.5.

THEOREM 22. *For every integer $l \geq 0$ and for any function ψ_j in W , the Riemann–Hilbert problem (378) has a unique solution consisting of an analytic function Ψ in H , normalized by $\Psi(\infty) = 0$, and of complex numbers $a_{j1}, \dots, a_{jl}$ and real numbers a_{j0} .*

Dundučenko [48] represents the solution of the conjugation problem ((378) with $l = 0$) by a “Schwarz formula”. Sorokin [248] gives a solution of RH problems on bounded multiply connected circular regions in closed form. The formulas involve infinite series of functions which are constructed by reflection at the boundary circles. It is not quite clear how these formulas can be used for computational purposes.

Just as for simply-connected regions (see Section 4.1), for multiply-connected regions which differ only slightly from circular regions an approximation of the conformal mapping can be obtained in terms of the conjugation operator (Sorokin [248], Wegmann [292]). To be specific, let H be an unbounded circular region whose boundary consists of circles C_j with centers z_j and radii R_j , and let ψ_j be twice continuously differentiable real functions. Then one can define, for small real numbers τ , neighboring regions G_τ with boundary curves parameterized by

$$\eta_j(s) = z_j + R_j e^{-is} + \tau \psi_j(s) e^{-is}. \quad (385)$$

On the other hand, one can consider the RH problem

$$\operatorname{Re}(e^{it} \Psi|_j + a_{j1} e^{it} + a_{j0}) = \psi_j \quad (386)$$

as a boundary problem for an analytic function Ψ in H with $\Psi(\infty) = 0$. The region G_τ is conformally equivalent to a circular region H_τ bounded by circles $C_{\tau j}$ whose centers and radii are in first order of τ given by

$$z_{\tau j} = z_j + \tau a_{j1}, \quad R_{\tau j} = R_j + \tau a_{j0}. \quad (387)$$

The boundary values of the conformal mapping $\Phi_\tau : H_\tau \rightarrow G_\tau$ are in first order of τ ,

$$\Phi_{\tau|j} \approx z_{\tau j} + R_{\tau j} e^{-it} + \tau \Psi|_j. \quad (388)$$

The mapping from a circular region to a slit region can easily be calculated by means of an RH problem of the kind (378) with $l = 0$. Let α_j be fixed angles. When the boundary circles of H have centers z_j and radii R_j , $j = 1, \dots, m$, and Ψ solves the boundary problem

$$\operatorname{Re}(e^{-i\alpha_j} \Psi|_j) + a_{j0} = R_j \sin(t + \alpha_j), \quad (389)$$

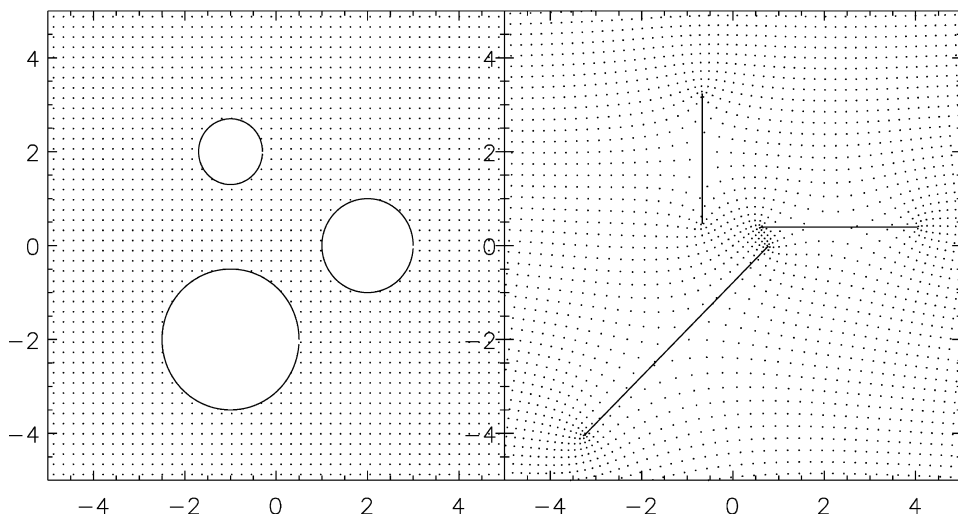


Fig. 31. Mapping of the circular region of Example 17 (left panel) to the complex plane with slits of prescribed inclination (right panel).

then the function

$$\Phi(z) = z + i\Psi(z) \quad (390)$$

maps H conformally to the complex plane with m straight slits which form the angles α_j with the real axis.

EXAMPLE 17. Circular region is bounded by circles with centers $z_1 = 2$, $z_2 = -1 - 2i$, $z_3 = -1 + 2i$ and radii $R_1 = 1$, $R_2 = 1.5$, $R_3 = 0.7$. The slits form with the real axis the angles $\alpha_1 = 0^\circ$, $\alpha_2 = 45^\circ$, $\alpha_3 = 90^\circ$.

Figure 31 shows the mapping of the circular region of Example 17 to the complex plane with slits of prescribed inclination.

12.5. The Newton method

Wegmann [293] developed a Newton method for multiply-connected regions. The procedure is quite analogous to that for simply- and doubly-connected regions as described in Sections 4.3 and 11.3. Some peculiarities must be noted, however.

A function F , analytic at ∞ , has a Laurent series representation $F(z) = b_0 + b_1/z + b_2/z^2 + \dots$. The *residue at infinity* is defined as (note the minus sign!)

$$\text{Res}_\infty F := -b_1. \quad (391)$$

Let G be an m -connected unbounded region with boundary curves Γ_j , $j = 1, \dots, m$. By linearization of the boundary correspondence equations (368) linear equations

$$\Phi|_j + D\Phi|_j + \Phi'|_j(Dz_j + DR_j e^{-it}) = \eta_j + \dot{\eta}_j DS_j \quad (392)$$

are obtained which connect the necessary changes $D\Phi$, Dz_j , DR_j , DS_j . These equations give the motivation for the method which will be described in the following.

The parameter functions η_j are assumed to be differentiable with Hölder continuous derivatives $\dot{\eta}_j(s) \neq 0$. The derivatives can be represented in the form

$$\dot{\eta}_j(s) = r_j(s) \exp(i\theta_j(s)) \quad (393)$$

with Hölder continuous real functions $r_j > 0$ and θ_j . When the functions S_j are Hölder continuous, the functions

$$v_j(t) := \theta_j(S_j(t)) + t + \frac{\pi}{2} = \arg[ie^{it} \dot{\eta}_j(S_j(t))] \quad (394)$$

are also Hölder continuous and 2π -periodic. There are uniquely defined Hölder continuous functions w_j , real numbers λ_j and an analytic function Y in H which has boundary values

$$Y|_j = w_j + iv_j + i\lambda_j \quad (395)$$

and vanishes at infinity. (See Vekua's "problem D" [269, p. 261].) The Y , w_j and λ_j satisfy the equation

$$\frac{1}{e^{i\lambda_j} \dot{\eta}_j} = i \frac{e^{w_j}}{r_j} \frac{e^{it}}{\exp(Y|_j)}. \quad (396)$$

With w_j and λ_j the auxiliary complex functions

$$F_j := \frac{r_j}{e^{i\lambda_j} e^{w_j}} \frac{\eta_j}{\dot{\eta}_j} = \exp(-w_j - i\theta_j - i\lambda_j) \eta_j \quad (397)$$

are calculated. Then an analytic function $\mathcal{E}$ in H , complex numbers a_j , real numbers α_j and real functions g_j are determined such that

$$e^{it} \mathcal{E}|_j + a_j e^{it} + \alpha_j = \operatorname{Im} F_j + i g_j \quad (398)$$

is satisfied. The function $\mathcal{E}$ is required to behave near infinity like

$$\mathcal{E}(z) = z + O\left(\frac{1}{z}\right). \quad (399)$$

By taking the real part of equation (398) the functions g_j are eliminated and an RH problem of the kind (378) for the function $\mathcal{E}$ remains. In view of Theorem 22, problem (398) has a unique solution $\mathcal{E}$, a_j , α_j and g_j .

With the numbers a_j, α_j and the functions g_j , the necessary changes in S_j, z_j and R_j are calculated by the formulas

$$DS_j = -\frac{e^{w_j}}{r_j}(g_j + \operatorname{Re} F_j) \quad (400)$$

and

$$Dz_j = a_j - \operatorname{Res}_\infty Y, \quad DR_j = \alpha_j. \quad (401)$$

Based on (400) and (401) one can devise an iterative method: When in the k th iteration functions $S_j^{(k)}$ and circle parameters $z_j^{(k)}$ and $R_j^{(k)}$ are given, insert these values into the formulas (394) etc. and calculate a shift according to (400) and (401); then change the data

$$S_j^{(k+1)} = S_j^{(k)} + DS_j, \quad z_j^{(k+1)} = z_j^{(k)} + Dz_j, \quad (402)$$

$$R_j^{(k+1)} = R_j^{(k)} + DR_j, \quad (403)$$

and perform the next step.

Wegmann [293] proved that this iteration converges to the solution of the conformal mapping problem whenever the initial functions $S_j^{(0)}$ and the circle parameters are sufficiently close to the solution. Convergence is quadratic.

The main computational effort must be spent on the solution of the boundary value problems (395) and (398). Both can be transformed into RH problems on circular regions and solved by the method of successive conjugation described in Section 12.4.

The residue term in (401) comes as a surprise. It can only be explained by the proof of convergence in [293]. On the other hand, one can convince oneself easily by numerical experiments that the method does not converge quadratically when the residue term in (401) is omitted.

Wegmann describes in [293] also how the method must be modified to deal with bounded multiply-connected regions. The outer boundary, however, requires special treatment and nonnegligible extra programming work.

EXAMPLE 18. Unbounded region bounded by three ellipses which are parameterized by (375) with $w_1 = 2i, w_2 = -2i, w_3 = -1, a_1 = 1.5, b_1 = 0.8, \alpha_1 = 0.5, a_2 = 2, b_2 = 1, \alpha_2 = -0.8, a_3 = 1.5, b_3 = 0.8, \alpha_3 = -0.5$.

The calculation for Example 18 was done with 64 grid points on each curve. It was started with $S_j^{(0)}(t) = t$ and with circle parameters $z_1 = -0.3 + 2.5i, z_2 = -0.2 - 2.5i, z_3 = -1.5, R_1 = 0.9, R_2 = 1.2, R_3 = 0.5$. At the end of the iteration the circle parameters are $z_1 = -0.13971 + 2.0470i, R_1 = 1.1214, z_2 = 0.91011 - 1.9955i, R_2 = 1.5101, z_3 = -1.2737 + 0.02541i, R_3 = 0.55958$. Figure 32 shows the image of an equidistant grid in H . The convergence, measured by δ_k defined in (376), and the accuracy α_k of the calculation are indicated on the right panel.

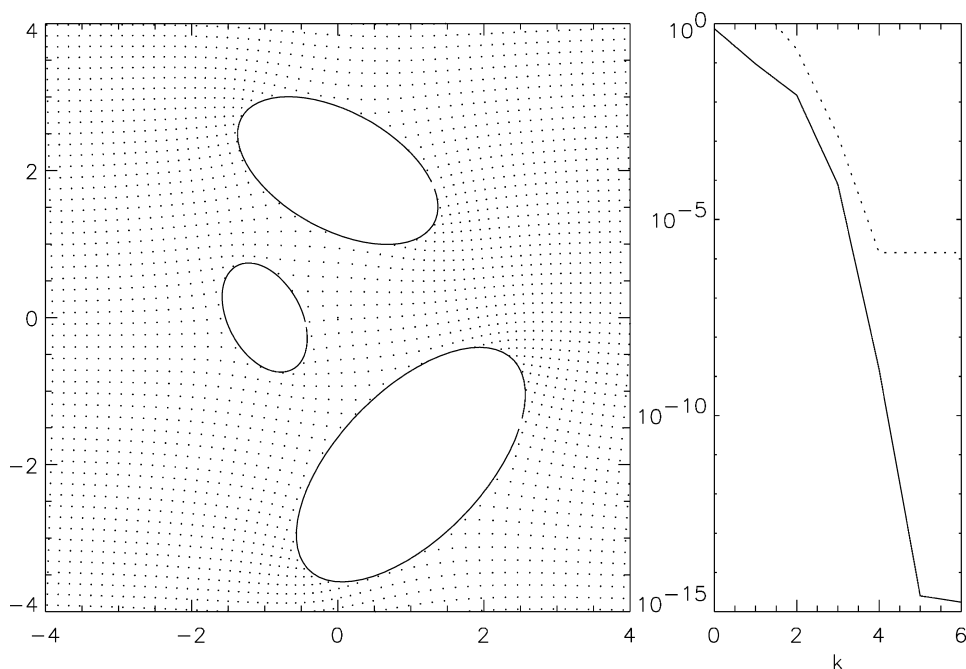


Fig. 32. Mapping of a circular region to the ellipse region of Example 18 by the Newton method. Right panel: δ_k solid, α_k dotted.

EXAMPLE 19. A region bounded from outside by an inverted ellipse which is parameterized by (162) with parameter $p = 0.5$. The inner curves are ellipses

$$\eta_j(s) = w_j + a_j \cos s + b_j i \sin s \quad (404)$$

with axes $a_1 = 0.3, b_1 = 0.2, a_2 = 0.2, b_2 = 0.4$ and centers $w_1 = -0.1 + 0.5i, w_2 = 0.1 - 0.3i$.

Example 19 is calculated with $N_j = 128$ points on each boundary. The interpolation conditions $S_0(t_1) = 0, S_0(t_{40}) = 1.9, S_0(t_{80}) = 3.6$ are applied at three of the grid points $t_k := (k - 1)2\pi/N_0$. Figure 33 shows the image of an equidistant grid in the unit circle and the convergence of the iteration. The calculated parameters of the circles are $z_1 = -0.23374 + 0.56475i, R_1 = 0.26034, z_2 = 0.10730 - 0.35282i, R_2 = 0.38068$.

12.6. Other methods

Gaier [69] gives a comprehensive survey of characterizations of slit mappings by extremal properties. Schiffer and Hawley [239] characterized the conformal mapping F of an m -connected region G to a circular region as the solution to a certain extremum problem

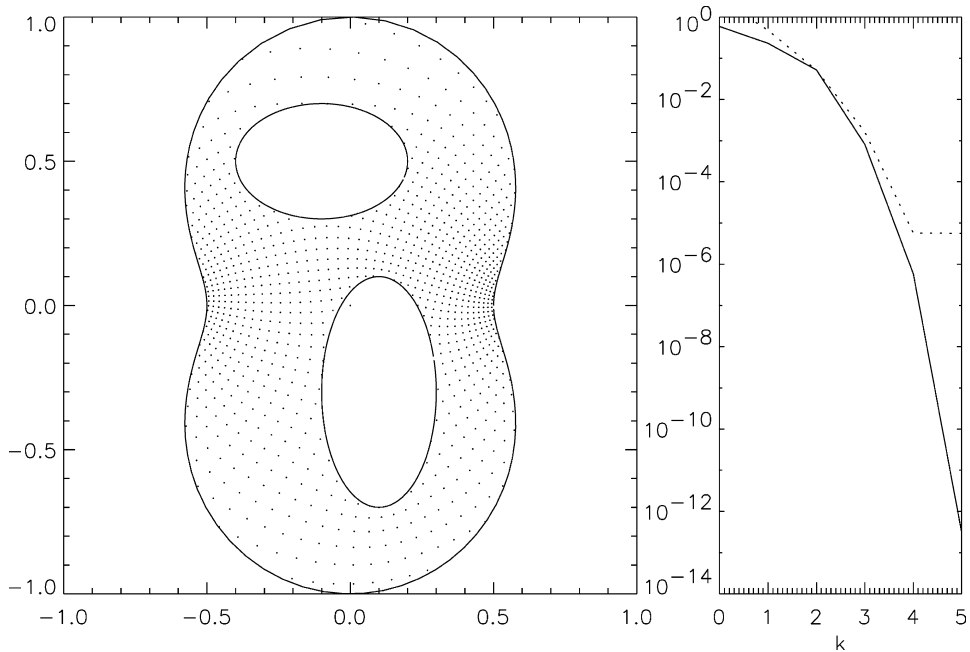


Fig. 33. Mapping of a circular region to the bounded region of Example 19 by the Newton method. Right panel: δ_k solid, α_k dotted.

which is solved by the harmonic function $\log|F'|$. This has apparently not yet been exploited for numerical calculations, but it is interesting from the theoretical point of view. It has been used by Dittmar [45] to give a new proof for the convergence of the Koebe method described in Section 12.2.

The Bergman and Szegő kernels of multiply-connected regions are discussed in great detail by Nehari [189, p. 367]. Both kernels can be built up by bilinear series of orthonormalized functions as in (93) and (105). For the Bergman kernel only analytic functions are admitted which have a single-valued integral, while for the Szegő kernel all analytic functions are admitted. A relation analogous to (109) holds between the two kernels with the modification that a linear combination of functions w_j occurs, where $\operatorname{Re} w_j = \omega_j$ is the harmonic measure of the boundary component Γ_j with respect to G . For a bounded region G a suitable set of functions for the construction of k_B by orthonormalization is given by z^l , $l = 0, 1, 2, \dots$, and $(z - a_j)^{-l}$, $j = 1, \dots, m$, $l = 2, 3, \dots$, with points a_j inside Γ_j (Gaier [65, p. 244]). Note that the omission of the terms $(z - a_j)^{-1}$ ensures that all functions have a single valued integral.

There are several ways to calculate from the kernel function k_B the conformal map from G to the complex plane with parallel slits (or the disk with circular slits) (see Gaier [65, pp. 242–243]).

Bergman and Chalmers [15] describe a procedure, based on the Bergman kernel function, for the mapping of a triply-connected domain to an annulus with a circular hole.

An orthonormalization method, quite analogous to the method described for doubly-connected regions in Section 10.2, was presented by Kokkinos et al. [143] for the construction of the mapping of a bounded multiply-connected region to an annulus with circular slits. This was extended by Kokkinos [142] to a unified method which includes several types of mappings to slit regions, in particular the mapping of an unbounded m -connected region to the exterior of a circle with $m - 1$ concentric circular slits. The boundary of G is assumed to consist of piecewise analytic Jordan curves. For a good convergence it is essential to include “singular basis functions” which reflect the behavior of the conformal mapping near the corners (see also Section 3.2).

A useful tool for multiply-connected regions is the *Ahlfors map* which is defined as follows: Let G be an m -connected bounded region with C^2 boundary and $a \in G$. Then the Ahlfors map f_a for G at a is the uniquely defined analytic function which solves the problem: f maps G into the unit disk D , is analytic in G , $f(a) = 0$, $f'(a)$ is real and $f'(a)$ is maximum. The mapping f_a is an m -to-one branched covering map of G to D . Each boundary component of G is mapped by f_a one-to-one to the unit circle (see Bell [9], Nehari [189, p. 378], Grunsky [89, p. 147]). For $m = 1$ the Ahlfors map f_a agrees with the Riemann mapping function.

Bell [9] uses a relation, discovered by Garabedian [80], between the Ahlfors map and the Szegő kernel function to compute the Ahlfors map numerically by a Kerzman–Stein method. He treats in his book [10] topics like the Riemann mapping theorem, the Ahlfors map, numerical conformal mapping, the Bergman and Szegő kernels and projections, and boundary value problems of potential theory.

The construction of the Bergman or Szegő kernels via orthonormalization of rational functions is not quite easy, and “the results of such numerical nightmares are usually disappointing” (Bell [12, p. 367]). Bell [11] points out that for an m -connected region, the Szegő kernel $S(z, w)$ is a combination of $S(z, a_j)$ at only m points a_j and the Ahlfors map f_a , i.e., of functions of a single variable only. Since the Bergman kernel can be expressed in terms of the Szegő kernel, a similar statement holds for the Bergman kernel (Chung [29]). The basic functions $S(z, a_j)$ are solutions to explicit Kerzman–Stein integral equations and as such, are easy to compute (Bell [12]). The Ahlfors map can also be calculated by this type of equations. The coefficients of the expressions $S(z, a_j)\overline{S(z, a_k)}$ are obtained by simple algebra.

DeLillo et al. [42] describe a Fornberg-type method for the mapping of an unbounded circular domain to an unbounded multiply-connected region. The formulation in [42] is asymmetric insofar as the curves Γ_1 and Γ_2 are treated differently from the other curves. The method could probably be simplified by using a symmetric formulation.

Harrington [99] proved the existence of a conformal mapping of multiply-connected regions to regions with arbitrarily specified boundary shapes. This generalizes results of Courant et al. [31] (see also Gaier [65, p. 183], Courant [32, p. 178]). Harrington’s proof in [99], based on homotopy, is constructive and has been used for the actual calculation of such mappings. But there are apparently no published results.

An asymptotic expansion for the conformal mapping of the multiply-connected domain exterior to some symmetrical thin regions to the slitted plane has been calculated by Homentcovschi [112].

References

- [1] L.V. Ahlfors, *Remarks on the Neumann–Poincaré integral equation*, Pacific J. Math. **2** (1952), 271–280.
- [2] K. Amano, *A charge simulation method for the numerical conformal mapping of interior, exterior and doubly-connected domains*, J. Comput. Appl. Math. **53** (1994), 353–370.
- [3] K. Amano, *A charge simulation method for numerical conformal mapping onto circular and radial slit domains*, SIAM J. Sci. Comput. **19** (1998), 1169–1187.
- [4] C. Anderson, S.E. Christiansen, O. Møller and H. Tornehave, *Conformal Mapping*, Selected Numerical Methods, C. Gram, ed., Regnecentralen, Copenhagen (1962).
- [5] C. Anderson and M. Dillon Daleh, *Rapid computation of the discrete Fourier transform*, SIAM J. Sci. Comput. **17** (1996), 913–919.
- [6] K.E. Atkinson, *The numerical solution of boundary integral equations*, The State of the Art in Numerical Analysis, I.S. Duff et al., eds, Oxford, Clarendon Press (1997), 223–259.
- [7] P.D. Banzuri, *On the Riemann–Hilbert-problem for doubly connected domains*, Soobshch. Akad. Nauk Gruzin. SSR (1975), 549–552 (in Russian, English summary).
- [8] R.S. Baty and P.J. Morris, *Conformal grid generation for high aspect ratio simply and doubly connected regions*, Internat. J. Numer. Methods Engrg. **38** (1995), 3817–3830.
- [9] S. Bell, *Numerical computation of the Ahlfors map of a multiply connected planar domain*, J. Math. Anal. Appl. **120** (1986), 211–217.
- [10] S.R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, CRC Press, Boca Raton, FL (1992).
- [11] S. Bell, *Complexity of the classical kernel functions of potential theory*, Indiana Univ. Math. J. **44** (1995), 1337–1369.
- [12] S. Bell, *Recipes for classical kernel functions associated to a multiply connected domain in the plane*, Complex Var. **29** (1996), 367–378.
- [13] S. Bergman(n), *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Math. Ann. **86** (1922), 238–271.
- [14] S. Bergman, *The Kernel Function and Conformal Mapping*, Math. Surveys 5, Amer. Math. Soc., New York (1950).
- [15] S. Bergman and B. Chalmers, *A procedure for conformal mapping of triply-connected domains*, Math. Comp. **21** (1967), 527–542.
- [16] H. Bergström, *An approximation of the analytic function mapping a given domain inside or outside the unit circle*, Mém. Publ. Soc. Sci. Arts Lettr. Hainaut (1958), 193–198.
- [17] J.-P. Berrut, *A Fredholm integral equation of the second kind for conformal mapping*, J. Comput. Appl. Math. **14** (1986), 99–110.
- [18] J.-P. Berrut and M.R. Trummer, *Equivalence of Nyström’s method and Fourier methods for the numerical solution of Fredholm integral equations*, Math. Comp. **48** (1987), 617–623.
- [19] G. Beylkin, *On the fast Fourier transform of functions with singularities*, Appl. Comput. Harmon. Anal. **2** (1995), 363–381.
- [20] L. Bieberbach, *Zur Theorie und Praxis der konformen Abbildung*, Rend. Circ. Mat. Palermo **38** (1914), 98–112.
- [21] F. Bisshopp, *Numerical conformal mapping and analytic continuation*, Quarterly Appl. Math. **41** (1983), 125–142.
- [22] H. Braß, *Zur numerischen Berechnung konjugierter Funktionen*, Numer. Meth. Approx. Theory, Vol. 6, L. Collatz, G. Meinardus, H. Werner, eds, Birkhäuser, Basel (1982), 43–66.
- [23] J. Burbea, *A procedure, for conformal maps of simply connected domains by using the Bergman function*, Math. Comp. **24** (1970), 821–829.
- [24] J. Burbea, *A numerical determination of the modulus of doubly connected domains by using the Bergman curvature*, Math. Comp. **25** (1971), 743–756.
- [25] G.F. Carey and A. Muleshkov, *Some conformal map constructs for numerical grids*, Comm. Numer. Methods Engrg. **11** (1995), 127–135.
- [26] N.V. Challis and D.M. Burley, *A numerical method for conformal mapping*, IMA J. Numer. Anal. **2** (1982), 169–181.

- [27] S. Chakravarty and D. Anderson, *Numerical conformal mapping*, Math. Comp. **33** (1979), 953–969.
- [28] S. Christiansen, *Condition number of matrices derived from two classes of integral equations*, Math. Methods Appl. Sci. **3** (1981), 364–392.
- [29] Y.-B. Chung, *An expression of the Bergman kernel function in terms of the Szegő kernel*, J. Math. Pures Appl. **75** (1996), 1–7.
- [30] J.W. Cooley and J.W. Tukey, *An algorithm for the machine calculation of complex Fourier series*, Math. Comp. **19** (1965), 297–301.
- [31] R. Courant, B. Manel and M. Shiffman, *A general theorem on conformal mapping of multiply connected domains*, Proc. Natl. Acad. Sci. USA **26** (1940), 503–507.
- [32] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience, New York (1950).
- [33] J.H. Curtiss, *Solution of the Dirichlet problem by interpolating harmonic polynomials*, Bull. Amer. Math. Soc. **68** (1962), 333–337.
- [34] T.K. DeLillo, *On some relations among numerical conformal mapping methods*, J. Comput. Appl. Math. **19** (1987), 363–377.
- [35] T.K. DeLillo, *A note on Rengel's inequality and the crowding phenomenon in conformal mapping*, Appl. Math. Lett. **3** (1990), 25–27.
- [36] T.K. DeLillo, *On the use of numerical conformal mapping methods in solving boundary value problems for the Laplace equation*, Adv. Comp. Methods Partial Differential Equations **7** (1992), 190–194.
- [37] T.K. DeLillo, *The accuracy of numerical conformal mapping methods: A survey of examples and results*, SIAM J. Numer. Anal. **31** (1994), 788–812.
- [38] T.K. DeLillo and A.R. Elcrat, *A comparison of some numerical conformal mapping methods for exterior regions*, SIAM J. Sci. Stat. Comput. **12** (1991), 399–422.
- [39] T.K. DeLillo and A.R. Elcrat, *A Fornberg-like conformal mapping method for slender regions*, J. Comput. Appl. Math. **46** (1993), 49–64.
- [40] T.K. DeLillo and A.R. Elcrat, *Numerical conformal mapping methods for exterior regions with corners*, J. Comput. Phys. **108** (1993), 199–208.
- [41] T.K. DeLillo, A.R. Elcrat and J.A. Pfaltzgraff, *Numerical conformal mapping methods based on Faber series*, J. Comput. Appl. Math. **83** (1997), 205–236.
- [42] T.K. DeLillo, M.A. Horn and J.A. Pfaltzgraff, *Numerical conformal mapping of multiply connected regions by Fornberg-like methods*, Numer. Math. **83** (1999), 205–230.
- [43] T.K. DeLillo and J.A. Pfaltzgraff, *Extremal distance, harmonic measure and numerical conformal mapping*, J. Comput. Appl. Math. **46** (1993), 103–113.
- [44] T.K. DeLillo and J.A. Pfaltzgraff, *Numerical conformal mapping methods for simply and doubly connected regions*, SIAM J. Sci. Comput. **19** (1998), 155–171.
- [45] B. Dittmar, *Bemerkungen zu einem Funktional von M. Schiffer und N.S. Hawley in der Theorie konformer Abbildungen*, Math. Nachr. **97** (1980), 21–37.
- [46] T.A. Driscoll, *A nonoverlapping domain decomposition method for Symm's equation for conformal mapping*, SIAM J. Numer. Anal. **36** (1999), 922–934.
- [47] T.A. Driscoll and L.N. Trefethen, *Schwarz–Christoffel Mapping*, Cambridge University Press, Cambridge (2002).
- [48] L.E. Dundučenko, *On the Riemann–Hilbert problem for a multiply connected domain*, Soviet Math. Dokl. **12** (1971), 34–37.
- [49] A. Dutt and V. Rokhlin, *Fast Fourier transforms for nonequispaced data*, SIAM J. Sci. Comput. **14** (1993), 1368–1393.
- [50] S.W. Ellacott, *A technique for approximate conformal mapping*, Multivariate Approximation (Sympos. Durham 1977), Academic Press, London (1978), 301–314.
- [51] S.W. Ellacott, *On the approximate conformal mapping of multiply connected domains*, Numer. Math. **33** (1979), 437–446.
- [52] J. Elschner and E.P. Stephan, *A discrete collocation method for Symm's integral equation on curves with corners*, J. Comput. Appl. Math. **75** (1996), 131–146.
- [53] J. Elschner and I.G. Graham, *Quadrature methods for Symm's integral equation on polygons*, IMA J. Numer. Anal. **17** (1997), 643–664.

- [54] V. Erohin, *On the theory of conformal and quasiconformal mapping of multiply connected regions*, Dokl. Akad. Nauk SSSR **127** (1959), 1155–1157 (in Russian).
- [55] M.I. Falcão, N. Papamichael and N.S. Stylianopoulos, *Curvilinear crosscuts of subdivision for a domain decomposition method in numerical conformal mapping*, J. Comput. Appl. Math. **106** (1999), 177–196.
- [56] M.I. Falcão, N. Papamichael and N.S. Stylianopoulos, *Approximating the conformal maps of elongated quadrilaterals by domain decomposition*, Constr. Approx. **17** (2001), 589–617.
- [57] P.F. Fil'čakov, *Approximate Methods of Conformal Mapping*, Naukova Dumka, Kiev (1964) (in Russian).
- [58] V.P. Fil'čakova, *Conformal Mappings of Domains of Special Type*, Naukova Dumka, Kiev (1972) (in Russian).
- [59] J.M. Floryan and C. Zemach, *Schwarz–Christoffel mappings: A general approach*, J. Comput. Phys. **72** (1987), 347–371.
- [60] J.M. Floryan and C. Zemach, *On the generalized Schwarz–Christoffel mappings and their numerical implementation*, Z. Angew. Math. Mech. **69** (1989), T75–T77.
- [61] J.M. Floryan and C. Zemach, *Schwarz–Christoffel methods for conformal mapping of regions with a periodic boundary*, J. Comput. Appl. Math. **46** (1/2) (1993), 77–102.
- [62] B. Fornberg, *A numerical method for conformal mappings*, SIAM J. Sci. Stat. Comput. **1** (1980), 386–400.
- [63] B. Fornberg, *A numerical method for conformal mapping of doubly connected regions*, SIAM J. Sci. Stat. Comput. **5** (1984), 771–783.
- [64] M.S. Friberg, *A new method for the effective determination of conformal maps*, Thesis, Univ. Minnesota (1951).
- [65] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, Berlin (1964).
- [66] D. Gaier, *Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden*, Numer. Math. **19** (1972), 179–194.
- [67] D. Gaier, *Ableitungsfreie Abschätzungen bei trigonometrischer Interpolation und Konjugierten-Bestimmung*, Computing **12** (1974), 145–148.
- [68] D. Gaier, *Integralgleichung erster Art und konforme Abbildung*, Math. Z. **147** (1976), 113–129.
- [69] D. Gaier, *Konforme Abbildung mehrfach zusammenhängender Gebiete*, Jahresber. Deutsch. Math.-Ver. **81** (1978), 25–44.
- [70] D. Gaier, *Das logarithmische Potential und die konforme Abbildung mehrfach zusammenhängender Gebiete*, The Influence of His Work on Mathematics and the Physical Sciences, P.L. Butzer, E.B. Christoffel and F. Feher, eds, Birkhäuser, Basel (1981), 290–303.
- [71] D. Gaier, *Numerical methods in conformal mapping*, Computational Aspects of Complex Analysis, H. Werner et al., eds, Reidel, Boston (1983), 51–78.
- [72] D. Gaier, *On a polynomial lemma of Andrievskii*, Arch. Math. **49** (1987), 119–123.
- [73] D. Gaier, *On the convergence of the Bieberbach polynomials in regions with piecewise analytic boundary*, Arch. Math. **58** (1992), 462–470.
- [74] D. Gaier, *Conformal mapping of analytic corners in a generalized sense*, Analysis **12** (1992), 187–193.
- [75] D. Gaier, *Conformal modules and their computation*, Computational Methods and Function Theory, Penang, Malaysia (1994), 159–171.
- [76] D. Gaier, *Polynomial approximation of conformal maps*, Constr. Approx. **14** (1998), 27–40.
- [77] D. Gaier and W.K. Hayman, *Modules of long quadrilaterals and thick ring domains*, Rend. Mat. Appl. **10** (1990), 809–834.
- [78] D. Gaier and W. Hayman, *On the computation of modules of long quadrilaterals*, Constr. Approx. **7** (1991), 453–467.
- [79] D. Gaier and N. Papamichael, *On the comparison of two numerical methods for conformal mapping*, IMA J. Numer. Anal. **7** (1987), 261–282.
- [80] P. Garabedian, *Schwarz's lemma and the Szegő kernel function*, Trans. Amer. Math. Soc. **67** (1949), 1–35.
- [81] I.E. Garrick, *Potential flow about arbitrary biplane wing sections*, NACA Report 542 (1936).
- [82] W. Gautschi, *Attenuation factors in practical Fourier analysis*, Numer. Math. **18** (1972), 373–400.
- [83] G. Gillot, *Calcul numérique de la transformation conforme faisant correspondre l'intérieur d'un cercle à l'intérieur d'une courbe fermée*, C. R. Acad. Sci. Paris **262** (1966), 1243–1246.
- [84] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 2nd edn, Amer. Math. Soc., Providence, RI (1969).

- [85] I.G. Graham and K.E. Atkinson, *On the Sloan iteration applied to integral equations of the first kind*, IMA J. Numer. Anal. **13** (1993), 29–41.
- [86] E. Grassmann, *Numerical experiments with a method of successive approximation for conformal mapping*, J. Appl. Math. Phys. (ZAMP) **30** (1979), 873–884.
- [87] A.Z. Grinspan and E.B. Saff, *Estimating the argument of approximate conformal mappings*, Complex Var. **26** (1994), 191–202.
- [88] H. Grötzsch, *Über das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche*, Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **84** (1932), 15–36.
- [89] H. Grunsky, *Lectures on Theory of Functions in Multiply Connected Domains*, Vandenhoeck & Ruprecht, Göttingen (1978).
- [90] M.H. Gutknecht, *Existence of a solution to the discrete Theodorsen equation for conformal mappings*, Math. Comp. **31** (1977), 478–480.
- [91] M.H. Gutknecht, *Fast algorithms for the conjugate periodic function*, Computing **22** (1979), 79–91.
- [92] M.H. Gutknecht, *Solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods*, Numer. Math. **36** (1981), 405–429.
- [93] M.H. Gutknecht, *On the computation of the conjugate trigonometric rational function and on a related splitting problem*, SIAM J. Numer. Anal. **20** (1983), 1198–1205.
- [94] M.H. Gutknecht, *Numerical experiments on solving Theodorsen's integral equation for conformal maps with the fast Fourier transform and various nonlinear iterative methods*, SIAM J. Sci. Stat. Comput. **4** (1983), 1–30.
- [95] M.H. Gutknecht, *Numerical conformal mapping methods based on function conjugation*, J. Comput. Appl. Math. **14** (1986), 31–77.
- [96] N.D. Halsey, *Potential flow analysis of multielement airfoils using conformal mapping*, AIAA J. **17** (1979), 1281–1288.
- [97] N.D. Halsey, *Comparison of the convergence characteristics of two conformal mapping methods*, AIAA J. **20** (1982), 724–726.
- [98] J. Hammerschick, *Über die discrete Form der Integralgleichungen von Garrick zur konformen Abbildung von Ringgebieten*, Mitt. Math. Sem. Giessen **70** (1966).
- [99] A.N. Harrington, *Conformal mappings onto domains with arbitrarily specified boundary shapes*, J. Anal. Math. **41** (1982), 39–53.
- [100] M. Hartmann and G. Opfer, *Uniform approximation as a numerical tool for constructing conformal maps*, J. Comput. Appl. Math. **14** (1986), 193–206.
- [101] J.K. Hayes, D.K. Kahaner and R.G. Kellner, *An improved method for numerical conformal mapping*, Math. Comp. **26** (1972), 327–334.
- [102] J.K. Hayes, D.K. Kahaner and R.G. Kellner, *A numerical comparison of integral equations of the first and second kind for conformal mapping*, Math. Comp. **29** (1975), 512–521.
- [103] P. Henrici, *Einige Anwendungen der schnellen Fouriertransformation*, Int. Ser. Numer. Math. **32** (1976), 111–124.
- [104] P. Henrici, *Fast Fourier methods in computational complex analysis*, SIAM Rev. **21** (1979), 481–527.
- [105] P. Henrici, *Pointwise error estimates for trigonometric interpolation and conjugation*, J. Comput. Appl. Math. **8** (1982), 131–132.
- [106] P. Henrici, *A general theory of osculation algorithms for conformal mapping*, Linear Algebra Appl. **52/53** (1983), 361–382.
- [107] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, Wiley, New York (1986).
- [108] R. Hofmann, *Zur punktwisen konformen Abbildung unter Verwendung des harmonischen Maßes*, Beiträge Numer. Math. **1** (1974), 57–60.
- [109] H.P. Hoidn, *Osculation methods for the conformal mapping of doubly connected regions*, J. Appl. Math. Phys. (ZAMP) **33** (1982), 640–651.
- [110] H.P. Hoidn, *A reparametrisation method to determine conformal maps*, J. Comput. Appl. Math. **14** (1986), 155–161.
- [111] D. Homentcovschi, *Conformal mapping of the domain exterior to a thin region*, SIAM J. Math. Anal. **10** (1979), 1246–1257.
- [112] D. Homentcovschi, *Conformal mapping of multiply connected domains exterior to thin regions*, Z. Angew. Math. Phys. **33** (1982), 503–511.

- [113] D.M. Hough, *User's Guide of CONFPACK*, IPS Research Report 90-11, ETH, Zürich (1990).
- [114] D.M. Hough, J. Levesley and S.N. Chandler-Wilde, *Numerical conformal mapping via Chebyshev weighted solutions of Symm's integral equation*, J. Comput. Appl. Math. **46** (1993), 29–48.
- [115] D.M. Hough and N. Papamichael, *The use of splines and singular functions in an integral equation method for conformal mapping*, Numer. Math. **37** (1981), 133–147.
- [116] D.M. Hough and N. Papamichael, *An integral equation method for the numerical conformal mapping of interior, exterior and doubly-connected domains*, Numer. Math. **41** (1983), 287–307.
- [117] O. Hübner, *Funktionentheoretische Iterationsverfahren zur konformen Abbildung ein- und mehrfach zusammenhängender Gebiete*, Mitt. Math. Sem. Giessen **63** (1964).
- [118] O. Hübner, *Die Faktorisierung konformer Abbildungen und Anwendungen*, Math. Z. **92** (1966), 95–109.
- [119] O. Hübner, *Zur Numerik der Theodorsenschen Integralgleichung in der konformen Abbildung*, Mitt. Math. Sem. Giessen **140** (1979), 1–32.
- [120] O. Hübner, *Über die Anzahl der Lösungen der diskreten Theodorsen-Gleichung*, Numer. Math. **39** (1982), 195–204.
- [121] O. Hübner, *Über die Existenz einer Lösung der diskreten Theodorsen-Gleichung in der Nähe der Restriktion der Ränderzuordnungsfunktion*, Mitt. Math. Sem. Giessen **175** (1986), 27–42.
- [122] O. Hübner, *On the fast solution of a linear system arising in numerical conformal mapping*, J. Comput. Appl. Math. **16** (1986), 381–386.
- [123] O. Hübner, *The Newton method for solving the Theodorsen integral equation*, J. Comput. Appl. Math. **14** (1986), 19–30.
- [124] T. Inoue, *Algorithm for computing an inverse function of a conformal mapping by fundamental solution method*, Trans. Inform. Process. Soc. Japan **38** (1997), 377–379.
- [125] T. Inoue, *New charge simulation method for numerical conformal mapping of ring domains*, Japan J. Industr. Appl. Math. **14** (1997), 295–301.
- [126] T. Inoue, *Theoretical consideration of charge simulation method for numerical conformal mappings in a ring domain*, Int. J. Math. Math. Sci. **21** (1998), 289–298.
- [127] T. Inoue, *Numerical conformal mappings of the domain whose outer boundary is a unit circle*, Japan J. Industr. Appl. Math. **19** (2002), 249–256.
- [128] V.I. Ivanov and M.K. Trubetskov, *Handbook of Conformal Mapping with Computer-Aided Visualization* (with 1 IBM-PC floppy disk), CRC Press, Boca Raton, FL (1995).
- [129] D.C. Ives, *A modern look at conformal mapping including multiply connected regions*, AIAA J. **14** (1976), 1006–1011.
- [130] R.M. James, *A new look at two-dimensional incompressible airfoil theory*, Report MDC-J0918/01, Douglas Aircraft Company (1971).
- [131] A. Kaiser, *L^∞ -Konvergenz verschiedener Verfahren der sukzessiven Konjugation zur Berechnung konformer Abbildungen*, Dissertation, ETH, Zürich (1986).
- [132] L.W. Kantorowitsch and W.I. Krylow, *Approximation Methods in Higher Analysis*, Interscience, New York (1958).
- [133] N. Kerzman and E.M. Stein, *The Cauchy kernel, the Szegő kernel, and the Riemann mapping function*, Math. Ann. **236** (1978), 85–93.
- [134] N. Kerzman and M.R. Trummer, *Numerical conformal mapping via the Szegő kernel*, J. Comput. Appl. Math. **14** (1986), 111–123.
- [135] W. Kleiner, *Sur l'approximation de la représentation conforme par la méthode des points extrémaux de M. Leja*, Ann. Polon. Math. **14** (1964), 131–140.
- [136] M.E. Klonowska and W.J. Prosnak, *On an effective method for conformal mapping of multiply connected domains*, Acta Mech. **119** (1996), 35–52.
- [137] H. Kober, *Dictionary of Conformal Representations*, Pover Publication, New York (1952).
- [138] P. Koebe, *Über die konforme Abbildung mehrfach zusammenhängender ebener Bereiche, insbesondere solcher Bereiche, deren Begrenzung von Kreisen gebildet wird*, Jahresber. Deutsch. Math.-Verein. **15** (1906), 142–153.
- [139] P. Koebe, *Über die konforme Abbildung mehrfach zusammenhängender Bereiche*, Jahresber. Deutsch. Math.-Verein. **19** (1910), 339–348.
- [140] P. Koebe, *Über eine neue Methode der konformen Abbildung und Uniformisierung*, Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Kl. (1912), 844–848.

- [141] P. Koebe, *Abhandlung zur Theorie der konformen Abbildung: V. Abbildung mehrfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Math. Z. **2** (1919), 198–236.
- [142] C.A. Kokkinos, *A unified orthonormalization method for the approximate conformal mapping of simply and multiply-connected domains*, Proc. CMFT Conference, Nicosia, 1997, N. Papamichael et al., eds, (1999), 327–344.
- [143] C.A. Kokkinos, N. Papamichael and A.B. Sideridis, *An orthonormalization method for the approximate conformal mapping of multiply-connected domains*, IMA J. Numer. Anal. **9** (1990), 343–359.
- [144] Y. Komatu, *Ein alternierendes Approximationsverfahren fuer konforme Abbildung von einem Ringgebiete auf einen Kreisring*, Proc. Japan Acad. **21** (1945), 146–155 (1949).
- [145] W. von Koppenfels and F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1959).
- [146] Z. Kosma, *On an algorithm for conformal mapping of plane multiply-connected domains*, Bull. Pol. Acad. Sci. **35** (1987), 41–52.
- [147] W. Krabs und G. Opfer, *Eine Methode zur Lösung des komplexen Approximationsproblems mit einer Anwendung auf konforme Abbildungen*, Z. Angew. Math. Mech. **55** (1975), T208–T211.
- [148] R. Kreß, *Über die numerische Berechnung konjugierter Funktionen*, Computing **10** (1972), 177–187.
- [149] P.A. Krutitskii, *Equivalence of the problem of the directional derivative and the Riemann–Hilbert problem*, Differ. Equ. **28** (1992), 698–707.
- [150] R. Kühnau, *Numerische Realisierung konformer Abbildungen durch Interpolation*, Z. Angew. Math. Mech. **63** (1983), 631–637.
- [151] U. Kulisch, *Ein Iterationsverfahren zur konformen Abbildung des Einheitskreises auf einen Stern*, Z. Angew. Math. Mech. **43** (1963), 403–410.
- [152] P.K. Kythe, *Computational Conformal Mapping*, Birkhäuser, Boston (1998).
- [153] H. Lamb, *Hydrodynamics*, 6th edn, Cambridge University Press, Cambridge (1993).
- [154] L. Landweber and T. Miloh, *Elimination of corners in the mapping of a closed curve*, J. Engrg. Math. **6** (1972), 369–375.
- [155] M. Lanza de Cristoforis, *A note on conformal representation in Sobolev spaces*, Complex Var. **20** (1992), 121–133.
- [156] R. Laugesen, *Conformal mapping of long quadrilaterals and thick doubly connected domains*, Constr. Approx. **10** (1994), 523–554.
- [157] V.I. Lavrik and V.N. Savenkov, *A Handbook on Conformal Mappings*, Naukova Dumka, Kiev (1970) (in Russian).
- [158] M.A. Lawrentjew und B.W. Schabat, *Methoden der komplexen Funktionentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin (1967).
- [159] B. Lee and M.R. Trummer, *Multigrid conformal mapping via the Szegő kernel*, Electron. Trans. Numer. Anal. **2** (1994), 22–43.
- [160] R.S. Lehman, *Development of the mapping function at an analytic corner*, Pacific J. Math. **7** (1957), 1437–1449.
- [161] F. Leja, *Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme*, Ann. Polon. Math. **4** (1957), 8–13.
- [162] F.D. Lesley, *Hölder continuity of conformal mappings at the boundary via the strip method*, Indiana Univ. Math. J. **31** (1982), 341–354.
- [163] J. Levesley, D.M. Hough and S.N. Chandler-Wilde, *A Chebyshev collocation method for solving Symm's integral equation for conformal mapping: A partial error analysis*, IMA J. Numer. Anal. **14** (1993), 57–79.
- [164] D. Levin, N. Papamichael and A. Sideridis, *The Bergman kernel method for the numerical conformal mapping of simply connected domains*, J. Inst. Math. App. **22** (1978), 171–187.
- [165] B.C. Li and S. Syngellakis, *Numerical conformal mapping based on the generalised conjugation operator*, Math. Comp. **67** (1998), 619–639.
- [166] I. Lind, *Some problems related to iterative methods in conformal mapping*, Ark. Mat. **7** (1967), 101–143.
- [167] M.V. Lotfullin, *A numerical method of conformal mapping of simply connected domains*, Trudy Sem. Kraev. Zadacham **22** (1985), 148–151 (in Russian).
- [168] P. Luchini and F. Manzo, *Flow around simply and multiply connected bodies: A new iterative scheme for conformal mapping*, AIAA J. **27** (1989), 345–351.

- [169] P. Luchini and F. Manzo, *A fast conformal mapping algorithm with no FFT*, J. Comput. Phys. **101** (1992), 368–374.
- [170] D.E. Marshall and S. Rohde, *The Loewner differential equation and slit mappings*, Preprint (2003).
- [171] V.V. Maymeskul, E.B. Saff and N.S. Stylianopoulos, *L^2 -approximation of power and logarithmic functions with applications to numerical conformal mapping*, Numer. Math. **91** (2002), 503–542.
- [172] A. Mayo, *The fast solution of Poisson's and the biharmonic equations on irregular regions*, SIAM J. Numer. Anal. **21** (1984), 285–299.
- [173] A. Mayo, *Rapid methods for the conformal mapping of multiply connected regions*, J. Comput. Appl. Math. **14** (1986), 143–153.
- [174] W. McLean, *A spectral Galerkin method for a boundary integral equation*, Math. Comp. **47** (1986), 597–607.
- [175] D.I. Meiron, S.A. Orszag and M. Israeli, *Applications of numerical conformal mapping*, J. Comput. Phys. **40** (1981), 345–360.
- [176] R. Menikoff and C. Zemach, *Methods for numerical conformal mapping*, J. Comput. Phys. **36** (1980), 366–410.
- [177] K. Menke, *Extremalpunkte und konforme Abbildung*, Math. Ann. **195** (1972), 292–308.
- [178] K. Menke, *Bestimmung von Näherungen für die konforme Abbildung mit Hilfe von stationären Punktsystemen*, Numer. Math. **22** (1974), 111–117.
- [179] K. Menke, *Point systems with extremal properties and conformal mapping*, Numer. Math. **54** (1988), 125–143.
- [180] K. Menke, *Conformal mapping of doubly connected regions*, Complex Var. **14** (1990), 251–260.
- [181] S.G. Mikhlin, *Integral Equations and Their Applications*, Pergamon Press, New York (1957).
- [182] H. Mizumoto and H. Hara, *Determination of the moduli of ring domains by finite element methods*, Int. J. Differential Equations Appl. **3** (2001), 325–337.
- [183] G. Monegato and L. Scuderi, *Global polynomial approximation for Symm's equation on polygons*, Numer. Math. **86** (2000), 655–683.
- [184] G. Moretti, *Orthogonal grids around difficult bodies*, AIAA J. **30** (1992), 933–938.
- [185] A.H.M. Murid, M.Z. Nashed and M.R.M. Razali, *Numerical conformal mapping for exterior regions via the Kerzman–Stein kernel*, J. Integral Equations Appl. **10** (1998), 517–532.
- [186] A.H.M. Murid, M.Z. Nashed and M.R.M. Razali, *Some integral equations related to the Riemann map*, Proc. CMFT Conference, Nicosia, 1997, N. Papamichael et al., eds, (1999), 405–419.
- [187] N.I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen (1953).
- [188] S.R. Nasyrov and D.A. Fokin, *Proof of the convergence of an approximate method of conformal mapping of the exterior of an airfoil onto the exterior of a disk*, Comput. Methods. Math. Phys. **33** (1993), 1447–1457.
- [189] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York (1952).
- [190] J. von Neumann, *Functional Operators, Vol II. The Geometry of Orthogonal Spaces*, Ann. of Math. Stud., Vol. 22, Princeton Univ. Press (1950).
- [191] W. Niethammer, *Iterationsverfahren bei der konformen Abbildung*, Computing **1** (1966), 146–153.
- [192] J.L. Nieto, J. Hureau and M. Mudry, *A general method for practical conformal mapping of aerodynamic shapes*, Eur. J. Mech. B Fluids **10** (1991), 147–160.
- [193] J.A. Nitsche, *Finite-element methods for conformal mapping*, SIAM J. Numer. Anal. **26** (1989), 1525–1533.
- [194] S.T. O'Donnell and V. Rokhlin, *A fast algorithm for the numerical evaluation of conformal mappings*, SIAM J. Sci. Stat. Comput. **10** (1989), 475–487.
- [195] H. Ogata, D. Okano and K. Amano, *Numerical conformal mapping of periodic structure domains*, Japan J. Industr. Appl. Math. **19** (2002), 257–275.
- [196] D. Okano, H. Ogata and K. Amano, *A method of numerical conformal mapping of curved slit domains by the charge simulation method*, J. Comput. Appl. Math. **152** (2003), 441–450.
- [197] G. Opfer, *Die Bestimmung des Moduls zweifach zusammenhängender Gebiete mit Hilfe von Differenzenverfahren*, Arch. Ration. Mech. Anal. **32** (1969), 281–297.
- [198] G. Opfer, *Über einige Approximationsprobleme im Zusammenhang mit konformen Abbildungen*, L. Collatz and G. Meinardus, eds, Numer. Meth. der Approximationstheorie **1** (1972), 93–116.
- [199] G. Opfer, *Konforme Abbildung und ihre numerische Behandlung*, Überbl. Math. **7** (1974), 33–113.

- [200] G. Opfer, *Ueber die Approximation der Identität im Komplexen*, Numer. Meth. der Approximationstheorie **2** (1975), 93–100.
- [201] G. Opfer, *New extremal properties for constructing conformal mappings*, Numer. Math. **32** (1979), 423–429.
- [202] G. Opfer, *Conformal mappings onto prescribed regions via optimization techniques*, Numer. Math. **35** (1980), 189–200.
- [203] G. Opitz, *Zur Konvergenz bei genäherter konformer Abbildung*, Z. Angew. Math. Mech. **30** (1950), 337–346.
- [204] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York (1970).
- [205] A.M. Ostrowski, *Math. Miscellen XV: Zur konformen Abbildung einfach zusammenhängender Gebiete*, Jahresber. Deutsch. Math.-Verein. **38** (1929), 168–182.
- [206] A.M. Ostrowski, *On the convergence of Theodoresen's and Garrick's method of conformal mapping*, Nat. Bur. Standards Appl. Mat. Ser. **18** (1952), 149–163.
- [207] N. Papamichael, *Numerical conformal mapping onto a rectangle with applications to the solution of Laplacian problems*, J. Comput. Appl. Math. **28** (1989), 63–83.
- [208] N. Papamichael and C.A. Kokkinos, *Two numerical methods for the conformal mapping of simply-connected domains*, Comput. Methods Appl. Mech. Engrg. **28** (1981), 285–307.
- [209] N. Papamichael and C.A. Kokkinos, *Numerical conformal mapping of exterior domains*, Comput. Methods Appl. Mech. Engrg. **31** (1982), 189–203.
- [210] N. Papamichael and C.A. Kokkinos, *The use of singular functions for the approximate conformal mapping of doubly-connected domains*, SIAM J. Sci. Stat. Comput. **5** (1984), 684–700.
- [211] N. Papamichael, C.A. Kokkinos and M.K. Warby, *Numerical techniques for conformal mapping onto a rectangle*, J. Comput. Appl. Math. **20** (1987), 349–358.
- [212] N. Papamichael, I.E. Pritsker, E.B. Saff and N.S. Stylianopoulos, *Approximation of conformal mappings of annular regions*, Numer. Math. **76** (1997), 489–513.
- [213] N. Papamichael and E.B. Saff, *Local behaviour of the error in the Bergman kernel method for numerical conformal mapping*, J. Comput. Appl. Math. **46** (1993), 65–75.
- [214] N. Papamichael and N.S. Stylianopoulos, *A domain decomposition method for conformal mapping onto a rectangle*, Constr. Approx. **7** (1991), 349–379.
- [215] N. Papamichael and N.S. Stylianopoulos, *A domain decomposition method for approximating the conformal modules of long quadrilaterals*, Numer. Math. **62** (1992), 213–234.
- [216] N. Papamichael and N.S. Stylianopoulos, *On the theory and application of a domain decomposition method for computing conformal modules*, J. Comput. Appl. Math. **50** (1994), 33–50.
- [217] N. Papamichael and M.K. Warby, *Pole-type singularities and the numerical conformal mapping of doubly-connected domains*, J. Comput. Appl. Math. **10** (1984), 93–106.
- [218] N. Papamichael and M.K. Warby, *Stability and convergence properties of Bergman kernel methods for numerical conformal mapping*, Numer. Math. **48** (1986), 639–669.
- [219] N. Papamichael, M.K. Warby and D.M. Hough, *The determination of the poles of the mapping function and their use in numerical conformal mapping*, J. Comput. Appl. Math. **9** (1983), 155–166.
- [220] N. Papamichael, M.K. Warby and D.M. Hough, *The treatment of corner and pole-type singularities in numerical conformal mapping techniques*, J. Comput. Appl. Math. **14** (1986), 163–191.
- [221] C. Pommerenke, *Polynome und konforme Abbildung*, Monatsh. Math. **69** (1965), 58–61.
- [222] C. Pommerenke, *Über die Verteilung der Fekete-Punkte*, Math. Ann. **168** (1967), 111–127.
- [223] C. Pommerenke, *Über die Verteilung der Fekete-Punkte, II*, Math. Ann. **179** (1969), 212–218.
- [224] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin (1992).
- [225] R.M. Porter, *An interpolating polynomial method for numerical conformal mapping*, SIAM J. Sci. Comput. **23** (2001), 1027–1041.
- [226] I.I. Priwalow, *Randeigenschaften analytischer Funktionen*, 2. Auflage, VEB Deutscher Verlag der Wissenschaften, Berlin (1956).
- [227] W.J. Prosnak, *Conformal representation of arbitrary multiconnected airfoil sections*, Bull. Acad. Pol. Sci. **25** (1977), 25–36.
- [228] B.I. Rabinovich and Yu.V. Tyurin, *A recursive numerical method in the conformal mapping of doubly connected regions onto a circular annulus*, Soviet Math. Dokl. **28** (1983), 447–450.

- [229] B.I. Rabinovich and Yu.V. Tyurin, *On a recursive numerical method in conformal mapping*, Soviet Math. Dokl. **28** (1983), 415–418.
- [230] B.I. Rabinovich, Yu.V. Tyurin, A.A. Livshits and A.S. Leviant, *Recursive numerical algorithm for conformal mapping in two-dimensional hydrodynamics*, AIAA J. **34** (1996), 2014–2020.
- [231] M.R.M. Razali, M.Z. Nashed and A.H.M. Murid, *Numerical conformal mapping via the Bergman kernel* J. Comput. Appl. Math. **82** (1997), 333–350.
- [232] M.R.M. Razali, M.Z. Nashed and A.H.M. Murid, *Numerical conformal mapping via the Bergman kernel using the generalized minimum residual method*, Comput. Math. Appl. **40** (2000), 157–164.
- [233] L. Reichel, *On polynomial approximation in the complex plane with application to conformal mapping*, Math. Comp. **44** (1985), 425–433.
- [234] L. Reichel, *A fast method for solving certain integral equations of the first kind with application to conformal mapping*, J. Comput. Appl. Math. **14** (1986), 125–142.
- [235] B. Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, Gesammelte mathematische Werke und wissenschaftlicher Nachlaß, R. Dedekind and H. Weber, eds, Teubner, Leipzig (1876).
- [236] E.V. Sallinen, *Approximate determination of conformal mappings by means of methods of continuation and the Newton–Raphson method*, Latv. Mat. Ezhegodnik **29** (1985), 236–242 (in Russian).
- [237] S.B. Samli, *Über die Konstruktion von Polynomen, die ein gegebenes Gebiet näherungsweise auf einen Kreis abbilden*, Z. Angew. Math. Mech. **43** (1963), T32–T35.
- [238] J. Saranen and G. Vainikko, *Two-grid solution of Symm's integral equation*, Math. Nachr. **177** (1996), 265–279.
- [239] M. Schiffer and N.S. Hawley, *Connections and conformal mapping*, Acta Math. **107** (1962), 175–274.
- [240] R. Schinzinger and P.A.A. Laura, *Conformal Mapping: Methods and Applications*, Elsevier, Amsterdam (1991).
- [241] A. Seidl and H. Klose, *Numerical conformal mapping of a towel-shaped region onto a rectangle*, SIAM J. Sci. Stat. Comput. **6** (1985), 833–842.
- [242] S. Semmes, *Nonlinear Fourier analysis*, Bull. Amer. Math. Soc. **20** (1989), 1–18.
- [243] M. Skwarczyński, *A general description of the Bergman projection*, Ann. Polon. Math. **46** (1985), 311–315.
- [244] M. Skwarczyński, *Alternating projections in complex analysis*, Compl. Anal. Appl., Varna, 1983, Sofia (1985), 192–199.
- [245] E.J. Song, *Methods for solving Theodorsen's equation in numerical conformal mapping*, Master thesis, Nagoya Univ. (1988).
- [246] E. Song and H. Sugiura, *Automatic numerical conformal mapping based on the Wegmann's method*, Trans. Inform. Proc. Soc. Japan **35** (1994), 309–312 (in Japanese).
- [247] E. Song, H. Sugiura and T. Sakurai, *Analysis of instability and stabilization on Wegmann's method for conformal mapping*, Trans. Inform. Proc. Soc. Japan **32** (1991), 126–132 (in Japanese).
- [248] A.S. Sorokin, *The variational method of M.A. Lavrent'ev and the Hilbert problem for multiply connected circular domains*, Dokl. Math. **54** (1996), 547–550.
- [249] G. Steidl, *A note on fast Fourier transforms for nonequispaced grids*, Adv. Comput. Math. **9** (1998), 337–352.
- [250] F. Stenger and R. Schmidlein, *Conformal maps via Sinc methods*, Proc. CMFT Conf., Nicosia, 1997, N. Papamichael et al., eds, (1999), 505–549.
- [251] H. Sugiura and T. Torii, *Real fast Fourier transform on quasi-equidistant sample points*, J. Inform. Process. **15** (1992), 460–465.
- [252] H. Švecová, *On the Bauer's scaled condition number of matrices arising from approximate conformal mapping*, Numer. Math. **14** (1970), 495–507.
- [253] G.T. Symm, *An integral equation method in conformal mapping*, Numer. Math. **9** (1966), 250–258.
- [254] G.T. Symm, *Numerical mapping of exterior domains*, Numer. Math. **10** (1967), 437–445.
- [255] G.T. Symm, *Conformal mapping of doubly-connected domains*, Numer. Math. **13** (1969), 448–457.
- [256] G. Szegő, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, Math. Z. **9** (1921), 218–270.
- [257] G. Szegő, *On conjugate trigonometric polynomials*, Amer. J. Math. **65** (1943), 532–536.
- [258] T. Theodorsen, *Theory of wing sections of arbitrary shape*, NACA Report 411 (1931).

- [259] T. Theodorsen and I.E. Garrick, *General potential theory of arbitrary wing sections*, NACA Report 452 (1933).
- [260] A.D. Thomas, *Conformal mapping of nonsmooth domains via the Kerzman–Stein integral equation*, J. Math. Anal. Appl. **200** (1996), 162–181.
- [261] J.F. Thompson, Z.U.A. Warsi and C.W. Mastin, *Boundary fitted coordinate systems for numerical solution of partial differential equations – A review*, J. Comput. Phys. **47** (1982), 1–108.
- [262] J.F. Thompson, Z.U.A. Warsi and C.W. Mastin, *Numerical Grid Generation*, North-Holland, Amsterdam (1985).
- [263] R. Timman, *The direct and the inverse problem of airfoil theory: A method to obtain numerical solutions*, Report F 16, Nat. Luchtvl. Labor., Amsterdam (1951).
- [264] L.N. Trefethen, *Numerical computation of the Schwarz–Christoffel transformation*, SIAM J. Sci. Stat. Comput. **1** (1980), 82–102.
- [265] L.N. Trefethen, ed., *Numerical Conformal Mapping*, Elsevier, New York (1986).
- [266] M.R. Trummer, *An efficient implementation of a conformal mapping method based on the Szegő kernel*, SIAM J. Numer. Anal. **23** (1986), 853–872.
- [267] A.G. Ugodčikov, *Construction of Functions Yielding Conformal Mappings, by Means of Electro-Modelling and the Lagrange Interpolation Polynomials*, Naukova Dumka, Kiev (1966) (in Russian).
- [268] P.N. Vabishchevich and S.I. Pulatov, *A numerical algorithm for conformal mapping*, Mat. Model. **1** (1989), 132–139 (in Russian).
- [269] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press, London (1992).
- [270] B.A. Vertgeim, *Approximate construction of some conformal mappings*, Dokl. Akad. Nauk SSSR **119** (1958), 12–14 (in Russian).
- [271] E.A. Volkov, *Block method for solving the Laplace equation and for constructing conformal mappings*, CRC Press, Boca Raton, FL (1994).
- [272] A. Wahl, *Konforme Abbildung durch gebrochene rationale Funktionen*, Z. Angew. Math. Mech. **44** (1964), 397–399.
- [273] S.E. Warschawski, *Über einen Satz von O.D. Kellogg*, Nachr. Ges. Wiss. Gött. **25** (1932), 73–86.
- [274] S.E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc. **38** (1935), 310–340.
- [275] S.E. Warschawski, *On conformal mapping of variable regions*, Nat. Bur. Standards **18** (1952), 175–187.
- [276] S.E. Warschawski, *On a theorem of L. Lichtenstein*, Pacific J. Math. **5** (1955), 835–839.
- [277] S.E. Warschawski, *On the solution of the Lichtenstein–Gershgorn integral equation in conformal mapping: I. Theory*, Nat. Bur. Standards Appl. Math. Ser. **42** (1955), 7–29.
- [278] S.E. Warschawski, *On Hölder continuity at the boundary in conformal maps*, J. Math. Mech. **18** (1968–1969), 423–427.
- [279] E. Wegert, *An iterative method for solving nonlinear Riemann–Hilbert problems*, J. Comput. Appl. Math. **29** (1990), 311–327.
- [280] E. Wegert, *Iterative methods for discrete nonlinear Riemann–Hilbert problems*, J. Comput. Appl. Math. **46** (1993), 143–163.
- [281] E. Wegert, *Nonlinear Riemann–Hilbert problems – history and perspectives*, Proc. CMFT Conf., Nicosia, 1997, N. Papamichael et al., eds, (1999), 583–615.
- [282] R. Wegmann, *Convergence proofs and error estimates for an iterative method for conformal mapping*, Numer. Math. **44** (1984), 435–461.
- [283] R. Wegmann, *Ein Iterationsverfahren zur konformen Abbildung*, Numer. Math. **30** (1978), 453–466; as: *An iterative method for conformal mapping*, Numerical Conformal Mapping, L.N. Trefethen, ed., North-Holland, Amsterdam (1986), 7–18.
- [284] R. Wegmann, *On Fornberg’s numerical method for conformal mapping*, SIAM J. Numer. Anal. **23** (1986), 1199–1213.
- [285] R. Wegmann, *An iterative method for the conformal mapping of doubly connected regions*, J. Comput. Appl. Math. **14** (1986), 79–98.
- [286] R. Wegmann, *Discrete Riemann–Hilbert problems, interpolation of simply closed curves, and numerical conformal mapping*, J. Comput. Appl. Math. **23** (1988), 323–352.
- [287] R. Wegmann, *Conformal mapping by the method of alternating projections*, Numer. Math. **56** (1989), 291–307.

- [288] R. Wegmann, *Discretized versions of Newton type iterative methods for conformal mapping*, J. Comput. Appl. Math. **29** (1990), 207–224.
- [289] R. Wegmann, *An estimate for crowding in conformal mapping to elongated regions*, Complex Var. **18** (1992), 193–199.
- [290] R. Wegmann, *Crowding for analytic functions with elongated range*, Constr. Approx. **10** (1994), 179–186.
- [291] R. Wegmann, *Fast conformal mapping of an ellipse to a simply connected region*, J. Comput. Appl. Math. **72** (1996), 101–126.
- [292] R. Wegmann, *Constructive solution of a certain class of Riemann–Hilbert problems on multiply connected circular regions*, J. Comput. Appl. Math. **130** (2001), 139–161.
- [293] R. Wegmann, *Fast conformal mapping of multiply connected regions*, J. Comput. Appl. Math. **130** (2001), 119–138.
- [294] L. von Wolfersdorf, *Zur Unität der Lösung der Theodorsenschen Integralgleichung der konformen Abbildung*, Z. Anal. Anwendg. **3** (1984), 523–526.
- [295] Y. Yan and I.H. Sloan, *On integral equations of the first kind with logarithmic kernels*, J. Integral Equations Appl. **1** (1988), 549–579.
- [296] H. Yoshikawa, *On the conformal mapping of nearly circular domains*, J. Math. Soc. Japan **12** (1960), 174–186.
- [297] C. Zemach, *A conformal map formula for difficult cases*, J. Comput. Appl. Math. **14** (1986), 207–215.

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CHAPTER 10

Univalent Harmonic Mappings in the Plane*

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1. Introduction

Let D be a domain of the extended complex plane $\overline{\mathbb{C}}$. A harmonic mapping is a complex valued function $w = f = u + iv$ of $z = x + iy$, which satisfies $\Delta f = 4f_{z\bar{z}} \equiv 0$ on D , where

$$f_z = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

and where u and v are real-valued harmonic functions on D . Observe that we do not require f to be univalent on D . For instance, any analytic or anti-analytic function is a harmonic mapping. Since u and v are real parts of locally analytic functions defined on D , it follows that f admits the representation

$$f(z) = h(z) + \overline{g(z)}, \quad (1)$$

where h and g are locally analytic on D . For example, $f(z) = z - 1/\bar{z} + 2 \ln|z|$ is a univalent harmonic mapping from the exterior of the unit disk U onto $\mathbb{C} \setminus \{0\}$, where $h(z) = z + \log z$ and $g(z) = \log z - 1/z$. If D is a simply connected domain of $\mathbb{C}$, then h and g are (globally) analytic functions on D . On the other hand, $h' = f_z$ and $g' = \bar{f}_{\bar{z}}$ are always (globally) analytic functions on D .

Lewy [L2] has shown the equivalence of the local univalence of a harmonic map with the nonvanishing of its Jacobian.

$$J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2. \quad (2)$$

We have the following theorem.

THEOREM 1.1. *A harmonic mapping is locally univalent in a neighborhood of a point z_0 if and only if its Jacobian $J_f(z)$ does not vanish at z_0 .*

REMARK 1.1. (1) Theorem 1.1 fails to be true for harmonic mappings in higher dimensions. The following counter-example is due to Wood [W5]. Define

$$\begin{aligned} u(x_1, x_2, x_3) &= x_1^3 - 3x_1x_3^2 + x_2x_3, \\ v(x_1, x_2, x_3) &= x_2 - 3x_1x_3, \\ w(x_1, x_2, x_3) &= x_3. \end{aligned}$$

Then the mapping $f = (u, v, w)$ is harmonic and univalent on $\mathbb{R}^3$ but the Jacobian vanishes on the plane $x = 0$.

(2) Starkov [S6] has studied in detail the behavior of locally univalent harmonic maps.

In contrast to the linear space $H(D)$ of analytic functions, the product and the composition of two harmonic mappings are in general not harmonic. Furthermore, neither the reciprocal $1/f$ nor the inverse f^{-1} (whenever they exist) of a harmonic mapping f are

in general harmonic. However, the composition of a harmonic mapping with a conformal premapping is a harmonic mapping. Moreover, an affine transformation applied to a harmonic mapping is also harmonic. Reich [R2,R3] has given a complete description of the harmonic mappings f and g with the property that $g \circ f$ is also harmonic. In particular, as a special case, he obtains the following Choquet–Deny theorem [C1] for open harmonic maps with nonnegative Jacobian (sense-preserving harmonic mappings).

THEOREM 1.2. *Suppose f is a sense-preserving harmonic homeomorphism and is neither analytic nor affine. Then f^{-1} is also harmonic if and only if*

$$f(z) = D + Az + B \log \frac{C - e^{-Az/B}}{\overline{C - e^{-Az/B}}},$$

where A, B, C and D are nonzero complex constants and $|C| > \sup_z |e^{-Az/B}|$.

We now show that harmonic mappings can be characterized as solutions of a certain system of linear partial differential equations. Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic mapping on D . Then h' and g' are analytic functions on D and hence $a = \frac{g'}{h'}$ is either meromorphic on D or $a \equiv \infty$. We shall call the function $a(z)$ the second dilatation function of f . From the equation $g' = ah'$ we conclude that f is a solution of the system of linear partial differential equations

$$\overline{f_z(z)} = a(z) f_z(z). \quad (3)$$

For the converse we have the following theorem.

THEOREM 1.3. *Let $w = f(z)$ be a complex-valued function in $C''(D)$ such that its image $f(D)$ is not a linear segment nor a point. Then f is harmonic on D if, and only if, either $\bar{f}$ is analytic on D or f is a solution of (3), where $a(z)$ is a meromorphic function on D such that $a \not\equiv e^{i\alpha}$ for some real α .*

Since the absolute value $|f|$ of a harmonic map f is subharmonic, it follows that f satisfies the maximum modulus principle. Furthermore, if f is not a constant, we conclude from (1) that the inverse image of a point is a union of points and analytic arcs. We say that a continuous map f is light, if the image of each continuum is a continuum. There are harmonic mappings which are not light. For instance, $z + \bar{z}$ and $z + \bar{z} - z^2 + \bar{z}^2$ map the imaginary axis onto the origin and $z - 1/\bar{z}$ maps the whole unit circle ∂U onto the origin. Let p and q be two analytic polynomials of degree n and m . If $n \neq m$, then the harmonic polynomial $P = p + \bar{q}$ is light. Let $Z(f, D)$ be the zero set of a function f defined on D and denote by $NZ(f, D)$ its cardinality. Then the valency $V(f, D)$ of a function f on D is defined by $V(f, D) = \max\{NZ(f - w, D) : w \in \mathbb{C}\}$. Furthermore, we define the valency of f at a point $z_0 \in D$ by $V(f, z_0) = \inf\{V(f, |z - z_0| < r) : r > 0\}$. For an analytic function h the valency of h at z_0 is the order of the zero of $f(z) - f(z_0)$ at z_0 . A harmonic mapping f is light on D if and only if $V(f, z)$ is finite for all $z \in D$.

A complete characterization of the local behavior of a light harmonic mapping has been given by Lyzzaik in [L3]. In particular, he has shown the following:

THEOREM 1.4. *Let $f = h + \bar{g}$ be a light harmonic mapping defined on D and let $z_0 \in D$.*

- (1) *If $|a(z_0)| < 1$, then $V(f, z_0) = V(h, z_0)$.*
- (2) *If $|a(z_0)| > 1$, then $V(f, z_0) = V(g, z_0)$.*
- (3) *If $|a(z_0)| = 1$, $a(z) = c_0 + \sum_{k=m}^{\infty} c_k(z - z_0)^k$, $c_m \neq 0$ and $h(z) = a_0 + \sum_{k=n}^{\infty} a_k(z - z_0)^k$, $a_n \neq 0$, then we have $n + m \leq V(f, z_0) \leq n + 2m$.*

EXAMPLE 1.1. Consider the harmonic mapping $f(z) = \bar{z} + z^2/2$. Then we have $V(f, \mathbb{C}) = 4$, $V(f, 0) = 1$, $V(f, -1) = 2$ and $V(f, 1) = 3$. This shows that both bounds for $V(f, z_0)$ are sharp.

Suppose now that f is a univalent harmonic mapping defined on D . Then, either f is sense-preserving or sense-reversing. In the first case, the Jacobian J_f is strictly positive on D and the second dilatation function a of f is analytic and satisfies $|a(z)| < 1$ on D . If the second case holds, then $\bar{f}$ is sense-preserving. Suppose that f is a univalent harmonic sense-preserving mapping. Then we may interpret (3) as a generalized Cauchy–Riemann equation. Indeed, using real notation, equation (2) is equivalent to

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} -\operatorname{Im}\{p\} & \operatorname{Re}\{p\} \\ -\operatorname{Re}\{p\} & -\operatorname{Im}\{p\} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix},$$

where p is the analytic function $(1+a)/(1-a)$. Observe that $\operatorname{Re}\{p\} > 0$ on U and if $a(0) = 0$, then p belongs to the Toeplitz class $\mathfrak{P}$. One easily recognizes the Cauchy–Riemann equation if $a \equiv 0$. It follows that univalent sense-preserving harmonic mappings are locally quasiconformal. Observe that we allow $|a(z)|$ to approach 1 as z approaches the boundary ∂D of D . Therefore, the boundary behavior of univalent harmonic mappings is not similar to the boundary behavior of quasiconformal mappings. The following two examples show the difference.

EXAMPLE 1.2. The mappings $f(z) = z - 1/\bar{z} + C \ln|z|$, $|C| \leq 2$, are univalent sense-preserving and harmonic on the exterior Δ of the unit disk U and we have $f(\Delta) = \mathbb{C} \setminus \{0\}$. The whole unit circle is mapped onto the origin. For more details see [HS3].

EXAMPLE 1.3. The Carathéodory kernel theorem does not hold. The mappings

$$f_n(z) = \frac{1}{2} \log \frac{1+z}{1-z} + 2 \operatorname{Re} \int \frac{(n-1)iz}{(n-(n-1)iz)(1-z^2)} dz$$

are univalent and harmonic on U and $f_n(U)$ is the horizontal strip $\Omega = \{w; |\operatorname{Im} w| < \pi/4\}$. Furthermore, the sequence converges locally uniformly to the univalent harmonic mapping

$$f(z) = \frac{1}{2} \log \frac{1+z}{1-z} + 2 \operatorname{Re} \int \frac{iz}{(1-iz)(1-z^2)} dz.$$

The image $f(U)$ is the triangle with vertices $\frac{\pi}{2} + i\frac{\pi}{4}$, $-\frac{\pi}{2} + i\frac{\pi}{4}$ and $-i\frac{\pi}{4}$ which is not the kernel Ω . For more details see [HS3].

In general, we have the following theorem.

THEOREM 1.5. *The limit function f of a locally uniformly convergent sequence of univalent harmonic mappings f_n on D is either univalent on D , is a constant, or its image lies on a straightline.*

A sense-preserving harmonic mapping f on D is locally of the form

$$f(z) = f(z_0) + A(z - z_0)^n + \overline{B(z - z_0)^n} + o(|z - z_0|^n); \quad |B| < |A|, \quad n \in \mathbb{N}. \quad (4)$$

If $f(z_0) = 0$, then we say that f has a zero of order n at z_0 and we denote by $NMZ(f, D)$ the number of zeros of f , multiplicity counted. Conversely, a sense-reversing harmonic mapping f on D is locally of the form (4) with $|A| < |B|$. We say that a point z_0 is an irregular point if $|A| = |B|$. The following argument principle for harmonic mappings in D is shown in [DHL1] and [HS4].

THEOREM 1.6 (Generalized argument principle for harmonic mappings). *Let $f(z)$ be a harmonic mapping defined on the closure $\overline{D}$ of a Jordan domain D . Consider the sets $S_G(D) = \{z: |a(z)| > 1\} \cap \overline{D}$ and $S_L(D) = \{z: |a(z)| < 1\} \cap \overline{D}$. Fix $w \in \mathbb{C}$ such that $Z(f - w, \mathbb{C}) \cap (\partial S_L(D) \cup \partial S_G(D))$ is empty. Then we have*

$$NMZ(f - w, S_L(D)) - NMZ(f - w, S_G(D)) = \frac{1}{2\pi} \oint_{\partial D} d \arg(f - w). \quad (5)$$

It can be applied in order to get uniqueness results as, for example, in [BHH1] and [BHH2], or to prove univalence of certain mappings (see, e.g., [BHH1] and [HS4]). Clunie and Sheil-Small [CS1] used the argument principle to show the following result:

THEOREM 1.7. *Let f_n be a sequence of univalent harmonic mappings defined on D such that $f_n(z_0) = 0$ for some $z_0 \in D$ and suppose that they converge locally uniformly to f . Then $f(D)$ lies in the kernel of $\{f_n(D)\}$.*

The following Schwarz lemma is due to Heinz [H1].

THEOREM 1.8. *Let f be a harmonic mapping from the unit disk U into itself normalized by $f(0) = 0$. Then the sharp estimate*

$$|f(z)| \leq \frac{4}{\pi} \arctan(|z|), \quad z \in U,$$

holds. Equality occurs, for example, for the mapping $f(z) = \omega(z, E_1) - \omega(z, E_2)$, where $E_1 = \{e^{it} : |t| < \pi/2\}$, $E_2 = \{e^{it} : \pi/2 < |t| < \pi\}$ and where $\omega(z, E)$ denotes the harmonic measure of E evaluated at z .

A harmonic polynomial is of the form

$$f(z) = p_n(z) + \overline{p_m(z)} = \sum_{k=0}^n a_k z^k + \sum_{k=0}^m \overline{b_k z^k},$$

where p_N is a analytic polynomial of degree N . Harmonic polynomials and mappings do not inherit some fundamental properties of analytic polynomials and mapping. For example, $f_1(z) = z + \bar{z} + z^2 - \bar{z}^2$ vanishes on the whole imaginary axis and $f_1 - i$ has no root. Furthermore, the limit $\lim_{z \rightarrow \infty} f(z)$ does not exist in $\overline{\mathbb{C}}$. Other examples are, $f_2(z) = z + e^z + \overline{e^z}$, a two-valent entire transcendental harmonic function and $f_3(z) = z + \exp\{-z^2\} + \overline{\exp\{-z^2\}}$, an entire transcendental harmonic function which satisfies $\lim_{z \rightarrow \infty} f(z) = \infty$ [N1]. However, the following interesting result is due to Wilmhurst [W4].

THEOREM 1.9. *Let $f(z) = p_n(z) + \overline{p_m(z)}$ be a harmonic polynomial. Then either f has infinitely many roots (in which case $n = m$ and $|a_n| = |b_m|$) or f has at most N^2 roots, where $N = \max(n, m)$. This bound is sharp.*

Indeed, the harmonic polynomial $f(z) = \operatorname{Im}\{iz^n\} + i\operatorname{Im}\{(z-1)^n\}$ has n^2 roots. Wilmhurst conjectured that for $1 \leq m \leq n-1$ the maximal number of zeroes of f is $m(m-1) + 3n-2$. In [KS1] this conjecture is proved for $m=1$.

In Section 2, we shall give a survey on univalent harmonic mappings defined on a simply connected domain D of $\mathbb{C}$. Section 3 deals with univalent harmonic mappings defined on multiply connected domains. Recently, the first account ever on the theory of planar harmonic mappings was published [D1].

For recent survey articles on harmonic mappings between Riemannian manifolds, see, for example, [EL1, ES1, J1, J2, S1] and [S2].

2. Univalent harmonic mappings on a simply connected domain

2.1. Motivation

Univalent harmonic mappings are closely related to minimal surfaces. Let Ω be a domain in the (u, v) -plane and let S be a nonparametric surface over Ω . In other words, we suppose that the surface can be expressed by the function $s = s(u, v)$. Moreover, we suppose that the surface is regular, i.e., that the function $s(u, v)$ belongs to $C'(\Omega)$. Then the following characterization holds:

THEOREM 2.1. *A nonparametric regular surface S over a domain Ω is a minimal surface if and only if there is a univalent harmonic mapping $f = u + iv$ from a domain D onto Ω such that $s_z^2 = -af_z^2 = -\bar{f}_z f_z$ holds, where a is defined in (3).*

Without loss of generality we can assume that the univalent harmonic mapping f is sense-preserving, i.e., that its second dilatation function a has modulus less than one. It is interesting to note that the normal vector $\mathbf{n}$ of the surface S , called the Gauss map, depends only on the second dilatation function a . Indeed, we have

$$\mathbf{n} = (2 \operatorname{Im}\{\sqrt{a}\}, 2 \operatorname{Re}\{\sqrt{a}\}, 1 - |a|) / (1 + |a|). \quad (6)$$

Observe that $\mathbf{n}$ is vertical if and only if $a = 0$ and it is horizontal if and only if $|a| = 1$. The differential condition for $s(u, v)$ implies that the second dilatation function a is the square of an analytic function on D . Applying the theory of univalent harmonic mappings, Hengartner and Schober [HS5] gave sharp estimates for the Gaussian curvature of nonparametric minimal surfaces over some given domain Ω . On the other hand, Weitsman [W2] used the theory of minimal surfaces to prove uniqueness results for univalent harmonic mappings (see Section 2.5). For more details on minimal surfaces see, e.g., [N3] and [O3].

2.2. Univalent harmonic mappings defined on the plane

There are very few harmonic mappings which are univalent on the whole plane $\mathbb{C}$. Indeed, Clunie and Sheil-Small [CS1] have shown:

THEOREM 2.2. *The only univalent harmonic mappings defined on the plane are the affine transformations*

$$f(z) = Az + \overline{Bz} + C, \quad |A| \neq |B|. \quad (7)$$

The proof is based on the fact that the second dilatation function a of f is constant on $\mathbb{C}$ and we have $|a| \neq 1$. It follows then that $\Phi = f - \overline{af}$ is a univalent analytic function on $\mathbb{C}$. Hence, we have $\Phi = cz + d$ and Theorem 2.2 follows.

Theorem 2.2 says that there are no univalent mappings from the plane onto a proper subdomain of $\mathbb{C}$. Since, in general, the inverse of a univalent harmonic mapping is not harmonic, it is natural to ask if there are other univalent harmonic mappings whose image are $\mathbb{C}$. The answer is *no* [CS1]. A proof given in [BH2] uses the following lemma which is of independent interest.

LEMMA 2.1. *Let $f = h + \bar{g}$ be a univalent harmonic and orientation-preserving mapping from a domain D onto the domain Ω . Suppose that z_1 and z_2 , $z_1 \neq z_2$, are two points in D such that the line segment $\gamma = \{w_t = tf(z_1) + (1-t)f(z_2); 0 \leq t \leq 1\}$ belongs to Ω . Then we have*

$$|h(z_2) - h(z_1)| > |g(z_2) - g(z_1)|.$$

THEOREM 2.3. *The only univalent harmonic mappings f satisfying $f(D) = \mathbb{C}$ are of the form (7).*

REMARK 2.1. (1) Clunie and Sheil-Small [CS1] have shown that if f is a univalent sense-preserving harmonic mapping defined on the unit disk U , then each circle $\{w; |w - f(0)| = r|f_z(0)|\}$, $r \geq 2\pi\sqrt{6}/9$, contains at least one point of $\mathbb{C} \setminus f(U)$. The constant $2\pi\sqrt{6}/9 \approx 1.710$ is best possible.

(2) Theorem 2.3 contains the famous Bernstein's theorem which says that the only minimal surfaces over the whole plane are planes.

2.3. The classes S_H and S_H^0

Let D be a proper simply connected domain of $\mathbb{C}$ and f a univalent harmonic mapping from D to $\mathbb{C}$. Since the composition of a univalent harmonic mapping with a conformal premapping is a univalent harmonic map, we may assume that D is the unit disk U and that f is orientation-preserving on U . Furthermore, since f_z does not vanish on U (Theorem 1.2), we may normalize f by the linear transformation $(f(z) - f(0))/f_z(0)$. Then f admits the unique representation

$$f = h + \bar{g} = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}. \quad (8)$$

Observe that $b_1 = a(0)$.

DEFINITION 2.1. The class S_H consists of all univalent harmonic and sense-preserving mappings $f = h + \bar{g}$ which are normalized by $g(0) = h(0) = 0$ and $f_z(0) = 1$.

Applying the affine postmapping $(w - \overline{a(0)w})/(1 - |a(0)|^2)$ to f we can transform f to a function whose dilatation function vanishes at the origin.

DEFINITION 2.2. The class S_H^0 consists of all mappings $f = h + \bar{g} \in S_H$ such that $f_{\bar{z}}(0) = 0$.

REMARK 2.2. The condition $f_{\bar{z}}(0) = 0$ is equivalent to the condition $a(0) = 0$ or to $g(z) = O(z^2)$ as $z \rightarrow 0$.

Since mappings in S_H^0 are K_r -quasiconformal on the disks $\{z; |z| \leq r\}$, $0 < r < 1$, where $K_r = (1+r)/(1-r)$, it follows that S_H^0 is compact with respect to the topology of locally uniform convergence. Furthermore, we have

$$\max_{f \in S_H} \max_{|z| \leq r} |f(z)| \leq 2 \max_{f \in S_H^0} \max_{|z| \leq r} |f(z)|$$

which shows that S_H is a normal family. Note that S_H is not compact. Indeed, the affine transformations $f_n(z) = z + \frac{n}{n+1}\bar{z}$ belong to the class S_H and the sequence converges locally uniformly to $f(z) = z + \bar{z}$ which is nowhere univalent. The following interesting distortion theorem is due to Clunie and Sheil-Small [CS1].

THEOREM 2.4. *If $f \in S_H^0$, then $|f(z)| \geq \frac{z}{4(1-z)^2}$. In particular, we have $\{w; |w| < 1/16\} \subset f(U)$.*

It is not known if the above estimate is sharp. There are some indications that perhaps the factor $1/4$ can be replaced by $2/3$. A possible candidate for the extremal function is the radial slit-mapping

$$k(z) = \frac{z - z^2/2 + z^3/6}{(1-z)^3} + \frac{\overline{z^2/2 + z^3/6}}{(1-z)^3}, \quad (9)$$

whose dilatation function is $a(z) = z$.

REMARK 2.3. (1) Let L be a linear continuous functional on the set $h(U)$ of all harmonic mappings defined on U . Then we have

$$L(f) = L(h + \bar{g}) = L_1(h) + \overline{L_2(g)},$$

where L_1 and L_2 belong to $H'(U)$, the topological dual space of $H(U)$.

(2) Since S_H^0 is compact, each real continuous functional attains its maximum and its minimum on S_H^0 . Hence, there are uniform bounds for the absolute value of the coefficients a_n and b_n in (10). Applying Schwarz's lemma to the dilatation $a(z)$, one gets immediately the sharp inequality $|b_2| \leq 1/2$. The best known estimate for a_2 is so far $|a_2| \leq 49$. It is conjectured that $|a_n| \leq (2n+1)(n+1)/6$ and $|b_n| \leq (2n-1)(n-1)/6$ and that equality is attained by the mapping given in (8). Another attractive conjecture is that $|a_n| - |b_n| \leq n$ holds for all $f \in S_H^0$ and all natural n . For further investigations see, e.g., [CS1].

Abu-Muhanna and Lyzzaik [AL1] have shown the following interesting result.

THEOREM 2.5. *Let $f = h + \bar{g}$ be a univalent harmonic map defined on the unit disk U . Then there is a universal $p > 0$ such that f belongs to the standard class h^p and that h and g belong to H^p .*

Using the estimate $|a_2| \leq 49$ and following the arguments given by Abu-Muhanna and Lyzzaik, we conclude that $f \in h^p$ for all $p \in (0, 10^{-4})$.

2.4. Univalent harmonic mappings onto convex domains

2.4.1. The Radó–Kneser (–Choquet) theorem. In 1926, Radó [R1] asked for a proof of the following result:

THEOREM 2.6. *Let f^* be a homeomorphism from the unit circle ∂U onto the boundary of a bounded convex domain Ω . Then the solution $f = u + iv$ of the Dirichlet problem $\Delta f \equiv 0$ on U and $f \equiv f^*$ on ∂U (the Poisson integral of f^*) is univalent on U .*

The same year, Kneser has shown in [K1] a much stronger result.

THEOREM 2.7. *Let f^* be a homeomorphism from the unit circle ∂U onto the boundary of a bounded Jordan domain Ω . Then the solution $f = u + iv$ of the Dirichlet problem $\Delta f \equiv 0$ on U and $f \equiv f^*$ on ∂U is univalent on U if and only if $f(U) = \Omega$.*

Theorem 2.7 implies Theorem 2.6. Indeed, we have $f = \int f^* d\omega$, where the harmonic measure $d\omega$ is a probability measure. Therefore we have $\Omega \subset f(U) \subset \overline{\partial\Omega}$. If Ω is a bounded convex domain, then we conclude that $f(U) = \Omega$ and Theorem 2.6 follows.

REMARK 2.4. (1) In 1945, Choquet [C1] gave another proof for Theorem 2.6, using the Poisson integral. One may also use the following arguments which were introduced by Clunie and Sheil-Small in [CS1]. It is enough to show that $J_f \neq 0$ on U . Let $f = h + \bar{g}$ and define ϕ_α by $\phi_\alpha(z) = e^{i\alpha}h(z) - e^{-i\alpha}g(z)$. Then ϕ_α is a pointwise horizontal translation of $e^{i\alpha}f$. In other words, we have $\phi_\alpha(z) = e^{i\alpha}f(z) - 2\operatorname{Re}\{e^{-i\alpha}g(z)\}$. The mappings ϕ_α are convex in the horizontal direction and hence, conformal for all real α . Therefore, ϕ'_α does not vanish on U which implies that the Jacobian J_f does not vanish on U .

(2) Theorem 2.6 is false if Ω is not convex. This was already observed by Choquet [C1].

(3) Theorem 2.6 does not hold if Ω is an unbounded convex domain.

(4) An extension of Theorems 2.6 and 2.7 to multiply connected domains will be given in Section 3.

(5) Theorems 2.6 and 2.7 do not hold in $\mathbb{R}^n$, $n \geq 3$. Laugesen gave in [L1] an example of a homeomorphism $f^* = (f_1^*, f_2^*, f_3^*)$ from the unit sphere of $\mathbb{R}^3$ onto itself such that the Poisson integral $f = (f_1, f_2, f_3)$ maps the unit ball onto itself, but is not a univalent harmonic mapping.

(6) A conformal mapping from U onto itself is uniquely determined by the correspondence of three boundary points. Theorem 2.6 shows that there are many univalent harmonic mappings from U onto U with a given correspondence of three boundary points.

(7) Duren and Schober [DS1, DS2] used Theorem 2.6 to develop a variation for univalent harmonic mappings from the unit disk U onto a fixed convex domain Ω . In particular, they gave for the case $\Omega = U$, sharp estimates for the coefficients and the distortion of the partial derivatives. A somewhat different approach is due to Wegmann [W1].

DEFINITION 2.3. Let Ω be a simply connected Jordan domain of $\mathbb{C}$ and let Φ be a conformal mapping from U onto Ω . A function f^* from ∂U onto $\partial\Omega$ is called a *quasihomeomorphism* from ∂U onto $\partial\Omega$ if f^* is the pointwise limit of a sequences of homeomorphisms from ∂U onto $\partial\Omega$. In other words, f^* is a quasihomeomorphism on ∂U if and only if either $\psi(t) = \arg \Phi^{-1} \circ f^*(e^{it})$ (which exists a.e. on ∂U) or $-\psi(t)$ is nondecreasing on $[0, 2\pi]$, $\psi(2\pi) = \psi(0) + 2\pi$ and $f^*(\partial U) \subset \partial\Omega$. A continuous quasihomeomorphism from ∂U onto $\partial\Omega$ will be called a *weak homeomorphism* from ∂U onto $\partial\Omega$.

Note that a quasihomeomorphism can be constant on an interval of ∂U and may have jumps; but it never can change the orientation. It follows immediately that Theorem 2.7 holds also for weak homeomorphisms on ∂U . On the other hand, if f^* is a quasihomeomorphism from ∂U onto the boundary $\partial\Omega$ of a convex domain Ω , such that its range consists of at least three different points, then f is univalent on U and its image is the interior of the closed convex hull of $f^*(\partial U)$.

2.4.2. The class K_H

DEFINITION 2.4. A harmonic mapping f defined on the unit disk U belongs to the class K_H (K_H^0 resp.) if $f \in S_H$ ($f \in S_H^0$ resp.) and if $\Omega = f(U)$ is a convex domain.

Using the fact that the associated functions ϕ_α defined in (9) are univalent mappings onto domains convex in the horizontal direction if and only if f is univalent and $f(U)$ is a convex domain, Clunie and Sheil-Small [CS1] gave sharp estimates for the Fourier coefficients of f . They also have shown the remarkable result that $\{w; |w| < \frac{1}{2}\} \subset f(U)$ whenever $f \in K_H^0$ which is already best possible for normalized conformal mappings onto convex domains.

2.4.3. Other special classes. We finish Section 2.4 with some remarks on in S_H (S_H^0 resp.) which are either close-to-convex or typically real. Recall that a domain Ω is close-to-convex if the complement of $\overline{\Omega}$ can be written as a union of noncrossing open half-lines. If $f = h + \bar{g} \in K_H$, then ϕ_α defined in (9) maps U onto a close-to-convex domain for all $\alpha \in \mathbb{R}$. It follows then (see [CS1]) that $h(z) - \zeta g(z)$ is a univalent close-to-convex mapping on U for all fixed ζ , $|\zeta| \leq 1$. Conversely, Clunie and Sheil-Small [CS1] have shown the following interesting result.

THEOREM 2.8. *Let h and g be analytic in U and suppose that $|g'(0)| < |h'(0)|$. If $h(z) - \zeta g(z)$ is a univalent close-to-convex mapping defined on U for all fixed ζ , $|\zeta| = 1$, then $f = h + \bar{g}$ is a univalent harmonic mapping from U onto a close-to-convex domain.*

Observe that the univalence of f follows directly from the univalence of the mappings $h(z) - \zeta g(z)$.

A harmonic mapping f on U is called typically real if $f(z)$ is real if, and only if, z is real. For example, a univalent harmonic mapping whose Fourier coefficients are real is typically real. Furthermore, $f = h + \bar{g}$ is typically real if and only if $\phi = h - g$ is typically real. Sharp coefficient estimates have been given in [CS1]. However, there are univalent (orientation-preserving) harmonic mappings $f = h + \bar{g}$ on U which are real on the real axis with $h'(0) > 0$ and $g'(0) > 0$ but fail to be typically real (see, e.g., [BHH2]). Furthermore, there are univalent (orientation-preserving) harmonic nontypically real mappings $f = h + \bar{g}$ satisfying $f(0) = 0$ and $f_z(0) = 1$ and whose image $f(U)$ is symmetric with respect to the real axis. However, if the second dilatation function a has real coefficients, then, in both cases, f is typically real.

2.5. Mapping problems

Recall that harmonic and orientation preserving-mappings defined on the unit disk U are solutions of the system of linear elliptic partial differential equation

$$\overline{f_z(z)} = a(z) f_z(z), \quad a \in H(U), |a| < 1,$$

on U . It is natural to ask the question if for each given dilatation $a(z)$, $a \in H(U)$, $|a| < 1$, and for each given simply connected domain Ω there is a univalent solution of (3) which maps U onto Ω . Unfortunately the answer is no. Indeed, it has been shown in [HS2] that if a is a finite Blaschke product, there is no univalent harmonic mapping from U onto any bounded strictly convex domain. However, the following result has been given in [HS1].

THEOREM 2.9. *Let Ω be a given bounded domain of $\mathbb{C}$ such that its boundary $\partial\Omega$ is locally connected. Suppose that $a \in H(U)$ satisfies $|a| < 1$ on U . Choose w_0 in Ω . Then there exists a univalent solution of (3) having the following properties:*

- (i) $f(0) = w_0$, $f_z(0) > 0$ and $f(U) \subset \Omega$;
- (ii) *there is a countable set E on ∂U such that the unrestricted limits $f^*(e^{it}) = \lim_{z \rightarrow e^{it}} f(z)$ exist on $\partial U \setminus E$ and they are on $\partial\Omega$;*
- (iii) *the functions $f_-^*(e^{it}) = \text{ess lim}_{s \uparrow t} f^*(e^{is})$ and $f_+^*(e^{it}) = \text{ess lim}_{s \downarrow t} f^*(e^{is})$ exist on ∂U ;*
- (iv) *the cluster set of f at e^{it} is the line segment from $f_-^*(e^{it})$ to $f_+^*(e^{it})$.*

REMARK 2.5. (1) If $|a| \leq k < 1$ then E is empty and f admits a continuous extension to $\overline{\Omega}$. Furthermore, we have $f(U) = \Omega$. If in addition, Ω is a Jordan domain, then f extends to a homeomorphism from $\overline{U}$ onto $\overline{\Omega}$.

(2) There is no analogue theorem for multiply connected domains.

The uniqueness problem of mappings f as defined in Theorem 2.9 is still open. There are several kinds of uniqueness theorems for k -quasiconformal mappings. But none of them applies to our case. Suppose that the boundary $\partial\Omega$ is smooth enough. If one knows that two mappings f and F satisfy Theorem 2.9 and that $f_z(0) = F_z(0)$ then one can conclude that $f = F$ (see, e.g., [GD1, GD2] and [B1]). Moreover if, in addition to the conditions of Theorem 2.9, one has $|a| \leq k < 1$ and Ω is a strictly starlike domain, then f is uniquely determined [BHH1]. Uniqueness also holds for symmetric Ω if a has real coefficients [BHH2]. Using minimal surfaces, Weitsman [W2] proves the uniqueness of the mapping in Theorem 2.9 for convex domains Ω and $a(z) = z^n$.

Suppose now that the second dilatation function is a finite Blaschke product of degree N . We say that a prime end $q \in \partial\Omega$ is a point of convexity (with respect to Ω) if there exists a neighborhood V of $f^{-1}(q)$ and a line segment L containing q as an interior point such that $L \setminus \{q\}$ lies in the exterior of $f(U \cap V)$.

DEFINITION 2.5. (1) We call $\beta(\tau)$ a regulated function on the interval $[a, b]$ if the one-sided limits $\beta(\tau + 0)$ and $\beta(\tau - 0)$ exist for all $t \in [a, b]$.

(2) Let Ω be a simply connected domain of $\overline{\mathbb{C}}$ and suppose that the boundary $\partial\Omega$ is locally connected (every prime end is a singleton). Let ϕ be a conformal mapping from U onto Ω . We call Ω a regulated domain if for each prime end $q = w(\tau) = \phi(e^{i\tau})$ of $\partial\Omega$ the direction angle of the forward (half-)tangent at $w(\tau)$,

$$\beta(q) = \lim_{s \downarrow \tau} \arg[w(s) - w(\tau)] = \lim_{s \downarrow \tau} \arg[w(s) - q], \quad (10)$$

exists and defines a regulated function.

We have the following mapping theorem [BH3].

THEOREM 2.10. *Let*

$$a(z) = e^{i\gamma} \frac{1}{z^{n_0}} \prod_{k=1}^m \left[\frac{1 - p_k z}{z - \bar{p}_k} \right]^{n_k},$$

$n_0 \geq 0, n_k > 0, 0 < |p_k| < 1$ for $1 \leq k \leq m$, $p_k \neq p_j$ if $i \neq j$ and $\sum_{k=0}^m n_k = N$, be a given finite Blaschke product of degree N and let Ω be a regulated Jordan domain of $\mathbb{C}$ whose boundary has at most $N + 2$ points of convexity. Let $f^*(e^{it})$ be a positively oriented weak homeomorphism from the unit circle ∂U onto $\partial\Omega$ satisfying

$$\operatorname{Im}(\sqrt{a(e^{it})} df^*(e^{it})) = 0. \quad (11)$$

Then the Poisson integral f of f^* is a univalent solution of

$$\bar{f}_{\bar{z}} = a(z) f_z(z)$$

which maps U onto Ω , if and only if

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{a(z) - a(e^{it})}{z - e^{it}} df^*(e^{it}) \equiv 0 \quad (12)$$

holds for all z in U .

The condition (12) can be replaced by any set of $[\frac{N}{2}]$ linear functionals which together with the condition (11) form a linearly independent set.

Suppose that the Blaschke product a contains a single factor. Then (12) is trivially satisfied and we have the following theorem.

THEOREM 2.11. *Let Ω be a regulated Jordan domain. Then there exists a univalent solution f of*

$$\overline{f_{\bar{z}}(z)} = e^{i\gamma} \frac{z - p}{1 - \bar{p}z} f_z(z)$$

which maps U onto Ω if and only if $\partial\Omega$ has exactly three points of convexity. Moreover, f is unique.

Consider now the case of $a(z) = e^{i\gamma} z^2$. Then condition (12) in Theorem 2.10 reduces to the single equation

$$\int_0^{2\pi} e^{it} df^*(e^{it}) = 0,$$

which together with relation (11) gives the following result [BH3].

THEOREM 2.12. *Let Ω be a regulated Jordan domain and let $a(z) = e^{i\gamma} z^2$.*

- (1) *If $\partial\Omega$ contains three points of convexity, then there exist always at least one solution of $\tilde{f}_{\bar{z}} = a(z) f_z(z)$ which maps U onto Ω and there are at most six different ones.*
- (2) *Suppose $\partial\Omega$ has four points of convexity, w_1, w_2, w_3 and w_4 oriented in the positive direction. Denote by L_k the Euclidean length of the boundary arc which joins w_k to w_{k+1} , where $w_5 = w_1$. Then there is a solution of $\tilde{f}_{\bar{z}} = a(z) f_z(z)$ which maps U onto Ω , if and only if $L_1 + L_3 = L_2 + L_4$.*
- (3) *If $\partial\Omega$ contains more than four points of convexity then there is no univalent solution of $\tilde{f}_{\bar{z}} = a(z) f_z(z)$ which maps U onto Ω .*

To each solution f in Theorem 2.12 corresponds a minimal surface whose normal vector covers the upper half-sphere exactly once. Theorem 2.12 can be expressed in terms of minimal surfaces as follows.

COROLLARY 2.1. *Let Ω be a simply connected regulated bounded domain of $\mathbb{C}$.*

- (1) *If $\partial\Omega$ has three points of convexity, then there exists always a nonparametric regular minimal surface over Ω whose Gauss map has the property that its image is the upper half-sphere covered exactly once. There are at most three essentially different ones.*
- (2) *Suppose $\partial\Omega$ has four points of convexity, w_1, w_2, w_3 and w_4 oriented in the positive direction. Then there exists a regular nonparametric minimal surface S over Ω whose Gauss map has the property that its image is the upper half-sphere covered exactly once if and only if the w_k divide $\partial\Omega$ in four parts such that the sum of the Euclidean length of the opposite sides are equal. There is essentially only one with this property.*
- (3) *In all other cases there are no regular nonparametric minimal surfaces having the above property.*

2.6. Boundary behavior

If the second dilatation function a of a univalent harmonic mapping f satisfies $|a(z)| \leq k < 1$ for all $z \in U$, then f is a quasiconformal map and its boundary behavior is the same as for conformal mappings. However if $|a|$ approaches one as z tends to the boundary, then the boundary behavior of f is quite different. It may happen that the boundary values are constant on an interval of ∂U or that there are jumps as the following example shows.

EXAMPLE 2.1. The Poisson integral f of the boundary function

$$f^*(e^{it}) = \begin{cases} 1 & \text{if } |t| < \pi/3, \\ e^{2\pi i/3} & \text{if } \pi/3 < t < \pi, \\ e^{-2\pi i/3} & \text{if } -\pi/3 > t > -\pi, \end{cases}$$

is a univalent harmonic mapping from the unit disk onto the triangle with vertices 1, $e^{2\pi i/3}$ and $e^{-2\pi i/3}$.

THEOREM 2.13 [BH3]. *Let*

- (i) Ω be a bounded domain of $\mathbb{C}$ such that its boundary $\partial\Omega$ is locally connected,
 - (ii) $a(z) \in H(U)$, $|a| < 1$, on U admitting a continuous extension to the boundary interval $J = \{e^{it}, \beta < t < \gamma\}$, $\beta < \gamma < \beta + 2\pi$, such that $|a(e^{it})| \equiv 1$ on J ,
 - (iii) $f(z)$ be a univalent solution of (3), such that $f(U) \subset \Omega$ and that $f^*(e^{it}) = \lim_{z \rightarrow e^{it}} f(z) \in \partial\Omega$ a.e.
- Then we have

$$f^*(e^{it}) - \overline{a(e^{it})f^*(e^{it})} + \int \overline{f^*(e^{it})} da(e^{it}) \equiv \text{const} \quad (13)$$

on J .

To prove Theorem 2.13, one shows that

$$\begin{aligned} d\mu_r(t) &= df(re^{it}) - \overline{a(re^{it})} df(re^{it}) \\ &= ire^{it}(1 - |a(re^{it})|^2)h'(re^{it})dt \end{aligned}$$

converges weakly to the identical zero measure on J as r tends to one.

COROLLARY 2.2. *Let Ω , a , f and f^* be as in Theorem 2.8. Then either f^* jumps at e^{it} , or is constant in a right or left neighborhood of e^{it} or the curvature is strictly negative at $f^*(e^{it})$.*

Abu-Muhanna and Lyzzaik [AL1] gave a prime-end theory for univalent harmonic mappings. In particular, they have shown that no continuum of ∂U can be mapped onto a cusp. For the extension of this latter result we shall assume that our domain is regulated, i.e., $\partial\Omega$ has a left and right tangent at each point.

DEFINITION 2.6. Let Ω be a simply connected regulated domain of $\mathbb{C}$ and let f be a univalent harmonic orientation-preserving mapping from U onto Ω . Let q be a prime end of $\partial\Omega$.

- (1) If q does not belong to a jump of f^* , we define $\gamma(q)$ and $\delta(q)$ by $(f^*)^{-1}(q) = J(q) = \{e^{it}, \gamma(q) \leq t \leq \delta(q)\}$.
- (2) If q is an interior point of a jump, i.e., $q = \lambda f^*(e^{i(t+0)}) + (1 - \lambda)f^*(e^{i(t-0)})$, $0 < \lambda < 1$, then define $\gamma(q) = \delta(q) = t$.

- (3) If q is the end point $f^*(e^{i(t-0)})$ of a jump, then define $\gamma(q)$ as in (1) and put $\delta(q) = t$.
- (4) If q is the end point $f^*(e^{i(t+0)})$ of a jump, then put $\gamma(q) = t$ and define $\delta(q)$ as in (1).

Observe that the cluster sets $C(f^*, e^{i\gamma(q)})$ and $C(f^*, e^{i\delta(q)})$ contain q but they may also contain other points if a jump appears. Furthermore, if $J(q) = (f^*)^{-1}(q)$ is a continuum then $|a| \equiv 1$ on $J(q)$. Finally, from (13) it follows that

$$\beta(q) = \lim_{h \downarrow 0} \arg[f^*(e^{i(\delta(q)+h)}) - f^*(e^{i(\delta(q)-0)})] = -\frac{1}{2} \arg a(e^{i\delta(q)}) \bmod \pi \quad (14)$$

exists at each prime end $q \in f^*(J)$.

THEOREM 2.14 [BH3]. *Let Ω be a regulated domain and let f , a and J be as in Theorem 2.13. Let q be a prime end in $f^*(J)$ such that $J(q) = (f^*)^{-1}(q) \subset J$. Denote by $\alpha(q)$ the opening angle at q as seen from the inside of Ω . Set $A(t) = \arg a(e^{it})$, $e^{it} \in J$, as a continuous function and define $\Delta A(q) = \frac{1}{2}[A(\delta(q)) - A(\gamma(q))]$. Then we have the following relation between $\alpha(q)$ and $\Delta A(q)$.*

- (1) For $0 \leq \alpha(q) < \pi$ we have $\alpha(q) = -\Delta A(q)$.
- (2) For $\pi \leq \alpha(q) \leq 2\pi$ we have either $\alpha(q) = -\Delta A(q)$ or $\alpha(q) = -\Delta A(q) + \pi$.

Theorem 2.14 states that the total change of $-\frac{1}{2} \arg a(e^{it})$ over the interval $J(q) = f^{-1}(q)$ is either equal to the opening angle $\alpha(q)$ as seen from the inside of the domain or, if $\pi \leq \alpha(q) \leq 2\pi$, it can also be $\alpha(q) - \pi$.

Suppose now that the second dilatation function a is finite Blaschke product of degree N and that f is a solution of (3). Then (13) holds on the whole unit circle which explains the condition (11) in Theorem 2.10. Moreover, it follows that the image $f(U)$ has to be a regulated domain.

DEFINITION 2.7. A prime end $q_0 \in \partial\Omega$ is said to be a complete resting point of f^* if $-\Delta A(q_0) = \alpha(q_0)$.

REMARK 2.6. If the prime end q is an interior point of a linear segment of $f^*(I)$, then either q is an interior point of a jump of f in which case $\Delta A(q) = 0$ or the inverse image $f^{-1}(q)$ is not a singleton and we have $\Delta A(q) = -\pi$.

(2) Each prime end with an opening angle $\alpha(q)$ strictly less than π is a complete resting point of f^* . In particular if $\alpha(q) = 0$, then $(f^*)^{-1}(q)$ is a singleton yet q is still a complete resting point of f^* . On the other hand, if $\alpha(q) > \pi$, it may happen that $(f^*)^{-1}(q)$ is an interval of ∂U with nonempty interior but q is not a complete resting point.

Our next result is a direct consequence of Theorem 2.14 that combines the number of complete resting points with the degree of the Blaschke product.

THEOREM 2.15. *Let Ω be a regulated domain of $\mathbb{C}$ and let $a(z)$ be a Blaschke product of degree N on U . Let f be a univalent solution of (3) with respect to this Blaschke product*

which maps U onto Ω . Then $\partial\Omega$ has at most $N + 2$ points of convexity. Furthermore, the number of complete resting points of f^* is equal to $N + 2$.

In particular (see also [HS2]), if Ω is a bounded convex domain and if a is a finite Blaschke product containing N factors, then f^* is piecewise constant on ∂U and $f(U)$ is a polygon with $N + 2$ edges. Finally, Sheil-Small [S4] considered harmonic mappings defined on U whose boundary functions $f^*(e^{it})$ are step function.

2.7. Univalent log-harmonic mappings

Studying minimal surfaces whose Gauss map (normal vector) are periodic, we are lead to univalent harmonic mappings with periodic partial derivatives. We may restrict ourself to periods of $2\pi i$.

Let D be the left half-plane $\{z; \operatorname{Re} z < 0\}$ and consider the set $\mathcal{F}$ of all univalent harmonic and orientation-preserving mapping $F = U + iV$ defined on D such that

$$F(z + 2\pi i) \equiv F(z) + 2\pi i \quad (15)$$

on D and

$$\operatorname{Re}\{F(-\infty)\} = \lim_{x \rightarrow -\infty} \operatorname{Re}\{F(x + iy)\} = -\infty. \quad (16)$$

It follows then that each $F \in \mathcal{F}$ admits the representation

$$F(z) = z + 2\beta x + H(z) + \overline{G(z)}, \quad (17)$$

where

- (1) $\operatorname{Re} \beta > -1/2$,
- (2) H and G are analytic in D ,
- (3) $G(-\infty) = \lim_{x \rightarrow -\infty} G(x + iy) = 0$,
- (4) $H(-\infty) = \lim_{x \rightarrow -\infty} H(x + iy)$ exists and is finite, and
- (5) $H(z + 2\pi i) \equiv H(z) + 2\pi i$ and $G(z + 2\pi i) \equiv G(z) + 2\pi i$ on D .

Furthermore, the second dilatation function $A = G'/H'$ of F satisfies the properties:

- (1) $A \in H(D)$ and $|A| < 1$ on D ,
 - (2) $A(z + 2\pi i) \equiv A(z)$ and
 - (3) $A(-\infty) = \lim_{x \rightarrow -\infty} A(x + iy)$ exists and is finite.
- (18)

Observe that β defined in (17) depends only on $a(-\infty)$. Suppose now that the domain Ω has the property

$$\Omega = \{w = u + iv; -\infty < u < u_0(v), v \in \mathbb{R}\}, \quad (19)$$

where u_0 satisfies $u_0(v + 2\pi) \equiv u_0(v)$.

The following mapping theorem in [AH2] corresponds to Theorem 2.9.

THEOREM 2.16. *Let Ω be given by (19) and let A satisfy (18). Then there exists a univalent solution F of (3) such that*

- (i) F is of the form (17),
- (ii) $H(-\infty)$ exists and is real,
- (iii) $F(D) \subset \Omega$,
- (iv) $\lim_{z \rightarrow it} F(z)$ exists and lies on $\partial\Omega$ for almost all t , and
- (v) F is uniquely determined if Ω is strictly convex in the horizontal direction, i.e., if each horizontal line intersects $\partial\Omega$ in exactly one point of $\mathbb{C}$.

Again, if $|a| \leq k < 1$ on D , then $f(D) = \Omega$. The proof uses the transformation $f(\zeta) = \exp(F(\log(\zeta)))$, $\zeta \in U$, or equivalently, $F(z) = \log f(e^z)$. Observe that f is univalent on U if and only if F is univalent on D and that f is a solution of the nonlinear elliptic partial differential equation

$$\bar{f}_z = (a \bar{f}/f) f_z, \quad a \in H(U) \text{ and } |a| < 1, \quad (20)$$

where $a(\zeta) = A(e^z)$. Any nonconstant solution of (20) is called a log-harmonic mapping. Such mappings have been studied in several papers, as, for example, [AB1, AH1, AH2] and [AH3]. In many cases, it is easier to work with log-harmonic mappings than with harmonic maps of the form (17) even if the differential equation is nonlinear. For instance, it has been shown in [AH1] that f is a log-harmonic automorphism on U satisfying $f(0) = 0$ and $f_z(0) > 0$ if and only if there is a normalized starlike conformal mapping ϕ and a β , $\operatorname{Re} \beta > -1/2$, such that

$$f(z) = A|z|^{2\beta+1} \frac{\phi(z)}{|\phi(z)|}, \quad A \neq 0, z \in U, \quad (21)$$

with the branch $1^{2\beta} = 1$. Using the transformation $F(z) = \log f(e^z)$, we conclude that $F \in \mathcal{F}$ is a sense-preserving automorphism on the left half-plane D if and only if there is a β , $\operatorname{Re} \beta > -1/2$, and a probability measure μ defined on the Borel σ -algebra over $[0, 2\pi)$ such that $F(z) = z + 2\beta x + c - 2i \int_0^{2\pi} \arg[1 - e^{it+z}] d\mu(t)$, where c is a constant.

Finally, log-harmonic polynomials have been studied in [AH4].

2.8. Constructive methods

There are several constructive methods for conformal mappings from a simply connected domain Ω containing the origin onto the unit disk U or from U onto Ω . Some of them are based on extremal problems. For example, define $N(\Omega) = \{f \in H(\Omega); f(0) = 0, f'(0) = 1\}$ and let Φ be the Riemann mapping from U onto Ω (Φ conformal, $\Phi(0) = 0$ and $\Phi'(0) > 0$). Then the unique solution $\hat{f}(z)$ of the extremal problem

$$\min_{f \in N(\Omega)} \int_{\Omega} |f'|^2 dx dy$$

is the conformal mapping $\Phi'(0)\Phi^{-1}(z)$ which maps Ω onto the disk centred at the origin with radius $\Phi'(0)$. Another extremal problem is

$$\min_{f \in N(\Omega)} \sup_{z \in \Omega} |f(z)|$$

which has the same solution as in the previous optimization problem.

Other methods use the boundary correspondence together with the Cauchy–Riemann equations (e.g., Theodorsen’s method). While such methods may be modified for K -quasiconformal mappings, they are not applicable for univalent harmonic maps since collapsing may appear. Observe also that most known methods give approximations of the Riemann mapping Φ^{-1} . The mapping Φ can then be obtained by inverting Φ^{-1} . Such a procedure does not apply for univalent harmonic maps. Indeed, knowing the mapping f^{-1} we do not know how to retrieve f .

The following method was first introduced for conformal mappings by Opfer [O1,O2]. It requires that $|a| \leq k < 1$ on U and Ω to be a strictly starlike domain (i.e., each radial line from the origin hits the boundary $\partial\Omega$ in exactly one finite point). In this case, $\partial\Omega$ admits the parametric representation $\omega(t) = R(t)e^{it}$, $0 \leq t < 2\pi$. The Minkowski functional $v(w)$ is defined by

$$v(w) = \begin{cases} 0 & \text{if } w = 0, \\ |w|/R(t) & \text{if } w = |w|e^{it} \neq 0. \end{cases}$$

If E is an arbitrary subset of $\mathbb{C}$, define $\mu(E) = \sup_{w \in E} v(w)$. Furthermore, for any complex-valued function f defined on a domain D , we put $\mu(f) = \mu(f(D))$. The following result has been shown in [BHH1].

THEOREM 2.17. *Let $a \in H(U)$ and suppose that $|a| \leq k < 1$ on U . Denote by N_a the set of all solutions f of (3) which are of the form $f(z) = z + a(0)z + o(|z|)$ as z tends to zero. Denote by F the unique univalent solution of (3) which is normalized by $F(0) = 0$ and $F_z(0) > 0$ and which maps the unit disk U onto the strictly starlike domain Ω . Then there is a unique function $\hat{f} \in N_a$ which solves the extremal problem*

$$\min_{f \in N_a} \mu(f).$$

Furthermore, we have $\hat{f} = F/F_z(0)$.

To approximate f , we proceed in the following way (for more details see [BHH1]).

- (1) Approximate $a(z)$ by a polynomial $a_1(z)$, $|a_1| < 1$.
- (2) Define

$$p_1(z) = z + \overline{\int_0^z a_1(s) ds},$$

$$p_n(z) = z^n + n \overline{\int_0^z s^{n-1} a_1(s) ds}$$

and

$$q_n(z) = i \left[z^n - n \int_0^z s^{n-1} a_1(s) ds \right].$$

(3) Put

$$V_N = \left\{ p_1 + \sum_{n=2}^N \lambda_n p_n + \sum_{n=2}^N \mu_n q_n \right\}, \quad \lambda_n \text{ and } \mu_n \text{ are real,}$$

and let $\hat{f}_N$ be a solution of

$$\min_{f \in V_N} \mu(f).$$

Then $\hat{f}_N$ converges locally uniformly to the mapping $\hat{f} = F/F_z(0)$.

(4) Define $\zeta_k = e^{2\pi i k/M}$, $1 \leq k \leq M$. Then the solution of the mathematical program

$$\begin{aligned} \min C \\ v \left(p_1(\zeta_k) + \sum_{n=2}^N \lambda_n p_n(\zeta_k) + \sum_{n=2}^N \mu_n q_n(\zeta_k) \right) \leq C, \\ \lambda_n \in \mathbb{R}, \mu_n \in \mathbb{R}, 2 \leq n \leq N \text{ and } 1 \leq k \leq M, \end{aligned} \quad (22)$$

approximates the univalent harmonic mapping $\hat{f} = F/F_z(0)$.

(5) If, in addition, Ω is a bounded convex domain, then (22) becomes a standard linear program.

An analogous constructive method for univalent harmonic mappings defined on the exterior of the unit disk have been studied in [HN1].

Lately, a constructive method of univalent harmonic mappings on U or the exterior of U were derived in [BH3, BH4] and [BHN1] for the case that the second dilatation function a is a finite Blaschke product. The method is based on the relations (11) and (12) in Section 2.6.

3. Univalent harmonic mappings on multiply connected domains

3.1. Univalent harmonic mapping defined on the exterior of the unit disk

Let K be a compact of $\mathbb{C}$ such that K and its complement $\mathbb{C} \setminus K$ are connected. We are interested in orientation-preserving univalent harmonic mappings f defined on the domain $D = \mathbb{C} \setminus K$ which keep infinity fixed. Applying the conformal premapping Φ from the exterior Δ of the unit disk U onto D normalized by $\Phi(\infty) = \infty$ and fixed $\arg \Phi'(\infty)$ to f , we may assume without loss of generality, that $D = \Delta$, $f(\infty) = \infty$, $f_z(\infty) = 1$ and that f is orientation preserving. So far, f can be written in the form

$$f(z) = z + \overline{Bz} + 2C \ln |z| + h(z) + \overline{g(z)}, \quad (23)$$

where

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

are analytic functions on $\Delta \cup \infty$ and where $|B| = |a(\infty)| < 1$. Furthermore, applying a translation, we may assume that $a_0 = 0$.

DEFINITION 3.1. The class Σ_H consists of all univalent harmonic and orientation-preserving mappings f defined on Δ which are of the form (23) and for which $a_0 = 0$.

Applying the affine postmapping $\Psi(w) = [w - \overline{a(\infty)w}]/[1 - |a(\infty)|^2]$ to f we can transform f to a function whose dilatation function vanishes at infinity.

DEFINITION 3.2. The class Σ_H^0 consists of all mappings f in Σ_H such that $f_{\bar{z}}(\infty) = 0$.

In contrast to conformal mappings, there is no elementary isomorphism between S_H and Σ_H . Another difference is the fact that there are univalent harmonic mappings from Δ onto the whole plane minus a point. The following theorem characterizes such mappings.

THEOREM 3.1 [HS4]. A harmonic function F is a univalent harmonic and sense-preserving mapping from Δ onto $\mathbb{C} \setminus \{p\}$ if and only if F is of the form

$$F(z) = A \left[z + cd\bar{z} + 2(c+d) \ln |z| - cd \frac{1}{z} - \frac{1}{\bar{z}} \right] + p,$$

where $A \in \mathbb{C} \setminus \{0\}$, $|c| < 1$ and $|d| \leq 1$.

The corresponding second dilatation function a is of the form

$$a(z) = \frac{\bar{A}}{A} \left(\frac{cz+1}{z+\bar{c}} \right) \left(\frac{dz+1}{z+\bar{d}} \right).$$

It is interesting to note [HS4] that there is no mapping in Σ_H such that $\mathbb{C} \setminus f(\Delta)$ is a continuum and such that f is a solution of

$$\overline{f_{\bar{z}}(z)} = \frac{cz+1}{z+\bar{c}} \frac{dz+1}{z+\bar{d}} f_z(z), \quad |c| < 1 \text{ and } |d| \leq 1.$$

In particular, no such mapping exists from Δ onto $\Delta_r = \{z; r < |z|\}$ for any $r > 0$. However, for all other dilatation functions $a(z)$ we can find a solution of

$$\overline{f_{\bar{z}}(z)} = a(z) f_z(z), \quad a \in H(\Delta), |a| < 1,$$

which belongs to Σ_H and whose image is Δ_r for some $r > 0$ [HS4].

Finally, let us mention that some extremal problems concerning mappings in Σ_H or Σ_H^0 have been solved in [HS3, HS4] and [SH1].

3.2. Univalent harmonic ring mappings

Fix $r \in (0, 1)$ and let $A(r, 1)$ be the annulus $\{z; r < |z| < 1\}$. In this section, we consider univalent harmonic mappings from $A(r, 1)$ onto $A(R, 1)$ for some $R \in [0, 1)$. If f is conformal, then $R = r$ and f is a rotation $f(z) = e^{i\gamma}z$ if $|z| = 1$ transforms onto itself. However, there are univalent harmonic (and orientation-preserving) mappings from $A(r, 1)$ onto $A(0, 1)$. For instance,

$$f(z) = \frac{z - r^2/\bar{z}}{1 - r^2}$$

is such a mapping. But there are many other ones as we shall see in Remark 3.2.4. On the other hand, $R(r)$ cannot be arbitrarily close to one. Nitsche [N2] has given the following elegant proof of this fact.

THEOREM 3.2. *For each $r \in (0, 1)$, there is an $R_0(r) \in (0, 1)$ such that if $f = u + iv$ is a univalent harmonic mapping from $A(r, 1)$ onto $A(R, 1)$, then $R \leq R_0(r)$.*

PROOF [N2]. Define $\gamma = \{z; |z| = (1 + r)/2\}$. Then by Harnack's inequality, there is a constant $K(\gamma) > 1$ such that $h(z_2) \leq Kh(z_1)$ for all positive harmonic functions h on $A(r, 1)$ and all z_1 and $z_2 \in \gamma$. Define $h = 1 - u$. Then h is a positive harmonic function on $A(r, 1)$. Next, there is a $z_1 \in \gamma$ such that $h(z_1) < 1 - R$ and there is a $z_2 \in \gamma$ such that $1 + R < h(z_2)$. Hence, $1 + R < h(z_2) \leq Kh(z_1) < K(1 - R)$ which implies that $R < (K - 1)/(K + 1) < 1$. $\square$

REMARK 3.1. The proof of Nitsche does not use the univalence of f but rather the fact, that $f(\gamma)$ contains a point in the region $\{w; R < \operatorname{Re} w < 1\}$ and a point in $\{w; -1 < \operatorname{Re} w < -R\}$.

(2) The same proof can also be applied to other image domains as, for example, $\Omega = U \setminus [-R, R]$ or $\Omega = \{w; R < |w| \text{ and } |\operatorname{Re} w| < 1\}$.

CONJECTURE 3.1. $R_0(r) \leq \frac{2r}{1+r^2}$. Equality holds for the mapping $f(z) = \frac{z+r^2/\bar{z}}{1+r^2}$.

It is unlikely that the above proof leads to the sharp bound for $R_0(r)$. Two different quantitative results in this direction are given in [L4] and [W3]. Lyzzaik uses smooth doubly-connected coverings of the plane in which he embeds a sheared transformation of f . Using the extremal module property of the Grötzsch domain $B(s) = U \setminus \{x; 0 \leq x \leq s\}$. He proves the following theorem.

THEOREM 3.3. *We have $R_0(r) \leq s$, where s is given by the Grötzsch domain conformally equivalent to $A(r, 1)$.*

Weitsman's estimate follows by real analytic methods via the Green function of the annulus:

THEOREM 3.4. *We have $R_0(r) \leq \frac{1}{\frac{r^2}{2}(\log r)^2 + 1}$.*

It is worthwhile to note that Lyzzaik's estimate, though it applies well to the whole interval $0 < r < 1$, it is asymptotically good for r close to 0. Weitsman's result, though not good for small values of r , it is asymptotically exact for r close to 1.

3.3. Extensions of Kneser's theorem

In this section, we extend Kneser's result, Theorems 2.6 and 2.7 for multiply connected domains of $\mathbb{C}$.

Let D be a Jordan domain of finite connectivity N in $\mathbb{C}$ whose boundary is $\partial D = \bigcup_{k=0}^N C_k$, where C_0 is the outer boundary of D . Applying an appropriate conformal premapping, we may assume without loss of generality, that each component C_k is an analytic Jordan curve. Let Ω be a domain of $\mathbb{C}$ of connectivity N such that the outer boundary S_0 is a Jordan curve and such that each inner boundary component S_k ; $1 \leq k \leq N$, is either a Jordan curve or a Jordan arc or a singleton. Denote by Φ_0 (Ψ_0 resp.) the conformal mapping from the unit disk U onto the bounded component of $\mathbb{C} \setminus C_0$ ($\mathbb{C} \setminus S_0$ resp.) and let Φ_k (Ψ_k resp.) be the conformal mapping from the exterior Δ of U onto the unbounded component of $\mathbb{C} \setminus C_0$ ($\mathbb{C} \setminus S_0$ resp.) keeping infinity fixed.

DEFINITION 3.3. Let D and Ω be as mentioned above. A function f^* from ∂D into $\partial \Omega$ is called an *orientation-preserving continuous weak homeomorphism* from ∂D onto $\partial \Omega$ if f^* is continuous on ∂D and $f^*(C_k) = S_k$; $0 \leq k \leq N$ and if S_k is not a singleton then

- (1) $d \arg \Psi_k^{-1} \circ f^* \circ \Phi_k \geq 0$ and
- (2) $\frac{1}{2\pi} \int_{\partial D} d \arg \Psi_k^{-1} \circ f^* \circ \Phi_k = 1$.

Modifying the proof of Kneser given in Theorem 2.7, we get [DH1] the theorem (see also [L5]):

THEOREM 3.5. *Let D and Ω be as in Definition 3.3 and let f^* be an orientation-preserving continuous weak homeomorphism from ∂D onto $\partial \Omega$. Then the solution f of the Dirichlet problem, $f = \int_{\partial D} f^* d\omega$ is univalent in D if and only if $f(D) = \Omega$.*

The next result is an extension of Theorem 2.6 to multiply connected domains.

THEOREM 3.6. *Let D be a Jordan domain of finite connectivity N and suppose that $\partial D = \bigcup_{k=0}^N C_k$, where C_0 is the outer boundary of D . Let Ω be a bounded convex domain of $\mathbb{C}$ and suppose that f^* is a weak homeomorphism from C_0 onto $\partial \Omega$ (see Definition 2.3). Let f be a harmonic mapping defined on D which satisfies*

- (i) $f = h + \bar{g}$, h and $g \in H(D)$,

- (ii) $\lim_{z \rightarrow \zeta} f(z) = f^*(\zeta)$ for all $\zeta \in C_0$ and
 - (iii) the image of each inner boundary component C_k of D is a singleton $\{p_k\}$.
- Then f is univalent on D .

REMARK 3.2. (1) There is at least one harmonic mapping which satisfies the conditions of Theorem 3.6.

(2) Theorem 3.6 does not hold neither for unbounded convex domains nor for nonconvex domains and there is no analogous result for harmonic mappings in higher dimensions.

(3) In general, one cannot prescribe the image points $f(C_k) = \{p_k\}$. However, if $N = 1$ and if D is the annulus $A(r, 1) = \{w; r < |w| < 1\}$, then $p_1 = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{it}) dt$. In [L6] a description of the possible values of p_1 is given.

(4) There are univalent harmonic mappings which satisfy Theorem 3.6 without having the property (1). For instance, suppose that $D = A(r, 1)$ and that $f^*(e^{it}) = e^{it}$. Then $f(z) = \frac{z-r^2/\bar{z}}{1-r^2} + 2C \ln |z|$ is univalent if and only if $|C| \leq r/(1-r^2)$.

(5) It is a natural question to ask whether Theorem 3.6 holds if we replace condition (3) for f by the following weaker condition (3'): The image of each inner boundary component C_k of D is a horizontal line segment. The answer is negative. Indeed, consider $D = A(\sqrt{11/26}, 1)$ and $f(z) = 4z - \frac{z}{3} - \frac{1}{6z} - \frac{2}{z}$. Then $f^* = f|_{C_0}$ is an orientation-preserving homeomorphism from C_0 onto $\partial\Omega$ and the inner boundary of D is mapped onto the horizontal slit $[-16/\sqrt{286}, 16/\sqrt{286}]$ but f is not univalent on D .

Theorem 3.6 together with Remark 3.2(3) gives the particular case:

THEOREM 3.7. Let $\Psi(t)$ be a nondecreasing function on $[0, 2\pi)$ such that

- (i) $\int_{[0, 2\pi)} d\Psi(t) = 2\pi$,
- (ii) $\int_0^{2\pi} e^{i\Psi(t)} dt = 0$ and
- (iii) the image $\Psi([0, 2\pi))$ contains at least three different points.

Then the solution of Dirichlet's problem,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(rR)^n}{R^{2n} - r^{2n}} \left[\left(\frac{ze^{-it}}{r} \right)^n - \left(\frac{r}{ze^{-it}} \right)^n \right] \right\} e^{i\Psi(t)} dt,$$

is univalent on $A(r, R)$.

In [HS6], necessary and sufficient conditions were given for a harmonic function f in a neighborhood of the unit circle ∂U to be univalent and orientation-preserving in an exterior neighborhood of ∂U , given that ∂U is mapped onto a single point.

3.4. Canonical harmonic-punctured plane mappings

It is well known that for any domain D of $\bar{\mathbb{C}}$ containing the point infinity there is a conformal mapping $j_\beta(z)$ such that the image $j_\beta(D)$ is a parallel slit domain with inclination β

with respect to the real axis and which satisfies $j_\beta(z) = z + o(1)$ as $z \rightarrow \infty$. If ∂D has countably many components then j_β is uniquely determined and we have

$$j_\beta(z) = e^{i\beta} [j_0(z) \cos \beta - i j_{\pi/2}(z) \sin \beta]. \quad (24)$$

If ∂D has uncountably many components then j_β may not be unique; but there is one representative for each β such that (23) holds (see, e.g., [A1]).

It is a natural question to ask whether there is for each domain D containing infinity a univalent harmonic mapping f such that $f(z) = z + o(1)$ in a neighborhood of infinity and such that each component of $\partial f(D)$ is a singleton. The next theorem gives an affirmative answer.

THEOREM 3.8. *Let D be a domain of $\overline{\mathbb{C}}$ containing the point infinity. Then there exists a univalent harmonic mapping $\widehat{F}$ from D onto $\overline{\mathbb{C}} \setminus \bigcup_{j \in J} \{p_j\}$ which is normalized at infinity by $f(z) = z + o(1)$. Furthermore, if ∂D has countably many component, then F is unique.*

REMARK 3.3. (1) The mapping $\widehat{F} = \widehat{H} + \overline{\widehat{G}}$ defined in Theorem 3.8. is called the canonical harmonic-punctured plane mapping.

(2) Denote by $\widehat{A}$ the second dilatation function of $\widehat{F}$. Then there is no other solution f of (4) with respect to $\widehat{A}$ which is univalent on D and satisfies $f(z) = z + o(1)$ as $z \rightarrow \infty$. Furthermore, if ∂D has N components then $\widehat{A}$ assumes in D every value in the unit disk U exactly at $2N$ points and it does not assume any other values at all. In other words, $\widehat{A}$ maps D onto a $2N$ -sheeted disk.

References

- [AB1] Z. Abdulhadi and D. Bshouty, *Univalent mappings in $H\overline{H}(D)$* , Trans. Amer. Math. Soc. **305** (1988), 841–849.
- [AH1] Z. Abdulhadi and W. Hengartner, *Spiral-like log-harmonic mappings*, Complex Var. Theory Appl. **9** (1987), 121–130.
- [AH2] Z. Abdulhadi and W. Hengartner, *Univalent harmonic mappings on the left half-plane with periodic dilatations*, Univalent Functions, Fractional Calculus and Their Applications, H. Srivastava and S. Owa, eds, Horwood, New York (1989), 3–28.
- [AH3] Z. Abdulhadi and W. Hengartner, *Univalent log-harmonic extensions onto the unit disk or onto an annulus*, Current Topics in Analytic Function Theory, H. Srivastava and S. Owa, eds, Scientific, River Edge, NJ (1992), 1–12.
- [AH4] Z. Abdulhadi and W. Hengartner, *Polynomials in $H\overline{H}$* , Complex Var. Theory Appl. **46** (2001), 89–107.
- [AL1] Y. Abu-Muhanna and A. Lyzzaik, *The boundary behaviour of harmonic univalent maps*, Pacific J. Math. **141** (1990), 1–20.
- [A1] L. Ahlfors, *Lecture notes on conformal mappings*, Summer Session 1951, transcribed by R. Ossermann (mimeographed), A&M College, Oklahoma (1951).
- [B1] B.V. Boyarskii, *Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients*, Mat. Sb. (N.S.) **43** (85) (1957), 451–503.
- [BHH1] D. Bshouty, N. Hengartner and W. Hengartner, *A constructive method for starlike harmonic mappings*, Numer. Math. **54** (1988), 167–178.
- [BH1] D. Bshouty and W. Hengartner, *Univalent solutions of the Dirichlet problem for ring domains*, Complex Var. Theory Appl. **21** (1993), 159–169.

- [BH2] D. Bshouty and W. Hengartner, *Univalent harmonic mappings in the plane*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **48** (1994), 12–42.
- [BH3] D. Bshouty and W. Hengartner, *Boundary values versus dilatations of harmonic mappings*, J. Anal. Math. **72** (1997), 141–164.
- [BH4] D. Bshouty and W. Hengartner, *Exterior univalent harmonic mappings with finite Blaschke dilatations*, Canad. J. Math. **51** (1999), 470–487.
- [BH5] D. Bshouty and W. Hengartner, *Univalent harmonic mappings in the plane*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **48** (1994), 12–42.
- [BHH2] D. Bshouty, W. Hengartner and O. Hossian, *Harmonic typically real mappings*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 673–680.
- [BHN1] D. Bshouty, W. Hengartner and M. Nicole, *A constructive method for univalent harmonic exterior maps with Blaschke dilatation*, Computational Methods and Function Theory (CMFT'1997), N. Papamichael, S. Ruscheweyh and E. Saff, eds, World Scientific, River Edge, NJ (1999), 99–115.
- [C1] G. Choquet, *Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques*, Bull. Sci. Math. **69** (2) (1945), 156–165.
- [CS1] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I **9** (1984), 3–25.
- [D1] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Univ. Press, Cambridge, UK (2004).
- [DH1] P. Duren and W. Hengartner, *Harmonic mappings of multiply connected domains*, Pacific J. Math. **180** (1997), 201–220.
- [DHL1] P. Duren, W. Hengartner and R. Laugesen, *The argument principle for harmonic functions*, Amer. Math. Monthly **103** (1996), 411–415.
- [DS1] P. Duren and G. Schober, *A variational method for harmonic mappings onto convex regions*, Complex Var. Theory Appl. **9** (1987), 153–168.
- [DS2] P. Duren and G. Schober, *Linear extremal problems for harmonic mappings of the disk*, Proc. Amer. Math. Soc. **106** (1989), 967–973.
- [EL1] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.
- [ES1] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
- [GD1] J.J. Gergen and F.G. Dressel, *Mapping by p -regular functions*, Duke Math. J. **18** (1951), 185–210.
- [GD2] J.J. Gergen and F.G. Dressel, *Uniqueness for p -regular mapping*, Duke Math. J. **19** (1952), 435–444.
- [H1] E. Heinz, *On one-to-one harmonic mappings*, Pacific J. Math. **9** (1959), 101–105.
- [HN1] W. Hengartner and L. Nadeau, *Univalent harmonic exterior mappings as solutions of an optimization problem*, Mat. Vesnik **40** (1988), 233–240.
- [HS1] W. Hengartner and G. Schober, *Harmonic mappings with given dilatation*, J. London Math. Soc. **33** (1986), 473–483.
- [HS2] W. Hengartner and G. Schober, *On the boundary behaviour of orientation-preserving harmonic mappings*, Complex Var. Theory Appl. **5** (1986), 197–208.
- [HS3] W. Hengartner and G. Schober, *Univalent harmonic mappings*, Trans. Amer. Math. Soc. **299** (1987), 1–31.
- [HS5] W. Hengartner and G. Schober, *Curvature estimates for some minimal surfaces*, Complex Analysis, Articles dedicated to Albert Pfluger, J. Hersch and A. Huber, eds, Birkhäuser, Basel (1988), 87–100.
- [HS4] W. Hengartner and G. Schober, *Univalent harmonic exterior and ring mappings*, J. Math. Anal. Appl. **156** (1991), 154–171.
- [HS6] W. Hengartner and J. Szynal, *Univalent harmonic ring mappings vanishing on the interior boundary*, Canad. J. Math. **44** (1991), 308–323.
- [J1] J. Jost, *Harmonic Maps Between Surfaces*, Lecture Notes in Math., Vol. 1062, Springer-Verlag, New York (1984).
- [J2] J. Jost, *Two-Dimensional Geometric Variational Problems*, Wiley, Toronto (1991).
- [K1] H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein **35** (1926), 123–124.
- [KS1] D. Khavinson and G. Swiatek, *On the number of zeroes of certain harmonic polynomials*, Proc. Amer. Math. Soc. **131** (2002), 409–414.
- [L1] R. Laugesen, *Injectivity can fail for higher-dimensional harmonic extensions*, Complex Var. Theory Appl. **28** (1996), 357–369.

- [L2] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), 689–692.
- [L3] A. Lyzzaik, *Local properties of light harmonic mappings*, Canad. J. Math. **44** (1992), 135–153.
- [L5] A. Lyzzaik, *Univalence criteria for harmonic mappings in multiply connected domains*, J. London Math. Soc. **58** (1998), 163–171.
- [L4] A. Lyzzaik, *Univalent harmonic mappings and a conjecture of J.C.C. Nitsche*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **53** (1999), 147–150.
- [L6] A. Lyzzaik, *Quasihomeomorphisms and univalent harmonic mappings onto punctured bounded convex domains*, Pacific J. Math. **200** (2001), 159–190.
- [N1] G. Neumann, Oral communication.
- [N2] J.C.C. Nitsche, *On the module of doubly-connected regions under harmonic mappings*, Amer. Math. Monthly **69** (1962), 781–782.
- [N3] J.C.C. Nitsche, *Vorlesungen über Minimalflächen*, Springer-Verlag, New York (1975).
- [O1] G. Opfer, *New extremal properties for constructing conformal mappings*, Numer. Math. **32** (1979), 423–429.
- [O2] G. Opfer, *Conformal mappings onto prescribed regions via optimization technics*, Numer. Math. **35** (1980), 423–429.
- [O3] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand, New York–Toronto (1986).
- [R1] T. Radó, *Aufgabe 41*, Jahresber. Deutsch Math.-Verein **35** (1926), 49.
- [R2] E. Reich, *The composition of harmonic mappings*, Ann. Acad. Sci. Fenn. Ser. A I **12** (1987), 47–53.
- [R3] E. Reich, *Local decomposition of harmonic mappings*, Complex Var. Theory Appl. **9** (1987), 263–269.
- [S2] R. Schoen, *The theory and applications of harmonic mappings between Riemannian manifolds*, AMS Progress in Mathematics Lecture Ser., Video Cassette (1992).
- [S1] R. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Lecture Notes in Pure Appl. Math., Vol. 143, Dekker, New York (1993), 179–200.
- [S3] T. Sheil-Small, *On the Fourier series of a finitely described convex curve and a conjecture of H.S. Shapiro*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 513–527.
- [S4] T. Sheil-Small, *On the Fourier series of a step function*, Michigan Math. J. **36** (1989), 459–475.
- [S5] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. **42** (1990), 237–248.
- [SH1] J. Sook Heui, *Univalent harmonic mappings on $\Delta = \{z : |z| > 1\}$* , Proc. Amer. Math. Soc. **119** (1993), 109–114.
- [S6] V. Starkov, *Harmonic locally quasiconformal mappings*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **49** (1995), 183–197.
- [W1] R. Wegmann, *Extremal problems for harmonic mappings from the unit disk to convex regions*, J. Comput. Appl. Math. **46** (1993), 165–181.
- [W2] A. Weitsman, *On univalent harmonic mappings and minimal surfaces*, Pacific J. Math. **192** (2000), 191–200.
- [W3] A. Weitsman, *Univalent harmonic mappings of annuli and a conjecture of J.C.C. Nitsche*, Israel J. Math. **124** (2001), 327–331.
- [W4] A.S. Wilmschurst, *The valence of harmonic polynomials*, Proc. Amer. Math. Soc. **126** (1998), 2077–2081.
- [W5] J.C. Wood, *Lewy's theorem fails in higher dimensions*, Math. Scand. **69** (1991), 166.

CHAPTER 11

Quasiconformal Extensions and Reflections

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Abstract

This survey chapter presents the developments of an important field of geometric complex analysis concerning quasiconformal extensions of univalent holomorphic functions and quasiconformal reflections as well as different applications of such quasiconformal maps. These directions are closely connected with various fields of Mathematics. An essential part is focused on the problems concerning quantitative estimation of reflections.

0. Preface

This survey chapter presents the developments of an important field of geometric complex analysis concerning quasiconformal extensions of univalent holomorphic functions, holomorphic dynamics and quasiconformal reflections as well as different applications of such quasiconformal maps. These directions are closely connected with various fields of mathematics. An essential part is focused on the problems concerning quantitative estimation of reflections.

1. Holomorphic motions and quasiconformal extension of univalent functions. Kühnau's problems

1.1. Some sufficient conditions for quasiconformal extension of univalent functions

1.1.1. It is important to know, especially in applications, whether a given locally univalent function f (i.e., with $f'(z) \neq 0$) is globally univalent in its domain. After the classical Grunsky coefficient conditions [Gru], the famous paper of Nehari [Ne1] has determined the fundamental role of differential operators in the problem of univalence. Most of the presently known conditions are presented, for example, in the survey papers [AvA, AvAE] and [AkS].

Let us consider now a somewhat different question. Various problems in complex analysis, complex dynamics, potential theory, approximation, the Fredholm eigenvalues theory concern continuous extension of conformal maps of plane regions onto the complete complex plane. The best possible situation occurs when this extension is quasiconformal.

In the seminal 1962 paper [AW], Ahlfors and Weill have essentially improved the Nehari theorem by showing that a large class of univalent functions f in the unit disk Δ can be extended to quasiconformal automorphisms $\hat{f}$ of the whole sphere $\widehat{\mathbb{C}}$; moreover, the quasiconformal extensions onto $\overline{\Delta}^*$ are provided explicitly.

Namely, let φ be a holomorphic function on Δ which satisfies

$$\sup_{\Delta} |(1 - |z|^2)^2 \varphi(z)| < 2 \quad (1.1)$$

(i.e., $|\varphi|$ is hyperbolically bounded by 2). It was shown in [AW] that for any two independent solutions η_1 and η_2 of the equation $\eta'' + (\varphi/2)\eta = 0$ in Δ , the maps $f(z) = \eta_1(z)/\eta_2(z)$ and

$$g(z) = \frac{\bar{z}\eta_1(z) + (1 - z\bar{z})\eta_1'(z)}{\bar{z}\eta_2(z) + (1 - z\bar{z})\eta_2'(z)} \quad (1.2)$$

are both continuous on $\overline{\Delta}$ and map Δ onto the complementary Jordan domains in $\widehat{\mathbb{C}}$ with a common boundary; in addition, these maps agree on the circle $\partial\Delta$. So we have the following theorem.

THEOREM 1.1 [AW]. *For each f holomorphic in Δ whose Schwarzian derivative S_f satisfies (1.1), the map*

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } |z| \leq 1, \\ g(1/\bar{z}) & \text{if } |z| \geq 1, \end{cases} \quad (1.3)$$

is a quasiconformal automorphism of $\widehat{\mathbb{C}}$.

Note that the extension (1.3) has the Beltrami coefficient

$$\mu_{\hat{f}}(z) = \begin{cases} -\frac{1}{2}(|z|^2 - 1)^2 S_{\hat{f}}(1/\bar{z}) 1/\bar{z}^4 & \text{if } |z| > 1, \\ 0 & \text{if } |z| < 1. \end{cases} \quad (1.4)$$

1.1.2. Duren and Lehto gave the following generalization of the Ahlfors–Weill theorem. It can be regarded also as an improvement of the corresponding univalence criterion of Nehari (see [Ne2]).

THEOREM 1.2 [DL]. *Let f be meromorphic in the unit disk, and let λ be a continuously differentiable nondecreasing function for $0 < r < 1$, $0 < \lambda < 1$, which satisfies the condition*

$$\frac{1}{1 - \lambda(r)} = O\left(\log \frac{1}{1 - r}\right).$$

If

$$|S_f(z)| \leq \frac{2\lambda(|z|)}{(1 - |z|^2)^2},$$

then f allows a quasiconformal extension to the whole sphere.

Another extension theorem of Lehto is the following theorem.

THEOREM 1.3 [Leh1]. *Let w be a quasiconformal map of a plane domain D , and let E be a compact set in D . Then there exists a quasiconformal map $\tilde{w}$ of the whole plane which is equal to w in E and is a linear conformal transformation in each component of the complement of D .*

This result was strengthened by Väisälä in the following way.

Let X and Y be metric spaces with distance written as $|a - b|$, and let $\eta: [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. An injective map $f: X \rightarrow Y$ is called η -quasisymmetric, if $|a - x| \leq t|b - x|$ implies

$$|f(a) - f(x)| \leq \eta(t)|f(b) - f(x)|$$

for every triple of points $a, b, x \in X$ and for every $t \geq 0$. A map η -quasisymmetric in a neighborhood of each point in X is called *locally η -quasisymmetric*. The basic theory of quasisymmetric maps is given in [TV] and [Vai2].

THEOREM 1.4 [Vai3]. *Suppose that G, G' are open set in $\mathbb{R}^n$, $n \geq 2$, that E is closed in G , and that $f: G \rightarrow G'$ is a homeomorphism such that $f|G \subset E$ is K -quasiconformal ($K \geq 1$) and $f|E$ locally η -quasisymmetric. Then f is K_1 -quasisymmetric with K_1 depending only on n, K, η .*

The assumption that the sets G and G' are open is essential: for example, the map $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x$ for $x \leq 0$ and $f(x) = x^2$ for $x \geq 0$ is a homeomorphism of $\mathbb{R}$ and quasisymmetric in $(-\infty, 0]$ and in $[0, \infty)$ but not in any neighborhood of the origin.

1.1.3. Using the chain rule for the Schwarzian derivatives

$$S_{f \circ g} = (S_f \circ g)(g')^2 + S_g, \quad (1.5)$$

one obtains a local version of Theorem 1.1 for an arbitrary hyperbolic simply connected domain $D \subset \widehat{\mathbb{C}}$ (cf. [Ahl3, Chapter 6]). Another approach based on the inverse mapping theorem for Banach spaces was provided by Bers in [Ber2].

Let D be a simply connected Jordan domain in $\widehat{\mathbb{C}}$, and let $\lambda_D(z)|dz|$ denote the hyperbolic Poincaré metric of D normalized by assumption that its Gaussian curvature equals -4 . We denote by $\mathbb{B}(D)$ the complex Banach space of hyperbolically bounded holomorphic functions on D with the norm

$$\|\psi\| = \sup_D \lambda_D^{-2}(z) |\psi(z)|.$$

If D contains the infinity point, then each $\psi \in \mathbb{B}(D)$ satisfies $\psi(z) = O(|z|^{-4})$ as $z \rightarrow \infty$. The elements of $\mathbb{B}(D)$ are the Schwarzian derivatives

$$S_w(z) = \left(\frac{w''(z)}{w'(z)} \right)' - \frac{1}{2} \left(\frac{w''(z)}{w'(z)} \right)^2 \quad (1.6)$$

of the locally univalent holomorphic functions w on D .

THEOREM 1.5 [Ber2]. *Let L be a quasicircle on $\widehat{\mathbb{C}}$ with the interior D and exterior D^* . Then, for some $\varepsilon > 0$, there exists an antiholomorphic homeomorphism τ (with $\tau(\mathbf{0}) = \mathbf{0}$) of the ball $V_\varepsilon = \{\varphi \in \mathbb{B}(D^*): \|\varphi\| < \varepsilon\}$ into $\mathbb{B}(D)$ such that every φ in V_ε is the Schwarzian derivative of some univalent function f which is the restriction to D^* of a quasiconformal automorphism $\hat{f}$ of Riemann sphere $\widehat{\mathbb{C}}$. This $\hat{f}$ can be chosen in such a way that its Beltrami coefficient is harmonic on D , i.e., of the form*

$$\mu_{\hat{f}}(z) = \lambda_D^{-2}(z) \overline{\psi(z)}, \quad \psi = \tau(\varphi). \quad (1.7)$$

If, in addition, L is the invariant curve of a quasi-Fuchsian group G (acting discontinuously on D and D^*), then τ transforms the G -automorphic holomorphic 2-forms in D into the forms in D^* , and the map $\hat{f}$ is compatible with G , i.e., fGf^{-1} is again a quasi-Fuchsian group.

This approach was extended in [Kru2] to invariant differential operators $\mathcal{D}f$ of arbitrary integer order $n \geq 2$.

Theorems 1.1 and 1.4 play a fundamental role in Teichmüller space theory and have various applications.

1.1.4. An alternative method for quasiconformal extension of univalent functions was provided by Becker [Bec1–Bec3]; see also [EE,Po1]. It relies on the *Löwner chains* determined by the *Löwner equation*

$$\frac{\partial f(z, t)}{\partial t} = zp(z, t) \frac{\partial f(z, t)}{\partial z}. \quad (1.8)$$

Here $z \in \Delta$ and $0 \leq t < \infty$. In the classical case, the function p has a more special form and is continuous at least on $S^1 \times [0, \infty)$. This equation determines an expanding flow. In the general case, the function p is only measurable on $\Delta \times [0, \infty)$, then it is assumed to be satisfied only almost everywhere.

THEOREM 1.6 [Bec1, Bec3]. *If $f(z, t)$ is a univalent solution to (1.8) with $p(z, t)$ satisfying the condition*

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1 \quad (1.9)$$

then, for each $t \geq 0$, the function $f_t(z) = f(z, t)$ maps Δ onto a Jordan domain bounded by a k -quasicircle (i.e., by a k -quasiconformal image of S^1), and the map $\hat{f}(z)$ defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0) & \text{if } r < 1, \\ f(e^{i\theta}, \log r) & \text{if } r \geq 1, \end{cases}$$

is a k -quasiconformal extension of $f(z, 0)$ onto $\widehat{\mathbb{C}}$ with $\hat{f}(\infty) = \infty$.

The assumption (1.9) means that the values of p must lie in a compact subset of the half-plane $\{w: \operatorname{Re} w > 0\}$.

As an important application of this theorem, Becker obtains that the inequality

$$\left| z \frac{f''(z)}{f'(z)} \right| \leq \frac{k}{|z|^2 - 1}, \quad k < 1, \quad (1.10)$$

for a holomorphic function f in the unit disk provides its k -quasiconformal extension to the whole sphere $\widehat{\mathbb{C}}$.

1.1.5. The following result of Krzyż concerns the extension of the functions $f \in \Sigma$, i.e., nonvanishing and univalent in Δ^* and normalized by $f(z) = z + b_1 z^{-1} + \dots$.

THEOREM 1.7 [Krz]. *A function $f \in \Sigma$ satisfying*

$$|f'(z) - 1| \leq k|z|^{-2}, \quad |z| < 1, \quad (1.11)$$

admits a k -quasiconformal extension onto $\widehat{\mathbb{C}}$.

Note that $z^2(f'(z) - 1) = \sum_{n=1}^{\infty} n b_n z^{-n+1}$, hence the assumption $\sum_{n=1}^{\infty} n |b_n| \leq k$ is also sufficient to have a k -quasiconformal extension.

The inequalities $\|S_f\|_{\mathbb{B}(\Delta)} \leq 2$ and (1.10), (1.11) with $k = 1$ (i.e., their limit cases as $k \rightarrow 1$) provide the sufficient conditions for the global univalence of f .

The existence of k -quasiconformal extensions in the above theorems follows also from the lambda-lemma of Mañé, Sad and Sullivan for holomorphic motions which are presented in the following section (cf. [Po2]). However, the technique of holomorphic motions does not provide the extensions explicitly.

1.1.6. We now list, following [Ahl4] and [Bec3], more general criteria for univalence and quasiconformal extendibility.

THEOREM 1.8. *Let $F(z) = z + b_1 z^{-1} + \dots$ be holomorphic in Δ^* . If $|c| \leq 1$, $c \neq 1$, and $k \leq 1$, then each of the following three conditions (where $|z| > 1$) implies that $F \in \Sigma$ and admits k -quasiconformal extension onto $\widehat{\mathbb{C}}$:*

$$\left| \frac{(1 - c|z|^{-2})^2}{1 - c} |z|^2 (F'(z) - 1) - c|z|^{-2} \right| \leq k, \quad (1.12)$$

$$\left| \frac{1 - c|z|^{-2}}{1 - c} (|z|^2 - 1) z \frac{F''(z)}{F'(z)} + c|z|^{-2} \right| \leq k, \quad (1.13)$$

$$\left| \frac{1}{1 - c} (|z|^2 - 1)^2 \frac{z}{2} S_F(z) - c|z|^{-2} \right| \leq k. \quad (1.14)$$

SKETCH OF THE PROOF. We set $a(t) = (e^t - ce^{-t})/(1 - c)$ and consider, for $|z| < 1$ and $0 \leq t < \infty$, the functions

$$f(z, t) = F\left(\frac{e^t}{z}\right) + \left(\frac{1}{a(t) - e^t}\right) \frac{1}{z}, \quad (1.15)$$

$$f(z, t) = F\left(\frac{e^t}{z}\right) + \left(\frac{1}{a(t) - e^t}\right) \frac{1}{z} F'\left(\frac{e^t}{z}\right), \quad (1.16)$$

$$f(z, t) = F\left(\frac{e^t}{z}\right) + \frac{\left(\frac{1}{a(t) - e^t}\right) \frac{1}{z} F'\left(\frac{e^t}{z}\right)}{1 - \frac{1}{2} \left(\frac{1}{a(t) - e^t}\right) \frac{1}{z} \frac{F''}{F'}\left(\frac{e^t}{z}\right)}, \quad (1.17)$$

respectively. Then $f(z, 0) = F(1/z)$ and

$$p(z, t) = \frac{\partial_t f(z, t)}{z \partial_z f(z, t)} = c_0(t) + c_1(t)z + \dots, \quad |z| < 1,$$

satisfies (1.9) and some other conditions which appear in the proof of Theorem 1.5. The desired k -quasiconformal extensions $\widehat{F}$ onto the disk Δ are given by

$$\widehat{F}(z) = f\left(\frac{|z|}{z}, \log \frac{1}{|z|}\right).$$

In particular, in the case $c = 0$ corresponding to the preceding theorems, one obtains, respectively:

$$\widehat{F}(z) = F\left(\frac{1}{z}\right) - \left(\frac{1}{\bar{z}} - z\right), \quad (1.18)$$

$$\widehat{F}(z) = F\left(\frac{1}{z}\right) - \left(\frac{1}{\bar{z}} - z\right)F'\left(\frac{1}{\bar{z}}\right), \quad (1.19)$$

$$\widehat{F}(z) = F\left(\frac{1}{z}\right) - \frac{\left(\frac{1}{\bar{z}} - z\right)F'\left(\frac{1}{\bar{z}}\right)}{1 + \frac{1}{2}\left(\frac{1}{\bar{z}} - z\right)\frac{F''\left(\frac{1}{\bar{z}}\right)}{F'\left(\frac{1}{\bar{z}}\right)}}, \quad (1.20)$$

where $|z| \leq 1$. □

Theorem 1.7 was strengthened by Becker and Pommerenke in a way that assuming $F(\Delta^*)$ to be a Jordan domain, one only needs to control the upper limits as $|z| \rightarrow 1+$ of the left-hand sides in (1.12)–(1.14) (see [Bec3, BeP]).

1.1.7. Applying the lambda-lemma and some other deep results, Sugawa established the following result in [Su2].

THEOREM 1.9. *For holomorphic functions in the unit disk with $f(0) = 0$ and $f'(0) \neq 0$, let $p(z)$ denote one of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ or $f'(z)$. If for some $k \in [0, 1)$,*

$$\left| p(z) - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}$$

for all $z \in \Delta$, then f is univalent and admits a k -quasiconformal extension onto $\widehat{\mathbb{C}}$.

Choosing $p(z) = (1 + kz^2)/(1 - kz^2)$, one can see that in the case $p(z) = zf'(z)/f(z)$ or $f'(z)$ the above result is best possible.

Theorem 1.8 improves, in particular, the result of [FKZ] (cf. also [Xi]). Sugawa's paper provides also criteria for quasiconformal extendibility for functions from more special subclasses. Their proof is based on the same technique.

Kühnau observed that in the case $p(z) = f'(z)$ the assertion of Theorem 1.9 can be established elementarily (see [Ku20]).

1.1.8. The following theorem of Kühnau and Blaar [KuB] concerns quasiconformal extensions of maps of the doubly connected domains (cf. [Os1]).

THEOREM 1.10. *If a holomorphic and univalent function f in the annulus $A_{1,R} = \{z: 1 < |z| < R\}$ satisfies the conditions $f(z) \neq 0$ and*

$$\left| 1 - z \frac{f'(z)}{f(z)} \right| \leq k < 1, \quad z \in A_{1,R},$$

then f has a k -quasiconformal extension onto $\widehat{\mathbb{C}}$ preserving the points 0 and ∞ .

1.1.9. Zhuravlev's criterion. One of the basic results in the theory of univalent functions is the *Golusin inequality* which states that if $f \in \Sigma$ and $z_n \in \Delta^*$, $\gamma \in \mathbb{C}$ ($n = 1, \dots, N$), then

$$\left| \sum_{m=1}^N \sum_{n=1}^N \gamma_m \gamma_n \log \frac{f(z_m) - f(z_n)}{z_m - z_n} \right| \leq \left| \sum_{m=1}^N \sum_{n=1}^N \gamma_m \bar{\gamma}_n \log \frac{1}{1 - (z_m \bar{z}_n)^{-1}} \right|$$

(see [Gol, Chapter 4], [Po1, Section 3.2]). It is equivalent to Grunsky's univalence criterion presented below in Section 1.5.

Choosing $N = 2$ and $z_1 = z$, $z_2 = \zeta$, $\gamma_1 = 1$, $\gamma_2 = -1$, one obtains

$$\left| \log \frac{f'(z)f'(\zeta)(z - \zeta)^2}{(f(z) - f(\zeta))^2} \right| \leq \log \frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)}, \quad z, \zeta \in \Delta^*. \quad (1.21)$$

Accordingly, for $f \in \Sigma$ having k -quasiconformal extensions to $\widehat{\mathbb{C}}$, we have (see [Ku1])

$$\left| \log \frac{f'(z)f'(\zeta)(z - \zeta)^2}{(f(z) - f(\zeta))^2} \right| \leq k \log \frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)}. \quad (1.22)$$

The question arises: *does any function $f \in \Sigma$ satisfying (1.22) with some $k < 1$ (i.e., for which (1.21) becomes a strict inequality) have quasiconformal extensions across S^1 (and therefore onto the whole plane $\widehat{\mathbb{C}}$)?*

The following theorem of Zhuravlev (see [Zh1], [KrK1, Part 1, Chapter 5]) gives an affirmative solution to this question.

THEOREM 1.11. *A meromorphic function f in the disk Δ^* admits a quasiconformal extension onto $\widehat{\mathbb{C}}$ if and only if there exists a constant q , $0 < q < 1$, such that*

$$\left| \frac{f'(z)f'(\zeta)(z - \zeta)^2}{(f(z) - f(\zeta))^2} \right| \leq \left(\frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)} \right)^q, \quad z, \zeta \in \Delta^*. \quad (1.23)$$

Moreover, if f has a k -quasiconformal extension, then one can choose $q = k$, getting the inequality (1.23) with such q . On the other hand, if the inequality (1.23) holds, then f has a k^* -quasiconformal extension with some k^* depending only on q .

The counterexamples of Kühnau and Krushkal for the Grunsky inequality work well also for the Golusin inequality and imply that in the general case $k^* > q$ (see, e.g., [Kru21, Section 2]).

Theorem 1.9 provides an alternative proof of the important result of Pommerenke [Po1] that the strengthened Grunsky inequality

$$\left| \sum_{m,n=1}^{\infty} c_{mn} x_m x_n \right| \leq k \sum_{n=1}^{\infty} \frac{|x_n|^2}{n}, \quad (1.24)$$

for every $\mathbf{x} = (x_1, x_2, \dots) \in l^2$ provides a k^* -quasiconformal extension with some k^* depending only on k .

Indeed, setting in (1.24) $x_n = z^n - \bar{z}^n$ and using the definition of the Grunsky coefficients c_{mn} (see (1.33)), one obtains (1.22) which provides the desired quasiconformal extension.

1.1.10. The relations between the Schwarzian derivative, univalence and quasiconformal extension were investigated by many authors from various points of view (see, e.g., [Ah, AG1, AnH1, AnH2, Ch1, Ch2, CO1, CO2, Ep2, Ge, Har1, Har2, KuB, Leh3, MSa, Os1, Os3, OS1, OS2, Po3, Pf3]).

The following two theorems relate to the above results.

THEOREM 1.12 [CO1]. *If f is a C^3 function on $(-1, 1)$, $f'(0) \neq 0$, and if, for some $0 \leq t < 1$, one has*

$$\frac{-4t}{1-t} \leq (1-x^2)^2 S_f(x) \leq \frac{4t}{1+t}, \quad x \in (-1, 1),$$

then f has a conformally natural (in the sense of Douady and Earle) $K(t) = (1+t)/(1-t)^3$ -quasiconformal extension to the plane $\mathbb{C}$ and to the space $\mathbb{R}^3$, preserving the upper half-plane and the upper half-space.

For a discussion on the naturally conformal maps see, e.g., [DE, GaL].

THEOREM 1.13 [Ch1]. *Let $p(x) \geq 0$ be an even function on $(-1, 1)$ with $(1-x^2)^2 p(x)$ nonincreasing for $x > 0$. Suppose that the even solution of the differential equation $y'' + py = 0$ is positive and is such that*

$$\int_0^1 y^{-2}(x) dx < \infty.$$

If a holomorphic function f in the disk Δ satisfies $|S_f(z)| \leq 2p(|z|)$, then $f(\Delta)$ is a quasidisk (i.e., the image of Δ under a quasiconformal homeomorphism of $\widehat{\mathbb{C}}$).

1.2. Holomorphic motions

1.2.1. Let E be a subset of $\widehat{\mathbb{C}}$ containing at least three points.

A *holomorphic motion* of E is a function $f : E \times \Delta \rightarrow \widehat{\mathbb{C}}$ such that:

- (a) for every fixed $z \in E$, the function $t \mapsto f(z, t) : \Delta \rightarrow \widehat{\mathbb{C}}$ is holomorphic in Δ ;
- (b) for every fixed $t \in \Delta$, the map $f(z, t) = f_t(z) : E \rightarrow \widehat{\mathbb{C}}$ is injective;
- (c) $f(z, 0) = z$ for all $z \in E$.

The remarkable lambda-lemma of Mañé, Sad and Sullivan [MSS] yields that such holomorphic dependence on the time parameter provides quasiconformality of f in the space parameter z . Moreover: (i) f extends to a holomorphic motion of the closure $\overline{E}$ of E ; (ii) each $f_t(z) = f(t, z) : \overline{E} \rightarrow \widehat{\mathbb{C}}$ is quasiconformal; (iii) f is jointly continuous in (z, t) .

Quasiconformality here means, in the general case, the boundedness of the distortion of the circles centered at the points $z \in E$ or of the cross-ratios of the ordered quadruples of points of E (cf., e.g., [GaL], [Kru20, Section 1.1]).

The Slodkowski lifting theorem ([S11], see also [AsM, Do]) solves the problem of Sullivan and Thurston on the extension of holomorphic motions from any set to a whole sphere:

EXTENDED LAMBDA-LEMMA. Any holomorphic motion $f : E \times \Delta \rightarrow \widehat{\mathbb{C}}$ can be extended to a holomorphic motion $\tilde{f} : \widehat{\mathbb{C}} \times \Delta \rightarrow \widehat{\mathbb{C}}$, with $\tilde{f}|E \times \Delta = f$.

The corresponding Beltrami differentials $\mu_{\tilde{f}_t}(z) = \partial_{\bar{z}} \tilde{f}(z, t) / \partial_z \tilde{f}(z, t)$ are holomorphic in t via elements of $L_\infty(\mathbb{C})$, and Schwarz's lemma yields

$$\|\mu_{\tilde{f}_t}\|_\infty \leq |t|, \quad (1.25)$$

or equivalently, the maximal dilatations $K(\tilde{f}_t) \leq (1 + |t|)/(1 - |t|)$. This bound cannot be improved in the general case.

Holomorphic motions have been important in the study of dynamical systems, Kleinian groups, holomorphic families of conformal maps and of Riemann surfaces as well as in many other fields (see, e.g., [As, AsM, Ber4, Ber5, BerR, Do, Ea1, Ea2, EF1, EF2, EGL, EK, EKK, EM, EMi, ErH, Kr2, Kru14, Kru18, MSu, MSS, Mar, Po2, PR, Ro, Shi2, S11–S13, Su1–Su3, Sul1–Sul3, Ta]).

1.2.2. There is an intrinsic connection between holomorphic motions and Teichmüller spaces, first mentioned by Bers and Royden in [BerR]. This leads to the equivariant forms of Slodkowski's theorem given in [EKK] and [S13]. One can also find in these papers the various applications of the equivariant extensions of lambda-lemma.

We shall use here other applications of holomorphic isotopies.

1.3. Which are the exact bounds for quasiconformal extensions of univalent functions?

We have seen that the lambda-lemma provides quasiconformal extensions of each complex holomorphic isotopy. It is important to find the exact bounds for the quasiconformality

coefficients of the extensions. Though the lambda-lemma provide a universal upper bound for those coefficients of the extensions, it is in the general case not the best possible.

The univalent functions on the disk often admit quasiconformal extensions with essentially smaller dilatations. This observation was made by Kühnau (see [Ku9, Ku11]), who formulated the problems of estimating the quasiconformality coefficients and the coefficients of naturally occurring quasireflections in the strongest form. Of course, such problems already appeared much earlier (see, e.g., [Bec3, Hu, Vu]).

These problems arose also in the context of the Fredholm eigenvalue theory and from the inner questions of Geometric Function Theory; originally they did not concern the dynamics.

The problems are more conveniently formulated using the hydrodynamic normalization of the maps. For each function

$$f(z) = z + a_0 + a_1 z^{-1} + \dots \quad (1.26)$$

univalent in Δ^* , i.e., $f \in \Sigma$, there is a natural family

$$f_r(z) = r f\left(\frac{z}{r}\right) = z + a_0 r + a_1 r^2 z^{-1} + \dots, \quad 0 \leq r \leq 1,$$

which determines an isotopy of f to an identity map in the topology of locally uniform convergence in Δ^* . When one takes the complex $t \in \Delta$, the function $f_t(z)$ defines a special type of holomorphic motions generated by a univalent function in the disk. It is natural to wait to get a much stronger estimate for these motions.

Let Σ_k denote again the subclass of functions in Σ which admit k' -quasiconformal extension to $\widehat{\mathbb{C}}$ (actually, to the unit disk Δ), with $k' \leq k$ for a given $k \in [0, 1)$, and $\Sigma_k(0)$ denote the class of functions $f \in \Sigma$, whose extensions $\tilde{f}$ satisfy the additional restriction $\tilde{f}(0) = 0$.

The above-mentioned problem of Kühnau states whether

$$f \in \Sigma_k \quad \text{implies} \quad f_r \in \Sigma_{kr^2}, \quad (1.27)$$

in particular, whether

$$f \in \Sigma \quad \text{implies} \quad f_r \in \Sigma_{r^2}. \quad (1.28)$$

This is the so-called r^2 -property of the disk. Most of the applications to conformal maps concern the case $k = 1$, i.e., the question (1.28).

If one requires moreover that the extensions fix the origin, i.e., that the initial function belongs to $\Sigma_k(0)$, then the order of dilatation can be essentially changed. This was a problem of Bshouty formulated earlier in the problem collection [CCH] (in an equivalent form), whether

$$f \in \Sigma_k(0) \quad \text{implies} \quad f_r \in \Sigma_{kr}(0). \quad (1.29)$$

All the above estimates are sharp, see, e.g., [Kru5].

As k tends to 1 in (1.29), one obtains that for $f \in \Sigma$ the function f_r admits an r -quasiconformal extension (this follows also from the lambda-lemma applied to the complex $r \in \Delta$).

THEOREM 1.14. *If $f \in \Sigma_k$, then, for any $t \in \Delta$, the map $f_t(z) = tf(t^{-1}z) \in \Sigma_{kt^2}$.*

As a direct consequence of this theorem, we mention the following corollary.

COROLLARY 1.15. *Let f map conformally the disk $\Delta_R = \{z: |z| < R\}$ and admit a k -quasiconformal extension to $\widehat{\mathbb{C}}$. Then f admits a k' -quasiconformal extension across the unit circle $\partial\Delta$, and the curve $L = f(|z| = 1)$ admits a k' -quasireflection, with $k' \leq kR^{-2}$.*

We provide below a sketch of the proof of Theorem 1.14 for $k = 1$, i.e., for the question (1.28). Most of the applications to conformal maps concern this case. The proof follows [Kru5] and relies upon the Royden–Gardiner theorem on coincidence of the Kobayashi and the Teichmüller metrics on Teichmüller spaces (see [Roy2, Gar]). The proof of relation (1.29) is given in [Kru4]. The proof of Theorem 1.14 for a $k < 1$ giving the affirmative answer to the question (1.27) requires much stronger arguments and will not be treated here.

We will talk about quasireflections in the following section.

1.4. Sketch of the proof of relation (1.28)

The given proof of (1.28), i.e., of the part of Theorem 1.14 concerning $k = 1$, is based on modifying the Royden–Gardiner arguments to the maps and metrics with isolated zeros.

For each holomorphic map h from Δ into the Teichmüller space $\mathbb{T}(\Gamma)$, the Finsler structure $F_{\mathbb{T}}$ of these spaces induces on the disks $h(\Delta)$ the conformal metric

$$\lambda(t)|dt| = F_{\mathbb{T}}(h(t), h'(t))|dt|. \quad (1.30)$$

The Teichmüller–Kobayashi metric $d_{\mathbb{T}}$ of $\mathbb{T}(\Gamma)$ (and its infinitesimal form $\lambda_0|dt|$) acts as a hyperbolic isometry on Teichmüller disks $\{\phi_{\Gamma}(t\mu_0): t \in \Delta\} \subset \mathbb{T}(\Gamma)$, which corresponds to extremal μ_0 from $M(\Gamma)$, $\|\mu_0\|_{\infty} = 1$.

The goal of the proof is to show that this metric becomes *supporting* for (1.30). More precisely, because of zeros, we must deal with the metrics

$$\lambda_m(t) = \frac{(m+1)|t|^m}{1 - |t|^{2(m+1)}}, \quad m \in \mathbb{N},$$

which belongs to C^2 on $\Delta \setminus \{0\}$ and has there constant Gauss curvature -4 . The role of these metrics is revealed by the next version of the Ahlfors–Schwarz lemma:

LEMMA 1.16. *Let $\lambda(t)|dt|$ be a conformal metric on the disk Δ such that $\lambda(t)$ is subharmonic,*

$$\lambda(t) = c|t|^m + o(|t|^m) \quad \text{as } t \rightarrow 0, \quad c \leq m+1,$$

and let λ admit at each point $t_0 \in \Delta \setminus \{0\}$, where $\lambda(t) \neq 0$, a C^2 -smooth supporting metric of curvature at most -4 . Then, for all $t \in \Delta$,

$$\lambda(t) \leq \lambda_m(t).$$

Using the properties of the Finsler structure $F_{(\Gamma)}(x, \xi)$ of the *finite-dimensional* Teichmüller spaces $\mathbb{T}(\Gamma)$ and the properties of extremal Beltrami differentials, we get the following lemma.

LEMMA 1.17. *For any holomorphic map $h: \Delta \rightarrow \mathbb{T}(\Gamma)$ such that*

$$h'(0) = h''(0) = \dots = h^{(m)}(0) = \mathbf{0}, \quad m \geq 1, \quad (1.31)$$

we have

$$\lambda(t) := F(h(t), h'(t)) \leq \lambda_m(t), \quad t \in \Delta.$$

This allows us to prove the following theorem.

THEOREM 1.18. *Let $h: \Delta \rightarrow \mathbb{T}(\Gamma)$ be a holomorphic map such that (1.6) holds. Then, for all $t \in \Delta$, we have*

$$d_{\mathbb{T}}(h(t), h(0)) \leq \rho(0, |t|^{m+1}). \quad (1.32)$$

Now the assertion of Theorem 1.14 for $k = 1$ follows from (1.32) applied to the holomorphic map

$$h(t) = S_{f_t}: \Delta \rightarrow \mathbb{T}(\Gamma), \quad h'(0) = \mathbf{0},$$

generated by the motions $f_t(z)$. Here

$$S_f = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2$$

is the Schwarzian derivative of f .

The proof for an arbitrary $k < 1$ involves the properties of the metrics on hyperbolic balls

$$B_k(\Gamma) = \{x \in \mathbb{T}(\Gamma): d_T(x, 0) < K\}, \quad 1 \leq K < \infty,$$

where k and K are related by $k = \tanh K$.

1.5. An improvement of Grunsky's and Milin's univalence criteria

Most of generally adopted methods to establish global univalence of a locally univalent holomorphic function are based either on infinite systems of inequalities (e.g., Grunsky's, Goluzin's or similar types) or after Nehari [Ne1], involving some invariant differential operators (Schwarzian derivative etc.). A sharpened form of corresponding inequalities ensures quasiconformal extension of holomorphic functions (see, e.g., [AvAE], [Po1] and references cited there).

As already mentioned in [Kru21, Section 2], due to Grunsky's univalence criterion, a meromorphic function in Δ^* with expansion (1.26) is univalent if and only if its Grunsky coefficients c_{mn} defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} c_{mn} z^m \zeta^n \quad (1.33)$$

(where the principal branch of the logarithmic function is chosen) satisfy the inequality

$$\left| \sum_{m,n=1}^{\infty} \sqrt{mn} c_{mn} x_m x_n \right| \leq 1 \quad (1.34)$$

for every $\mathbf{x} = (x_1, x_2, \dots) \in l^2$ with $\|\mathbf{x}\| = 1$.

Milin [Mi] proved an equivalent result:

THEOREM 1.19. *A function f with the series expansion (1.26) in a neighborhood of the infinity point is holomorphic and univalent in Δ^* if and only if its Grunsky coefficients c_{mn} satisfy the inequalities*

$$\sum_{n=1}^{\infty} n |c_{mn}|^2 \leq \alpha_m, \quad m = 1, 2, \dots, \quad (1.35)$$

where

$$\lim_{m \rightarrow \infty} \alpha_m^{1/m} \leq 1.$$

If, in addition, $\alpha_m = 1/m$ and equality holds at least in one inequality from (1.35), then all other inequalities also become equalities and, moreover, the image $f(\Delta^*)$ does not contain exterior points.

Both criteria are extended at least to the finitely connected domains. The next result provides their improvement.

THEOREM 1.20 [Kru15]. *Let a function f be holomorphic in a neighborhood of infinity with the series expansion of the form (1.26). Define its Grunsky coefficients c_{mn} by (1.33). Then the following conditions are equivalent:*

- (i) $\limsup_{m+n \rightarrow \infty} |c_{mn}|^{1/(m+n)} = \alpha \leq 1$;
- (ii) $\lim_{m \rightarrow \infty} (\sum_{n=1}^{\infty} n |c_{mn}|^2)^{1/2m} = \alpha$;
- (iii) f is holomorphic and univalent in the disk $\Delta_\alpha^* = \{z \in \widehat{\mathbb{C}}: |z| > \alpha\}$.

In the case when $\alpha < 1$, every condition from (i), (ii), (iii) ensures that f admits k -quasiconformal extension across the unit circle $S^1 = \{z: |z| = 1\}$ and the image curve $L = f(S^1)$ admits k -quasiconformal reflection with $k \leq \alpha^2$. This bound is sharp.

The proof of the first two assertions of the theorem actually follows the classical cases. The proof of (iii) concerning α^2 -quasiconformal extension is more complicated; it relies on Theorem 1.14.

1.6. r^2 -property and a dynamical characterization of the disk

In fact, Kühnau posed more general problems which concern quasiconformal extension of conformal maps across the level lines of Green's function of an arbitrary simply connected plane domain. Such a situation appears in Fredholm eigenvalue theory.

The point is that in a sense the inverse fact is also true; thus the r^2 -property is a dynamical characterization of the disk among simply connected domains. This is proved in [KrK2].

To formulate the result, we present some definitions.

Let D be a simply connected hyperbolic domain in $\widehat{\mathbb{C}}$.

The preimages of the circles $C_r = \{|z| = 1/r\}$, $0 < r < 1$, under a conformal map of D onto the disk Δ^* are called r -lines in D . These r -lines depend on the “center point” of D which corresponds to the point at infinity.

We say that such a domain D admits the r^2 -property if for every r and every center point there exists for every conformal map $f: D \rightarrow \widehat{\mathbb{C}}$ an r^2 -quasiconformal self-map of $\widehat{\mathbb{C}}$ whose restriction to the interior of the r -line of D (which contains the center point) is f .

Due to the previous subsection, every disk admits the r^2 -property as in [Kru5]. On the other hand, the main result of [KrK2] is the following theorem.

THEOREM 1.21. *Any simply connected domain D which is not a disk does not admit the r^2 -property. Moreover, to every given center point in D there is a conformal embedding f of D into $\widehat{\mathbb{C}}$ whose quasiconformal extensions across each r -line have only dilatations $k(f) > r^2$.*

The proof of this theorem also relies upon the properties of invariant metrics of the universal Teichmüller space as well as on the general theorem about the range of values of holomorphic functionals on the classes of quasiconformal maps given in [Kru21, Section 3.2]. Most of the difficulties appear in the investigation of the case when $a_1 = 0$.

It has been noticed by Kühnau in [KrK2] that although D does not have the r^2 -property if it is not a disk, it always has the (analogously defined) $\frac{2r^2}{1+r^4}$ -property. To prove this, we apply twice the r^2 -property [Kru5] of Δ^* (for the maps $\Delta^* \rightarrow D$ and $\Delta^* \rightarrow$ another simply connected domain).

Kühnau's next observation is that the assertion of Theorem 1.14 would follow if one knows that the ratio $q_f(r)/r^2$ strictly monotonically increases with r provided f is not a map of the form $f = z + a/z$, $|a| \leq 1$, where $q_f(r)$ is the least bound for the dilatations of all extensions of the map f_r (cf. with the analogous situation in connection with the reciprocal Fredholm eigenvalues of the r -lines [Ku1]). Then we have also

$$\lim_{r \rightarrow 0} \frac{q_f(r)}{r^2} = |a_1|,$$

where a_1 is the coefficient by z^{-1} in (1.26). This follows from $|a_1| \leq q_f(r)/r^2$ [KrK1, p. 100] and from the fact that

$$f(z) = z + a_1 r^2 \bar{z} + a_2 r^4 \bar{z}^2 + \dots$$

is a possible quasiconformal extension to $|z| \leq 1/r$; therefore

$$q_f(r) \leq |a_1| r^2 + 2|a_2| r^3 + 3|a_3| r^4 + \dots$$

1.7. Application to coefficient estimates for univalent functions: Zalcman's conjecture

Theorem 1.14 was applied in [Kru9] to solve some coefficient problems for univalent functions related to the Bieberbach and Zalcman conjectures.

Let S denote again the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ holomorphic and univalent in the unit disk Δ . There were several well-known conjectures implying the Bieberbach conjecture that $|a_n| \leq n$ for each $f \in S$. Most of them have been proved by the de Branges theorem [DB].

Still unsolved, however, is Zalcman's conjecture, which asserts that, for any $f \in S$,

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2. \quad (1.36)$$

This conjecture, which was posed many years ago as an approach to prove the Bieberbach conjecture, remains a very interesting open problem (for $n \geq 3$) even after de Branges' result. It has been proved only for certain special subclasses of S in [BrT, Ma].

As pointed out by Pfluger [Pfl], the quantities $\lambda a_n^2 - a_{2n-1}$ with $\lambda > 0$ arise naturally because such expressions are contained in the representation of the coefficients of the functions $(f(z^2))^{1/2}$ and $1/f(1/z)$.

Denote $\kappa_{\alpha t}(z) = z(1 - \alpha t z)^{-2}$ which represents an $|\alpha|$ -quasiconformal deformation of the Koebe function

$$\kappa_t(z) = z(1 - tz)^{-2}. \quad (1.37)$$

Here $|\alpha| \leq 1$ and $|t| = 1$.

Let us consider on S the functionals of the form

$$I_\lambda(f) = a_3^2 - a_5 + \lambda P(a_2, a_3, a_4, a_5), \quad \lambda \geq 0,$$

where $a_j = a_j(f)$ and P is a homogeneous polynomial of degree 4 such that $|I_\lambda(\kappa_\alpha)| = \alpha^4 + P(a_2(\kappa_\alpha), \dots, a_5(\kappa_\alpha))$ does not decrease for $0 < \alpha < 1$.

The following result is established in [Kru9] using the holomorphic isotopy

$$f_t(z) = \frac{1}{t} f(tz) = z + \sum_{n=2}^{\infty} a_n t^{n-1} z^n, \quad t \in \Delta.$$

THEOREM 1.22. *For any functional P of the above form, there exists $\lambda_0 > 0$ such that, for every $\lambda \leq \lambda_0$, the maximum of $|I_\lambda|$ on S is attained only at the Koebe function (1.37), i.e., $\max_S |I_\lambda(f)| = |I_\lambda(\kappa_t)|$.*

This proves Zalcman's conjecture for $n = 3$. Moreover, it implies that if $f \in S$, then for $\lambda \geq 1$ we have $|\lambda a_3^2 - a_5| \leq 9\lambda - 5$, with equality for κ_t only.

The arguments in the proof possess a nice geometric interpretation. The coefficients of f_t have the same homogeneity degree as the Zalcman functional $a_3 - a_5$. Using the properties of extremal quasiconformal maps, it is shown, roughly speaking, that for an extremal function f_0 of the Zalcman functional, the tangent map at $t = 0$ should be equal to $\kappa_\alpha(z) = z(1 - \alpha tz)^{-2}$; in other words, $f_{0t} = w^\sigma \circ \kappa_{\alpha t}$. Now the goal is to establish that $|\alpha| = 1$; then w^σ should reduce to a rotation.

The above construction can be described more easily in terms of holomorphic disks in the Teichmüller space. The existence of extensions with the Beltrami coefficients depending holomorphically on the time parameter in the whole disk, ensured by Ślodkowski's lifting theorem, is crucial for the proof.

1.8. Analytic dependence of conformal invariants on parameters

The classical problem of dependence of various quantities related to the Riemann surfaces on the moduli of these surfaces goes back to Klein and Poincaré and is intrinsically connected with uniformization of the surfaces.

Its solution for tori (and related ring domains) was given by Teichmüller [Te2] by applying his method which involves the extremal quasiconformal maps. The investigations of many authors in this direction were completed by general theorems of Bers (see [Ber1, Ber3]).

Gaier posed certain problems about the dependence of real conformal invariants on parameters of plane domains. This case has specific features. Earle and Mitra solved these problems in [EMi]; they applied the technique of holomorphic motions and the Teichmüller space theory.

The general question can be stated informally as follows.

Suppose a family of plane domains depends on some geometric parameters. Are the conformal invariants of the domains real analytic functions of these parameters?

Earle and Mitra gave a positive answer in a wide variety of cases. They have shown that the invariants vary real analytically when the boundary of a domain undergoes a holomorphic motion. Such a motion determines a quasiconformal variation of the plane.

The following results of [EMi] concern the module of an annulus and the length of the Poincaré geodesic in a given homotopy class. The parameter t defining a holomorphic motion can run, in the general case, over a connected complex Banach manifold V .

THEOREM 1.23. *Let E be a boundary of the nondegenerate ring domain D . If ϕ is a holomorphic motion of E over the parameter space V , then the set $\phi_t(D)$ is a nondegenerate ring domain for each t in V , and the function $t \mapsto \text{mod}(\phi_t(D))$ is real analytic in V .*

THEOREM 1.24. *Let D be a region in $\widehat{\mathbb{C}}$ whose boundary E contains at least three points, and let ϕ be a holomorphic motion of E over the simple connected parameter space V . For any closed Poincaré geodesic in D , the length of the closed Poincaré geodesic $\phi(\gamma)$ in $\phi(D)$ is a real analytic function of t in V .*

2. Quasireflections across curves

We now consider some other classical and modern topics of geometrical complex analysis, which involve the interplay of holomorphic or quasiconformal extension of the maps with the best polynomial approximation of holomorphic functions in one and two dimensions as well as the Teichmüller space theory.

2.1. Topological background

2.1.1. The classical Brouwer–Kerekjarto theorem ([Br,Ke], see also [Sm1]) says that every periodic homeomorphism of S^2 is topologically equivalent to a rotation, or to a product of a rotation and a reflection across a diametral plane. The first case corresponds to orientation preserving homeomorphisms (and then E consists of two points), the second one is orientation reversing, and then either the fixed point set E is empty (which is excluded in our situation) or it is a topological circle.

We are concerned with homeomorphisms reversing orientation. Such homeomorphisms of order 2 are topological involutions of S^2 with $f \circ f = \text{id}$ and are called topological reflections.

The Brouwer–Kerekjarto theorem was extended in [Sm2] to actions on the sphere S^3 (for suitable orders p of homeomorphisms). As for larger dimensions n , there are examples which show that the fixed point set of a topological periodic transformation on S^n is not necessarily a manifold.

The homological structure of the fixed point sets of periodic homeomorphisms on the sphere S^n is described by the Smith theory (see, e.g., [Sm1,Sm2]), which states that if the group $\mathbb{Z}_p$ acts on a homological n -sphere over $\mathbb{Z}_p$ then its fixed point set E is a homological r -sphere over $\mathbb{Z}_p$, for some $r \leq n$. In the case of smooth periodic maps, this set is a smooth submanifold, cf. [CF].

Smith's theory was applied to quasiconformal reflections with $f \circ f = \text{id}$ across closed hypersurfaces in $\mathbb{R}^n$, $n \geq 2$ (which are the boundaries of some Jordan domains), in [Ya].

2.1.2. We shall consider *arbitrary quasiconformal homeomorphisms of S^2 reversing orientation* and do not require that they be periodic of finite order.

2.2. Quasiconformal reflections

2.2.1. Let us consider *quasiconformal* reflections on the sphere $S^2 = \widehat{\mathbb{C}}$. The topological circles admitting such reflections are *quasicircles*, i.e., the topological circles which are locally *quasiintervals* (the images of straight line segments under quasiconformal maps of the sphere S^2). Qualitatively, any quasicircle L is characterized, due to [Ahl2], by uniform boundedness of the cross-ratios for all ordered quadruples (z_1, z_2, z_3, z_4) of the distinct points on L ; namely,

$$\frac{\overline{z_1 z_2} \overline{z_3 z_4}}{\overline{z_1 z_3} \overline{z_2 z_4}} \leq C < \infty$$

for any quadruple of points z_1, z_2, z_3, z_4 on L following this order. Using a fractional linear transformation, one can send one of the points, for example, z_4 , to infinity; then the above inequality assumes the form

$$\left| \frac{z_2 - z_1}{z_3 - z_1} \right| \leq C.$$

This is shown in [Ahl2] by applying the properties of quasimetric maps. Ahlfors has established also that if a topological circle L admits quasireflections (i.e., is a quasicircle), then there exists a differentiable quasireflection across L which is Lipschitz-continuous. This property is very useful in various applications.

It is evident that every quasicircle has zero two-dimensional Lebesgue measure.

Geometrically, a quasicircle is characterized by the property that, for any two points z_1, z_2 on L , the ratio of the chordal distance $|z_1 - z_2|$ to the diameters of the corresponding subarcs with these endpoints is uniformly bounded.

Other characterizations of quasicircles are given, for example, in [Ge, LV, Po1]. For an extension to higher dimensions see [MMPV, Mik, Ya].

Quasireflections across more general sets $E \subset \widehat{\mathbb{C}}$ also appear in certain questions and are of independent interest. Those sets admitting quasireflections are called *quasiconformal mirrors*.

2.2.2. One defines for each mirror E its *reflection coefficient*

$$q_E = \inf k(f) = \inf \|\partial_z f / \partial_{\bar{z}} f\|_\infty \quad (2.1)$$

and *quasiconformal dilatation*

$$Q_E = \frac{1 + q_E}{1 - q_E} \geq 1;$$

the infimum in (2.1) is taken over all quasireflections across E , provided those exist, and is attained by some quasireflection f_0 .

When $E = L$ is a quasicircle, the corresponding quantity

$$k_E = \inf\{k(f_*): f_*(S^1) = E\} \quad (2.2)$$

and the reflection coefficient q_E can be estimated in terms of one another; moreover, due to [Ahl3, Ku11], we have

$$Q_E = K_E^2 := \left(\frac{1+k_E}{1-k_E} \right)^2. \quad (2.3)$$

The infimum in (2.2) is taken over all orientation preserving quasiconformal automorphisms f_* carrying the unit circle onto L , and $k(f_*) = \|\partial_{\bar{z}} f_* / \partial_z f_*\|_\infty$.

It was established in [Kru14, Kru16] that any set $E \subset S^2$ which admits an orientation reversing quasiconformal homeomorphism f of the sphere S^2 keeping this set pointwise fixed, is necessarily a subset of a quasicircle (hence, any quasiconformal mirror $E \subset S^2$ obeys quasiconformal involutions of S^2 of order 2). Moreover, this quasicircle can be chosen to have the same reflection coefficient as the initial set E (cf. [Kru16, Ku14]). This result gives a complete answer to a question of Kühnau to describe all sets $E \in \widehat{\mathbb{C}}$ which admit quasiconformal reflections (see [Ku10, Ku16]); it yields also various quantitative consequences. In particular, this result provides that equality (2.3) is true for any subset E of S^2 (assuming $Q_E = \infty$ if E does not admit quasiconformal reflections). We shall discuss these results below.

We point out that the conformal symmetry on the extended complex plane is strictly rigid and reduces to reflection $z \mapsto \bar{z}$ within conjugation by transformations $g \in PSL(2, \mathbb{C})$. The quasiconformal symmetry avoids such rigidity and is possible over quasicircles. The results mentioned above show that, in fact, this case is the most general one, since for any set $E \subset \widehat{\mathbb{C}}$ we have $Q_E = \infty$, unless E is a subset of a quasicircle with the same reflection coefficient.

Let us mention also that a somewhat different construction of quasiconformal reflections across Jordan curves has been provided in [EN]; it relies on the conformally natural extension of homeomorphisms of the circle introduced by Douady and Earle [DE].

2.3. Fredholm eigenvalues

Though a general theory of quasireflections of S^2 is rather complete, there are no general methods for calculating the exact or approximate values of the reflection coefficients of particular curves and arcs. So far these coefficients have been established only for a few classes of curves. Even for polygons only a few special results are known (see, e.g., [Ku8, Ku11, We]). This remains an important open problem initiated by Kühnau.

Here, the least nontrivial Fredholm eigenvalue $\lambda_1 = \lambda_L$ plays a crucial role. The Fredholm eigenvalues are defined for a smooth closed bounded curve L to be the eigen-

values of the double-layer potential over L , in other words, of the equation

$$h(z) = \frac{\rho}{\pi} \int_L h(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta, \quad \zeta \in L,$$

where n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$. These values are important in various questions, see, e.g., [Ahl1, Ga, Roy1, Schi1, Schi2, Scho3, War], [KrK1, Part 2].

The indicated eigenvalue ρ_L can be defined for any oriented closed Jordan curve $L \subset \widehat{\mathbb{C}}$ by the equality

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)}, \quad (2.4)$$

where G and G^* are, respectively, the interior and exterior of L ; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. Due to [Ahl1],

$$q_L \geq \kappa_L := \frac{1}{\rho_L}. \quad (2.5)$$

In view of the invariance of all quantities in (2.5) under the action of the Möbius group $PSL(2, \mathbb{C})/\pm 1$, it suffices to consider the quasiconformal homeomorphisms of the sphere carrying S^1 onto L whose Beltrami coefficients $\mu_f(z) = \partial_{\bar{z}} f / \partial_z f$ have support in the unit disk Δ , and $f|_{\Delta^*}(z) = z + b_0 + b_1 z^{-1} + \dots$ (or in the upper half-plane $U = \{\operatorname{Im} z > 0\}$). Then q_L is equal to the minimum $k_0(f)$ of dilatations $k(f) = \|\mu\|_\infty$ of quasiconformal extensions of the function $f^* = f|_{\Delta^*}$ onto Δ .

By the Kühnau–Schiffer theorem [Ku7, Schi2] we have $\rho_L = 1/\kappa(f^*)$, where

$$\kappa(f^*) = \sup \left| \sum_{m,n=1}^{\infty} \sqrt{mn} \alpha_{mn} x_m x_n \right|$$

is the *Grunsky constant* of the map f^* ; here α_{mn} are the Grunsky coefficients obtained from the expansion of the principal branch of the function $\log[(f^*(z) - f^*(\zeta))/(z - \zeta)]$ into the double series on the bidisk $(\Delta^*)^2$, and the supremum is taken over the points $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\|^2 = 1$ (cf. Section 1.5).

It is well known that $\kappa(f^*) \leq k_0(f^*)$, but in the general case the strict inequality $\kappa(f^*) < k_0(f^*)$ holds (see, e.g., [Ku6, Ku9, Kru3], [Kru21, Section 2]). A complete characterization of maps for which $\kappa(f^*) = k_0(f^*) = q_{f^*(S^1)}$ is given in [Kru3, Kru6].

One of the standard ways of establishing the reflection coefficients q_L (respectively, the Fredholm eigenvalues ρ_L) consists of verifying whether the equality in (2.5) or the equality $\kappa(f^*) = k_0(f^*)$ hold for a given curve L (cf. [Kru6, Kru10, Ku8, Ku11, We]).

This is unknown even for the rectangles, for which such a question was stated by Kühnau already more than 20 years ago. It was established only (see [Ku6, Ku11, We]) that the answer is in the affirmative for the square and for close rectangles $\mathcal{R}$ whose moduli $m(\mathcal{R})$

vary in the interval $1 \leq m(\mathcal{R}) < 1.037$; moreover, in this case $q_L = 1/\rho_L = 1/2$. The method exploited by these authors relies on an explicit construction of an extremal reflection. Kühnau made also deep use of Fredholm eigenvalue theory.

Another way relies on Teichmüller space theory and on approximation of holomorphic functions of several complex variables. We discuss briefly some of these results in Section 3.

2.4. Results of Kühnau

2.4.1. After Ahlfors' seminal papers [Ahl1, Ahl2], the works of Kühnau in quasiconformal reflections and their applications gave the deepest contribution to this theory. He has outlined a wide program for investigations, setting many open problems and conjectures. We already quoted above his various results related to quasireflections; many other results will be listed in this and in the following sections.

2.4.2. Let L be a quasicircle on $\widehat{\mathbb{C}}$. Denote its interior and exterior by D and D^* , respectively, and define the functionals

$$Q_L = \frac{q_L + 1}{q_L - 1}, \quad \Lambda_L = \frac{\lambda_L + 1}{\lambda_L - 1}, \quad M_L = \sup \frac{\text{mod}(Q^*)}{\text{mod}(Q)},$$

where Q and Q^* are the quadrilaterals in D and D^* , respectively, with the common vertices $z_1, z_2, z_3, z_4 \in L$, and the infimum is taken over conformal moduli of all such quadrilaterals.

THEOREM 2.1 [Ku19]. *For each quasicircle L , we have*

$$M_L \leq \Lambda_L \leq Q_L.$$

2.4.3. Now suppose that $E = (z_1, \dots, z_n)$ be a given ordered system of $n \geq 4$ distinct points on the sphere $\widehat{\mathbb{C}}$. Consider the family $\mathcal{L}$ of all quasiconformal Jordan curves passing through these points (in the prescribed order). Let $\mathcal{L}_\alpha$ denote the subfamily of $\mathcal{L}$, which belongs to a fixed homotopy class α of admissible curves on the punctured sphere $\widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$. We set

$$Q_\alpha(z_j) = \inf_{L \in \mathcal{L}_\alpha} Q_L$$

and are interested to determine a minimal quasicircle $L_0 \in \mathcal{L}_\alpha$, provided each point z_{j_0} is *essential* for $Q_\alpha(z_j)$, i.e., that after removing this point the reflection coefficient becomes smaller. The existence of extremal curves is evident.

The problem is reduced to minimization of dilatations $K(h)$ in the corresponding homotopy class of quasiconformal homeomorphisms of the punctured spheres

$$h(\bar{z}): \widehat{\mathbb{C}} \setminus \{\bar{z}_1, \dots, \bar{z}_n\} \rightarrow \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\},$$

preserving orientation. Using Teichmüller's theorem on extremal quasiconformal maps (cf., e.g., [GaL,Kru1]), Kühnau has established that the desired quasicircle L_0 has the reflection coefficient $Q_{L_0} = Q_E$ and is composed of analytic subarcs of passing through the points z_k horizontal trajectories of the $\widehat{\mathbb{C}}$ -holomorphic integrable quadratic differential φdz^2 , which determines the extremal quasiconformal map h^{-1} . For details of the geometric description of the solution of initial problem, we refer to [Ku14].

In the case of four points ($n = 4$), an explicit complete description is given in [Ku15]; it involves the elliptic integrals of first kind. See also Kühnau's surveys [Ku11,Ku16].

2.5. Some qualitative estimates

2.5.1. Kühnau's investigations of the extremal quasireflections across the closed Jordan curves were extended by Werner. He established certain sharp bounds for the reflection coefficients given in the following theorems.

THEOREM 2.2 [We]. *Let P be a tangential circular polygon (whose sides touch a common circle) with the least interior angle $\alpha\pi$. Then*

$$q_{\partial P} = \frac{1}{\rho_{\partial P}} = 1 - \alpha. \quad (2.6)$$

In particular, this yields the exact values of the reflection coefficient and of the Fredholm eigenvalue for each rectilinear or circular triangle.

Let $\mathcal{R}$ be a rectangle whose horizontal and vertical sides have the lengths a and b , respectively. Consider its conformal module $\mu(\mathcal{R}) = b/a$.

THEOREM 2.3 [We]. *For the rectangles $\mathcal{R}$, we have:*

- (i) *if the module $\mu(\mathcal{R})$ satisfies $1 \leq \mu(\mathcal{R}) < 1.037$, then $q_{\partial \mathcal{R}} = 1/\rho_{\partial \mathcal{R}} = 1/2$;*
- (ii) *if $\mu(\mathcal{R}) > 2.76$, then $q_{\partial \mathcal{R}} > 1/2$;*
- (iii) *the dilatation $Q_{\partial \mathcal{R}}$ satisfies*

$$\frac{\pi}{4} < \frac{Q_{\partial \mathcal{R}}}{\mu(\mathcal{R})} < \pi, \quad \text{i.e., } Q_{\partial \mathcal{R}} = O(\mu(\mathcal{R})) \text{ as } \mu(\mathcal{R}) \rightarrow \infty.$$

The proofs of these theorems rely on the explicit geometric construction of extremal quasireflections and involve the properties of the elliptic integrals.

The equality in (i) was earlier established for the square in [Ku11]. The assertion (ii) was proved in [Ku8] for $\mu > 3.31$.

2.5.2. As was already indicated in Section 2.2.1, Ahlfors established that any quasicircle L admits a Euclidean M -bi-Lipschitz reflection f with $M = M(K_L)$. Here M denotes the Lipschitz constant of f , i.e.,

$$1/M |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq M |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C}.$$

Replacing the Euclidean distance by the hyperbolic distances in the corresponding regions D and $f(D)$, one can define the hyperbolic M -bi-Lipschitz reflections.

Certain estimates connecting the reflection coefficients with the Euclidean and hyperbolic dilatations of the corresponding quasiconformal deformations were obtained by Gehring and Hag in [GeH]. For example, they proved the following result.

THEOREM 2.4. *Let D be a K -quasidisk. If f is a hyperbolic M -bi-Lipschitz reflection in ∂D , then*

$$M \geq K^2.$$

If f is a Euclidean M -bi-Lipschitz reflection in ∂D , then

$$M \geq \csc\left(\frac{\pi}{K^2 + 1}\right).$$

2.5.3. Some explicit expressions and bounds for the reflection coefficients of other special curves are given, e.g., in [Ahl1, Har1, Kru5, Kru6, KuB]. In the following sections, we provide a general approach.

2.6. Quasiconformal extension and reflections across quasicircles

One of the consequences of Theorem 1.14 is the following result which solves a problem of Kühnau posed in [Ku11].

THEOREM 2.5 [Kru5]. *For each conformal map f of an annulus $A_{rR} = \{z: r < |z| < R\}$ ($0 < r < 1 < R < \infty$), the reflection coefficient of the image $L = f(S^1)$ of the unit circle is bounded by*

$$q_L \leq \frac{1 + (rR)^2}{r^2 + R^2}. \quad (2.7)$$

This bound is sharp, the extreme value is achieved by the Koebe function (1.37).

This result allows the following improvement for the maps f with k -quasiconformal extensions to $\widehat{\mathbb{C}}$:

$$q_L \leq k \frac{1 + (rR)^2}{R^2 + k^2 r^2}; \quad (2.8)$$

the bound is again the best possible.

2.7. Sketch of the proof of Theorem 2.5

It suffices to establish (2.7) for the maps which are C^2 -smooth in $\overline{A}_{rR}$. Having a quasi-conformal extension w^μ of f , we factorize $w^\mu = w^{\mu_2} \circ w^{\mu_1}$, where w^{μ_1} and w^{μ_2} are conformal in Δ_r^* and $w^{\mu_1}(\Delta_R)$, respectively.

Applying Theorem 1.14 to $w^{\mu_1}|_{\Delta^*}$, we obtain a k_1 -quasiconformal extension of these function to $\widehat{\mathbb{C}}$ across S^1 , with $k_1 \leq r^2$; we denote it by $\omega(z)$.

The map w^{μ_2} will be treated in a different way, using standard approximation.

We choose an everywhere dense countable set on the circle $C_R = \{w: |w| = R\}$. Let

$$E_n = \{z_1, \dots, z_n\}.$$

Fix $n \geq 4$; then we have a holomorphic map

$$t \mapsto (f_t(z_1) = w^{\mu_2} \circ \omega_t(z_1), \dots, f_t(z_n) = w^{\mu_2} \circ \omega_t(z_n)) \quad (2.9)$$

of the disk Δ onto domain

$$V_n := \{w = (w_1, \dots, w_n) \in \mathbb{C}^n: w_l \neq w_m \text{ if } l \neq m\},$$

which is lifted to a holomorphic map h_n of Δ into the Teichmüller space $\mathbb{T}(0, n)$ of spheres with n punctures:

$$h_n: \Delta \rightarrow \mathbb{T}(0, n), \quad h_n(t) = [\mu_{w^{\mu_2} \circ \omega_t}]_n.$$

(Here $[\mu]_n$ denotes the points of $\mathbb{T}(0, n)$ as the equivalence classes of the Beltrami coefficients under the canonical factorization of the maps which are homotopic on an n times punctured sphere.) The fact that this map is well defined follows from the conformality of the function $w^{\mu_2} \circ \omega_t$ on the initial annulus A_{rR} . Theorem 1.14 yields

$$h'_n(0) = \mathbf{0}.$$

Let $\tau_n(\mathbf{x}, \mathbf{y})$ and $d_n(\mathbf{x}, \mathbf{y})$ denote the Teichmüller metric and the Kobayashi metric of $\mathbb{T}(0, n)$, respectively, defined by the hyperbolic metric $\text{hyp}(0, t)$ in Δ of curvature -4 . Then from Royden–Gardiner’s theorem and Lemma 1.17, we have

$$\tau_n(h_n(0), h_n(t)) = d_n(h_n(0), h_n(t)) \leq \text{hyp}(0, |t|^2). \quad (2.10)$$

Setting $n = 4, 5, \dots$, we get a sequence of k_n -quasiconformal maps $\{W^{\nu_n}\}$, where ν_n is the extremal Beltrami coefficient from $\mathbf{M}(\mathbb{C})$ for $h_n(1/R)$. From (2.10),

$$k_n = \|\nu_n\|_\infty \leq 1/R^2.$$

By compactness we can extract from $\{W^{\nu_n}\}$ a subsequence which converges spherically uniformly to a k_2 -quasiconformal extension $\Omega(w)$ of $w^{\mu_2}|_{\omega(\Delta)}$ to $\widehat{\mathbb{C}}$, with $k_2 \leq R^{-2}$.

In addition, we obtain from above that both extensions $\omega(z)$ and $\Omega(w)$ have *extremal* Beltrami coefficients, that is, with minimal dilatations. By Strebel's frame mappings criterion [St1], these coefficients are unique and of Teichmüller type related to holomorphic quadratic differentials. Evaluating the dilatation of the composition

$$F = \Omega \circ \omega \circ 1/\bar{z} \circ \omega^{-1} \circ \Omega^{-1},$$

we obtain (2.7).

To establish (2.8), one should consider the hyperbolic Kobayashi balls in $\mathbb{T}(0, n)$ and use the strict assertion of Theorem 1.14.

As a consequence of Theorem 2.5, we obtain the following sharp result (see [Kru13]).

THEOREM 2.6. *For any analytic arc L whose defining function f extends to a holomorphic injection of the interior E_i of the ellipse with foci $-1, 1$ and semiaxes a, b , we have*

$$q_L \leq \frac{1}{a^2 + b^2}. \quad (2.11)$$

To prove (2.11), the map f is lifted to a conformal map $\tilde{f}$ of the two-sheeted cover of $\tilde{E}_i$ with branch points at foci onto the two-sheeted cover of $f(E_i)$ with branch points $f(-1)$ and $f(1)$. Then, conjugating $\tilde{f}$ by the Joukowski function

$$j(\tau) = \frac{1}{2} \left(\tau + \frac{1}{\tau} \right), \quad (2.12)$$

we get a holomorphic injection $f_j = j^{-1} \circ \tilde{f} \circ j$ of the annulus $r < |\zeta| < 1/r$, where $r = 1/(a + b)$. It maps the circle $\{|\zeta| = 1\}$ onto the curve $j^{-1}(\tilde{L})$, where $\tilde{L} = \tilde{f}([-1, 1])$ is a double cover of the initial arc L . Therefore, $q_{j^{-1}(\tilde{L})} = q_{\tilde{L}} = q_L$, which implies (2.7).

The estimates (2.7) and (2.11) show that the bounds for reflection coefficients depend only on the conformal module of the annulus A_{rR} , respectively on the defining parameters of the ellipse E , but not on the geometric characteristics of the curves L . The proof of (2.8) makes use of the complex metric geometry of Teichmüller spaces.

2.8. Asymptotical approach

It is natural to consider the asymptotical behavior of the reflection coefficients when the sets shrink to a point.

There is a well-known asymptotic estimate of Kühnau for small analytic arcs. His remarkable result is:

THEOREM 2.7 [Ku10, Ku13]. *Let L be a fixed analytic arc, and L_ε be the closed subarc of the length 2ε , with the midpoint z_0 . Let s be the arc length on L , and $k(s)$ the curvature at point $z = z(s)$. Then,*

$$q_{L_\varepsilon} = \frac{1}{12} \left| \frac{dk(z_0)}{ds} \right| \varepsilon^2 + O(\varepsilon^3) \quad \text{as } \varepsilon \rightarrow 0, \quad (2.13)$$

where the ratio $O(\varepsilon^3)/\varepsilon^3$ remains bounded as $\varepsilon \rightarrow 0$.

Under the additional condition that the reflections preserve the point at infinity, the estimate (2.13) is replaced by

$$q_{L_\varepsilon} = \frac{1}{4}k(z_0)\varepsilon + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

Note that the estimate (2.13) contains the derivative of the curvature in accordance with the fact that the fractional linear transformations change the curvature of a curve but preserve its reflection coefficient.

The proof of (2.13) involves some intrinsic properties of Fredholm eigenvalues, as well as Pick's formula: for any arc L defined by $z = z(t)$ ($0 \leq t \leq 1$),

$$\frac{dk}{ds} = \frac{\operatorname{Im} S_z(t)}{|z'(t)|^2},$$

here $S_z(t) = (z''/z')' - (z''/z')^2/2$ is the Schwarzian derivative of z .

It turns out that order two of decay holds for shrinking along every three times differentiable arc. More precisely, we have the following theorem.

THEOREM 2.8. *Let L be a $C^{3+\alpha}$ -smooth arc, and let L_ε be its subarc of the length 2ε with the midpoint z_0 . Then*

$$q_{L_\varepsilon} = M(z_0)\varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0,$$

or equivalently, denoting the endpoints of L_ε by $z_\varepsilon^\pm$,

$$q_{L_\varepsilon} = M(z_0)|z_\varepsilon^\pm - z_0|^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0.$$

Here the constant M and the upper bounds for the ratios $o(\varepsilon^2)/\varepsilon^2$ depend on the arc L .

Recall that $C^{m+\alpha}$ -smoothness of an arc L' means that its defining function $z = h(t) : [0, 1] \rightarrow L'$ satisfies

$$\|h\|_{C^{m+\alpha}[0,1]} := \sum_{j=0}^m \|h^{(j)}(t)\|_{C[0,1]} + \max_{0 \leq t_1, t_2 \leq 1} \left| \frac{h^{(m)}(t_1) - h^{(m)}(t_2)}{t_1 - t_2} \right| < \infty.$$

The proof of this theorem is given in [Kru14]. It involves the estimates of distortion of conformal maps and their derivatives in the closed disk (given, e.g., in [Da]) which are applied to the osculating circle (circle of curvature) of the arc C at its midpoint z_0 .

2.9. Reflection coefficients and Fredholm eigenvalues of convex domains

2.9.1. Main theorem. As was mentioned above, among the important and still unsolved problems for applications in this theory, it remains to develop general algorithms for an

exact or approximate calculation of the reflection coefficients of particular curves and their subarcs.

Our goal is to present a new method which enables us to solve the indicated problems for large classes of convex domains and of their fractional linear images. This method involves in an essential way the complex geometry of the universal Teichmüller space $\mathbb{T}$ and the Finsler metrics on holomorphic disks in $\mathbb{T}$ as well as the properties of holomorphic motions on such disks.

The following theorem implies an explicit representation of the domain invariants by their geometric characteristics.

THEOREM 2.9 [Kru18,Kru19]. *For every unbounded convex domain $D \subset \mathbb{C}$ with piecewise $C^{1+\delta}$ -smooth boundary L ($\delta > 0$) (and all its fractional linear images), the equalities*

$$q_L = \frac{1}{\rho_L} = \kappa(f) = \kappa(f^*) = k_0(f) = k_0(f^*) = 1 - |\alpha| \quad (2.14)$$

hold, where f and f^* denote the appropriately normalized conformal maps $\Delta \rightarrow D$ and $\Delta^* \rightarrow D^*$, respectively; $k_0(f)$ and $k_0(f^*)$ are the minimal dilatations of their quasiconformal extensions to $\widehat{\mathbb{C}}$, and $\pi|\alpha|$ is the opening of the least interior angle between the boundary arcs $L_j \subset L$. Here $0 < \alpha < 1$ if the corresponding vertex is finite and $-1 < \alpha < 0$ for the angle at the vertex at infinity.

Equalities (2.14) hold also for the unbounded concave domains (the complements of convex ones) which do not contain ∞ ; for those one must replace the last term by $|\beta| - 1$, where $\pi|\beta|$ is the opening of the largest interior angle of D .

In particular, for any closed unbounded curve L with the convex interior, which is $C^{1+\delta}$ smooth at all finite points and has at infinity the asymptotes approaching the interior angle $\pi\alpha < 0$, we have

$$q_L = \frac{1}{\rho_L} = 1 - |\alpha|. \quad (2.15)$$

Theorem 2.9 has various important consequences and extends to rather wide classes of bounded and unbounded domains. It distinguishes an ample class of domains, whose geometric properties provide the explicit values of intrinsic conformal and quasiconformal characteristics of these domains.

For a few special curves, similar equalities were established in [Ku8,Ku11,We] by applying geometric constructions giving explicitly the extremal quasireflections.

The proof of Theorem 2.9 involves a completely different approach; it relies on the properties of holomorphic motions.

The above geometric assumptions on the domains are essential. The assertion of Theorem 2.9 extends neither to the arbitrary unbounded nonconvex and nonconcave domains nor to the arbitrary bounded convex domains.

Indeed, any rectangle P is fractional-linearly equivalent to a circular tetragon P^* with one vertex at infinity, whose boundary consists of two infinite straight line intervals and two circular arcs; these four parts of ∂P^* are mutually orthogonal. By Theorem 2.3, we

have $q_{\partial P} > 1/2$ for any rectangle whose conformal module is greater than 2.76, though the interior angles of P^* are equal to $\pm\pi/2$.

To examine the second case, observe that, due to well-known examples from [Ku6] and [Kru3], there exist bounded convex domains $D \subset \mathbb{C}$ with real analytic boundaries, for which the Grunsky constant $\kappa(f)$ of the Riemann mapping function f of the exterior domain $D^* = \widehat{\mathbb{C}} \setminus \overline{D}$ relates to the minimal dilatation $k_0(f)$ of quasiconformal extensions of f by the strict inequality $\kappa(f) < k_0(f)$. However, $k_0(f) = q_{\partial D}$, while $\kappa(f) = 1/\rho_{\partial D}$ by the Kühnau–Schiffer theorem [Ku7, Schi2], and comparison with the first equality in (2.14) shows that it cannot hold for such domains. An approximation yields that Theorem 2.9 also fails for bounded convex rectilinear polygons.

2.9.2. Scheme of the proof of Theorem 2.9. The proof consists of three stages. First the equalities (2.14) are established for convex rectilinear polygons. Let P_{n+1} be such an $(n+1)$ -gon with the finite vertices $A_1, A_2, \dots, A_n$ and with vertex $A_\infty = \infty$, and let the interior angle at the vertex A_j be equal to $\pi\alpha_j$ ($0 < \alpha_j < 1$), $j = 1, \dots, n$, and be equal to $\pi\alpha_\infty$ at A_∞ , where $\alpha_\infty < 0$. Then $\sum_{j=1}^n (\alpha_j - 1) = -\alpha_\infty - 1$ (see, e.g., [KS]). The conformal map of the upper half-plane U onto P_{n+1} fixing the point $\zeta = \infty$ is represented by the Schwarz–Christoffel integral

$$f_*(\zeta) = d_1 \int_0^\zeta (\xi - a_1)^{\alpha_1-1} (\xi - a_2)^{\alpha_2-1} \dots (\xi - a_n)^{\alpha_n-1} d\xi + d_0, \quad (2.16)$$

where $a_j = f_*^{-1}(A_j) \in \mathbb{R}$, $j = 1, \dots, n$, and d_0, d_1 are the complex constants. Its logarithmic derivative $b_f = f''/f'$ is of the form

$$b_{f_*}(\zeta) = \sum_{j=1}^n \frac{\alpha_j - 1}{\zeta - a_j}.$$

Without loss of generality, one can normalize f_* by means of $f_*(\zeta) = \zeta - i + O(\zeta - i)$ as $\zeta \rightarrow i$.

Let us construct now the isotopy $w(\zeta, t) = w_t(\zeta) : U \times I_\alpha \rightarrow \widehat{\mathbb{C}}$ requiring that in the space coordinate $w_t(\zeta)$ is a solution to the equation $w''(\zeta) = t b_{f_*}(\zeta) w'(\zeta)$ with the initial conditions $w_t(i) = i$, $w_t(\infty) = \infty$, while the time parameter ranges over the interval

$$I_\alpha = \left\{ t : \frac{-1}{1 - |\alpha|} < t < \frac{1}{1 - |\alpha|} \right\}.$$

Then

$$b_{w_t}(\zeta) = \sum_{j=1}^n t \frac{\alpha_j - 1}{\zeta - a_j} = \sum_{j=1}^n \frac{\alpha_j(t) - 1}{\zeta - a_j},$$

where $\alpha_j(t) = t(\alpha_j - 1) + 1$. Such a solution is again represented via an integral of type (2.16) (replacing α_j by $\alpha(t)_j$ and with suitable constants d_{0t}, d_{1t}); moreover, the convexity

of the original polygon P_{n+1} and the admissible bounds for the possible values of its angles prescribed by convexity ensure the univalence of this integral on U for every $t \in I_\alpha$. One obtains that $w_t(U)$ is also a polygon, which is convex for $t > 0$ and concave for $t < 0$, while $w_0(U) = U$.

Passing to the Schwarzians $S_{w_t} = (w_t''/w_t')' - (w_t''/w_t')^2/2$, we obtain a real curve Ω in the universal Teichmüller space $\mathbb{T}$ modeled as a bounded domain in the Banach space $\mathbb{B} = \mathbb{B}(U)$ of hyperbolically-bounded holomorphic functions in U (cf. Section 1.1.3). The curve Ω is defined by the map

$$\mathbf{b}: t \mapsto S_{w_t} = tb_{f_*} - t^2 b_{f_*}^2/2, \quad I_\alpha \rightarrow \mathbb{T},$$

and is located in the holomorphic disk $\tilde{\Omega} = \mathbf{b}(G) \subset \mathbb{T}$, where $G \supset I_\alpha$ is a simply connected planar domain. This induces a holomorphic motion (complex isotopy) $w(\zeta, t): U \times G \rightarrow \hat{\mathbb{C}}$.

Using well-known results on holomorphic motions mentioned in Section 1 and the properties of the domain G , we obtain that the map (2.16) admits a quasiconformal extension $\hat{f}_*$ onto $\hat{\mathbb{C}}$ with dilatation $k(\hat{f}_*) \leq 1 - |\alpha|$; hence, $q_{\partial P_{n+1}} \leq 1 - |\alpha|$. On the other hand, due to Kühnau [Ku11], if a closed curve $L \subset \hat{\mathbb{C}}$ contains two analytic arcs with the interior intersection angle $\pi\alpha'$, then

$$\frac{1}{\rho_L} \geq |1 - |\alpha'||. \quad (2.17)$$

This yields (together with the Kühnau–Schiffer theorem mentioned above) the desired equalities (2.14) for P_{n+1} .

For an arbitrary convex domain D , these equalities follow after a suitable approximation by polygons.

2.9.3. Examples. We restrict ourselves to two illustrative examples.

EXAMPLE 1. Let $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\gamma_1 = [a_1, \infty], \quad \gamma_2 = e^{i\pi\alpha}[a_2, \infty], \quad a_1 \geq a_2 > 0, \quad 0 < \alpha \leq 1/2,$$

and

$$\gamma_3 = \{(x, y): y = h(x), \ a_2 \cos \pi\alpha \leq x \leq a_1\},$$

with a decreasing convex piecewise $C^{1+\delta}$ -smooth function h such that

$$h(a_2 \cos \pi\alpha) = a_2 \sin \pi|\alpha|, \quad h(a_1) = 0.$$

For any such curve, $q_L = 1/\rho_L = 1 - \alpha$.

EXAMPLE 2. Similarly, for the right semihyperbola $L = \{x^2/a^2 - y^2/b^2 = 1, x > 0\}$, we have

$$q_L = \frac{1}{\rho_L} = 1 - \frac{2}{\pi} \arctan \frac{b}{a}.$$

When $a = b$, i.e., the hyperbola is equilateral, we get $q_L = 1/\rho_L = 1/2$. By a fractional linear transformation this semihyperbola is carried onto one half of the lemniscate, and provides the value of its reflection coefficient (cf. [Ku11, p. 103]).

In either case, Theorem 2.9 yields the values of the Grunsky constants and of minimal dilatations for quasiconformal extensions to both sides of L .

2.9.4. Arbitrary unbounded rectilinear polygons. The arguments applied above in the proof of Theorem 2.9 can be partly used also in the case of arbitrary unbounded rectilinear polygons.

Consider an unbounded rectilinear polygon P_{n+1} with the angles $\alpha_1, \dots, \alpha_n, \alpha_\infty$ at the vertices $A_1, \dots, A_n, A_\infty = \infty$, respectively. The Schwarz–Christoffel integral representing a conformal map $f_*: U \rightarrow P_{n+1}$ is similar to (2.16). Consider again its logarithmic derivative

$$b_{f_*}(\zeta) = \sum_1^n \frac{\alpha_j - 1}{\zeta - a_j}.$$

The above arguments now give the following theorem.

THEOREM 2.10. *Let $t_0 > 1$ denote the supremum of the values $t > 0$, for which the solution to the equation*

$$w''(\zeta) = t b_{f_*}(\zeta) w'(\zeta), \quad \zeta \in U, \quad (2.18)$$

is univalent on U (and maps U onto the corresponding unbounded polygon P_{n+1}^t). Then the reflection coefficient of the original polygon P_{n+1} satisfies

$$q_{\partial P_{n+1}} \leq \frac{1}{t_0}. \quad (2.19)$$

This bound cannot be improved in the general case. The equality in (2.19) is attained by any polygon P_{n+1} which has the least interior angle $\alpha\pi$ with $\alpha = 1 - 1/t_0$; in this case

$$q_{\partial P_{n+1}} = \frac{1}{\rho_{\partial P_{n+1}}} = \frac{1}{t_0}.$$

Note that, due to the properties of univalent holomorphic functions, the solutions of (2.18) with $t = t_0$ also are univalent on U .

Theorem 2.10 can be extended to more general quasidisks.

3. The best polynomial approximation of holomorphic functions and general quasiconformal mirrors

3.1. Pluricomplex Green function and holomorphic extension

Now recall that the *pluricomplex Green function* $V_K^*(z)$ (with pole at infinity) of a compact set $K \subset \mathbb{C}^n$ is defined as the plurisubharmonic regularization of the function

$$V_K(z) = \sup\{u(z): u|_K \leq 0\},$$

where the supremum is taken over all plurisubharmonic functions u on $\mathbb{C}^n$, whose growth satisfies

$$u(z) = \log |z| + O(1) \quad \text{as } z \rightarrow \infty.$$

The regularization means that

$$V_K^*(z) = \limsup_{z' \rightarrow z} V_K(z').$$

If the compact K is *pluriregular*, i.e., such that

$$V_K^*(z)|_K \equiv 0, \tag{3.1}$$

then V_K^* is continuous and coincides with V_K on $\mathbb{C}^n$. Note that if K_1 and K_2 are compact subsets of $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$, respectively, then

$$V_{K_1 \times K_2} = \max\{V_{K_1}(z), V_{K_2}(w)\}, \quad z \in \mathbb{C}^{m_1}, w \in \mathbb{C}^{m_2}.$$

In particular,

$$V_{[-1,1]^n}(z) = \max_{1 \leq j \leq n} \{\log |h(z_j)|: z = (z_1, \dots, z_n) \in \mathbb{C}^n\},$$

where $h(t) = t + \sqrt{t^2 - 1}$ is the single-valued branch of inverse function to the Joukowski function (2.11) chosen so that $h(t) > 1$ for $t > 1$.

Due to the classical Bernstein–Walsh–Siciak theorem, the region $G \supset K$ in $\mathbb{C}^n$, to which a function g is extended from K holomorphically, is determined by the best uniform Chebyshev approximations of g on K . In other words, this region depends on the behavior of the quantities

$$e_m(g, K) = \inf \left\{ \max_{z \in K} |g(z) - p(z)|: p \in \mathcal{P}_m \right\}, \tag{3.2}$$

where $\mathcal{P}_m$ is the space of polynomials whose degrees do not exceed m , $m = 1, 2, \dots$. The result is exactly as follows:

THEOREM 3.1. *A continuous function g on a pluriregular compact $K \subset \mathbb{C}^n$ extends holomorphically to the region*

$$G_R = \{z \in \mathbb{C}^n: V_K(z) < \log R\}, \quad R > 1, \quad (3.3)$$

if and only if

$$\limsup_{m \rightarrow \infty} e_m^{1/m}(g, K) \leq \frac{1}{R}. \quad (3.4)$$

For details see, e.g., in [Kl,Sa,Si,Wal,Za].

3.2. Applications to holomorphic functions on the interval and quasireflections over arcs

Consider now an injective holomorphic function

$$x \mapsto f(x): I = [-1, 1] \rightarrow \mathbb{C}$$

that satisfies $f'(x) \neq 0$ on I . It defines an analytic arc $L = f([-1, 1])$.

Similar to Theorem 1.19 we associate with f the holomorphic function of two variables

$$F(x, \xi) = \log \frac{f(x) - f(\xi)}{x - \xi}: [-1, 1]^2 \rightarrow \mathbb{C}, \quad (3.5)$$

where again the single-valued branch of $\log w$ is chosen to be positive for $w > 1$, and $F(x, x) = \log f'(x)$. The holomorphy of F on $I \times I$ follows from the fact that f is holomorphic and injective on I . Theorems 2.6 and 3.1 imply the following one.

THEOREM 3.2. *The reflection coefficient q_L of each analytic arc $L = f([-1, 1])$ satisfies*

$$q_L \leq \frac{2}{r^2 + 1/r^2} = \frac{1}{a^2 + b^2}, \quad (3.6)$$

where

$$r = \limsup_{m \rightarrow \infty} e_m^{1/m}(F, [-1, 1]^2) < 1, \quad (3.7)$$

and a, b are the semiaxes of the ellipse with foci at $(-1, 1)$ such that $a + b = \frac{1}{r}$. The bound (3.6) is sharp.

In fact, from the Bernstein–Walsh–Siciak theorem we derive that the function $F(x, \xi)$ extends holomorphically to the domain

$$D_{1/r} = \left\{ (z, \zeta) \in \mathbb{C}^2: \max(|h(z)|, |h(\zeta)|) < \frac{1}{r} \right\},$$

where r is defined by (3.7). This domain is a product $E_{ri} \times E_{ri}$ of the interiors of two ellipses E_r with foci at $(-1, 1)$ and semiaxes a, b whose sum equals $1/r$.

The assumption $f'(x) \neq 0$ ensures that $f(z)$ must be holomorphic and univalent in E_{ri} . Now (3.6) follows from Theorem 2.6, and by this theorem the bound is sharp.

We illustrate this theorem by the following simple

EXAMPLE. Let L be the arc of parabola defined by the equation

$$y = (x + \alpha)^2, \quad -1 \leq x \leq 1, \quad \alpha > 1.$$

In this case, $f(x) = x + i(x + \alpha)^2$, $-1 \leq x \leq 1$, and

$$F(z, \zeta) = \log[1 + 2i\alpha + i(z + \zeta)].$$

This function has singularities on the whole complex line

$$z + \zeta + 2\alpha - i = 0,$$

which must lie in the complement of $E_{ri} \times E_{ri}$ in $\mathbb{C}^2$. Setting $\zeta = z$, we find the point $z_\alpha = -\alpha + i/2$ which belongs to the ellipse E_r and is singular for f . The desired value of r is defined from the equality

$$\left|1 + \alpha - \frac{i}{2}\right| + \left|1 - \alpha - \frac{i}{2}\right| = 2a.$$

Theorem 3.2 gives a universal bound for the reflection coefficients of all analytic arcs. To simplify its application in the concrete evaluations, we provide also a sufficient condition for univalence of extension of the defining function f of an arc L in the interior of some ellipses E with foci at $(-1, 1)$.

The function (3.5) is holomorphic on a domain $D_r = E_{ri} \times E_{ri} \supset [-1, 1]^2$ with r that will be specified below. Thus it admits an expansion into a double series in the Chebyshev polynomials

$$F(x, \xi) = \sum_{m,n=0}^{\infty} c_{mn} T_m(x) T_n(\xi).$$

Recall that, for $|x| > 1$,

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n].$$

The properties of these polynomials enable us to establish the following result.

THEOREM 3.3. Assume that

$$\limsup_{m+n \rightarrow \infty} |c_{mn}|^{1/(m+n)} = \gamma < 1. \quad (3.8)$$

Then $F(x, \xi)$ is holomorphic in the domain D_r for $r = \frac{1}{\gamma}$ (so q_L satisfies (3.7) with this r).

We present also two further sufficient conditions formulated in terms of the Schwarzian derivative S_f of the function f defining the arc L . These conditions are assumed to hold in the interior of some ellipse $E_r \supset [-1, 1]$.

Namely: a holomorphic function f is univalent in E_{ri} , provided

$$\int_{E_{ri}} \int |S_f(z)| dx dy \leq \arctan \frac{b}{a},$$

where a, b are again the semiaxes of E_r , or if

$$|S_f(z)| < \frac{\pi^2}{2a^2}, \quad z \in E_{ri}.$$

These inequalities are the particular cases of more general conditions for convex domains given by Elias [El] and Ryll-Nardzewski [RN].

The function $F(x, \xi)$ can be used also for quasiconformal extension of certain injective not necessarily holomorphic maps.

An evaluation of reflection coefficients of arbitrary quasiconformal (even of smooth non-analytic) arcs still remains an open problem.

3.3. General quasiconformal mirrors

3.3.1. In 1988, Kühnau posed the question which sets $E \subset \widehat{\mathbb{C}}$ admit quasiconformal reflections [Ku10]; see also [Ku16]. He conjectured that the answer must be similar to the cases of curves and arcs, namely, that any such set E must be a subset of a quasicircle. An affirmative answer to Kühnau's conjecture was given in [Kru14], where it was established that every set $E \subset \widehat{\mathbb{C}}$ with the reflection coefficient $q_E < 1$ is a subset of a quasicircle L whose quasiconformal dilatation satisfies $Q_L \leq Q_E^4$, or equivalently, the reflection coefficients of L and E are related by

$$q_L \leq \frac{(1 + q_E)^4 - (1 - q_E)^4}{(1 + q_E)^4 + (1 - q_E)^4} = 4q_E \frac{1 + q_E^2}{1 + 6q_E^2 + q_E^4}. \quad (3.9)$$

This yields also various quantitative consequences, see [Kru14]. As corollaries, one immediately obtains that similar to the case of quasicircles, the two-dimensional Lebesgue measure of any set $E \subset \widehat{\mathbb{C}}$ admitting quasireflections equals zero and that every set E admitting quasireflections also admits a differentiable Lipschitz-continuous quasireflection.

The proof of (3.9) is based on the uniformization of the punctured spheres and conformal maps of the polygons bounded by circular arcs.

3.3.2. The following theorem gives a complete solution to the problem.

THEOREM 3.4. *For any set $E \subset \widehat{\mathbb{C}}$ which admits quasireflections, there is a quasicircle $L \supset E$ with the same reflection coefficient; therefore,*

$$Q_E = \min\{Q_L: L \supset E \text{ quasicircle}\}. \quad (3.10)$$

Equality (3.10) was established earlier for finite sets as well as for a few examples of infinite (uncountable) sets, see, e.g., [Kru14, Ku20]. The proof for finite sets $E = \{z_1, \dots, z_n\}$ was given by Kühnau in [Ku14] using Teichmüller's theorem on extremal quasiconformal maps. This theorem is applied to the homotopy classes of quasiconformal homeomorphisms of the punctured spheres

$$h(\bar{z}): \widehat{\mathbb{C}} \setminus \{\bar{z}_1, \dots, \bar{z}_n\} \rightarrow \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}, \quad (3.11)$$

preserving orientation (cf. Section 2.4). The desired quasicircle L_0 with $Q_{L_0} = Q_E$ (taking now infimum in (2.1) over quasireflections from a given homotopy class) is composed of analytic subarcs of passing through the points z_k horizontal trajectories of the $\widehat{\mathbb{C}}$ -holomorphic integrable quadratic differential φdz^2 , which determines the extremal quasiconformal map for (3.11).

3.3.3. Examples. Let L_0 be the boundary of the angle $A_\alpha = \{z \in \mathbb{C}: 0 < \arg z < \pi\alpha\}$, i.e., $L_0 = \overline{\mathbb{R}}_+ \cup e^{i\alpha\pi}\overline{\mathbb{R}}_+$, $\mathbb{R}_+ = \{x: 0 < x < \infty\}$, $0 \leq \alpha \leq 1$. Denote $I_a = \{x: 0 \leq x \leq a\}$ ($a > 0$), and let $E_0 \subset L_0$ be the union of two intervals: $E_0 = I_{a_1} \cup e^{i\alpha\pi}I_{a_2}$. An elementary calculation shows $q_{L_0} = |1 - \alpha|$ (cf. [Ku11]), and the extremal quasireflection across L_0 is determined by an affine map from (a principal branch of) $\log z$. Hence,

$$q_{E_0} \leq |1 - \alpha|. \quad (3.12)$$

Combining with Kühnau's inequality (2.16) yields that we have in (3.12) the case of equality $q_{E_0} = |1 - \alpha|$.

Now, let E be an arbitrary set in L_0 , which contains E_0 . Then similarly to (3.12), $q_E \leq |1 - \alpha|$, and since $q_E \geq q_{E_0}$, we have that $q_E = |1 - \alpha|$ for every subset $E \subset L$ containing E_0 . Of course, similar estimates hold for circular lunes.

The above examples provide the sets $E \subset \widehat{\mathbb{C}}$, for which

$$q_E = \inf\{q_L: L \supset E \text{ quasicircle}\}.$$

This property allows us to extend the equality (2.3) to such sets and get $Q_E = K_E^2$.

Indeed, from (2.1), (2.3) and (3.10) we have

$$Q_E = \inf\{Q_L: L \supset E\} = \inf\{K_L^2: L \supset E\} = K_E^2, \quad L \text{ quasicircle},$$

since both infimums are attained on some quasicircle (whose existence is ensured by Theorem 3.4).

3.3.4. Remarks on the inverse problem. Let us now consider the following question: given a quasicircle $L \subset \widehat{\mathbb{C}}$, find the smallest closed subsets $L_0 \subset L$ such that

$$L_0 = \bigcap \{L' : L_0 \subset L' \subseteq L, Q_{L'} = Q_L\}.$$

Clearly, $Q_{L_0} = Q_L$. We call any such subset L_0 the *reflection kernel* of the quasicircle L .

Any finite set $E \subset \widehat{\mathbb{C}}$ is a reflection kernel of the corresponding quasicircle $L \supset E$ with $Q_L = Q_E$.

The above example in Section 3.3.3 shows that a circular lune L bounded by two circular arcs with the common endpoints a, b has two one-point reflection kernels $\{a\}$ and $\{b\}$.

The next example provides the reflection kernels, which are proper subarcs. Let a quasicircle L be *asymptotically conformal*, i.e., such that for each pair of its points a, b we have

$$\max_{z \in L(a,b)} \frac{|a - z| + |z - b|}{|a - b|} \rightarrow 1 \quad \text{as } |a - b| \rightarrow 0,$$

where z lies between $a, b \in L$ (cf. [Ca, Po1], [Kru21, Section 2.3]).

Assume that L goes around the origin and possesses p -multiple rotational symmetry, i.e., its defining function $\gamma : S^1 \rightarrow L$ satisfies

$$\gamma(e^{2m\pi i/p} z) = e^{2m\pi i/p} \gamma(z), \quad t \in [0, 2\pi], \quad m = 0, 1, \dots, p-1.$$

It follows from results of [St1] and [Kru10] that a conformal map of the unit disk Δ onto the interior of any asymptotically conformal curve L admits a unique extremal Teichmüller extension onto Δ^* , i.e., with the Beltrami coefficient of the form $\mu_0 = k_0|\varphi_0|/\varphi_0$, where φ_0 is an integrable holomorphic function in Δ^* . This yields $Q_L = (1 + k_0)/(1 - k_0)$.

If L has singularities on a dense subset, then the symmetry implies, in view of uniqueness of extremal extension of Teichmüller's type, that the intersection of L with any angular domain $\mathcal{A}_p = \{z : \alpha_0 \leq \arg z \leq \alpha_0 + 2\pi/p\}$ contains a reflection kernel of L .

The reflection kernels are closely connected with the essential boundary points of extremal quasiconformal maps at which the extremal dilatation is attained, cf. [St1].

It was first observed by Kühnau in [Ku12] (in somewhat other terms) that any proper subarc of an analytic curve cannot be its reflection kernel.

3.3.5. It is interesting to establish simple sufficient (not inevitably necessary) conditions for a set $E \in \widehat{\mathbb{C}}$ to admit quasireflections, or equivalently, to ensure the existence of quasicircles passing through this set. Such a result was established in [Kru14] involving holomorphic motions.

3.3.6. A theorem of Kühnau. The well-known Teichmüller's Verschiebungssatz describes the extremal quasiconformal map in the class of q -quasiconformal automorphisms f of the disk Δ ($0 < q < 1$) with $f|_{\partial\Delta} = \text{id}$ moving the origin to a maximal distance (or, equivalently, interplating two given inner points z_1, z_2); see [Te3]. The extremal map (with

minimal dilatation) is unique and is a composition of the corresponding conformal maps with a Q -quasiconformal affine map w_0 whose dilatation equals

$$Q = \frac{(1+q)^2}{(1-q)^2} \geq 1. \quad (3.13)$$

We shall need Reich's version of this theorem.

THEOREM 3.5 [Re2]. *Among all quasiconformal automorphisms f of the ellipse $\mathcal{E}$ with semiaxes $1+q^2$ and $1-q^2$ with given $q \in (0, 1)$ satisfying $f|_{\partial\mathcal{E}} = \text{id}$ and carrying the right focus of $\mathcal{E}$ into its left focus, the extremal map has dilatation given by (3.13).*

This extremal map again can be described geometrically.

Using this theorem, Kühnau found the exact value of the reflection coefficient of the set E which consists of the interval $[-2i, 2i]$ and a separate point $t > 0$, giving explicitly the extremal quasicircle $L \supset E$.

THEOREM 3.6 [Ku20]. *The reflection coefficient of E is given by*

$$Q_E = 1 + \sinh^{-2} \left\{ \frac{1}{2} \mu \left(\frac{2}{\sqrt{t^2 + 4}} \right) \right\}, \quad (3.14)$$

where $\mu(r)$ denotes the module of the extremal Grötzsch domain obtained by cutting the unit disk along the segment $[0, r]$, $r < 1$ (see, e.g., [Ahl3, LV]). The (unique) extremal quasicircle is the boundary of analytic triangle based on $[-2i, 2i]$ whose lateral sides are two symmetric well-defined analytic arcs joining the point t with $\pm 2i$ and such that the exterior angle at the vertex t and the interior angles at $\pm 2i$ are equal to each $\pi - 4 \arctan q$.

References

- [Ab] W. Abikoff, *Real Analytic Theory of Teichmüller Spaces*, Lecture Notes in Math., Vol. 820, Springer-Verlag, Berlin (1980).
- [Ah] D. Aharonov, *A necessary and sufficient condition for univalence of a meromorphic function*, Duke Math. J. **36** (1969), 599–604.
- [Ahl1] L. V. Ahlfors, *Remarks on the Neumann–Poincaré equation*, Pacific J. Math. **2** (1952), 271–280.
- [Ahl2] L. V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
- [Ahl3] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton (1966).
- [Ahl4] L. V. Ahlfors, *A remark on schlicht functions with quasiconformal extensions*, Collected Papers, Vol. 2, Birkhäuser, Basel (1982), 438–441.
- [AB] L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 383–404.
- [AW] L. V. Ahlfors and G. Weill, *A uniqueness theorem for the Beltrami equation*, Proc. Amer. Math. Soc. **13** (1962), 975–978.
- [AkS] L. A. Aksent'ev and P. L. Shabalin, *Sufficient conditions for univalent and quasiconformal extendibility of analytic functions*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, R. Kühnau, ed., Elsevier, Amsterdam (2002), 169–206.

- [An] J.M. Anderson, *The Grunsky inequalities*, Inequalities. Fifty years on from Hardy, Littlewood and Pólya, Proc. Int. Conf. Birmingham, UK, 1987, Lecture Notes in Pure and Appl. Math. **129** (1991), 21–28.
- [AnH1] J.M. Anderson and A. Hinkkanen, *A univalence criterion*, Michigan Math. J. **32** (1985), 33–40.
- [AnH2] J.M. Anderson and A. Hinkkanen, *Univalence criteria and quasiconformal extensions*, Trans. Amer. Math. Soc. **324** (1991), 823–842.
- [As] K. Astala, *Area distortion for quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [AG1] K. Astala and F.W. Gehring, *Injectivity criteria and the quasidisk*, Complex Var. **3** (1984), 45–54.
- [AsG2] K. Astala and F.W. Gehring, *Crickets, zippers, and the universal Teichmüller space*, Proc. Amer. Math. Soc. **110** (1990), 675–687.
- [ASm] K. Astala and G. Martin, *Holomorphic motions*, Papers on Analysis, Report Univ. Jyväskylä, Dept. Math. Stat. 83, Univ. Jyväskylä (2001), 27–40.
- [AvA] F.G. Avkhadiev and L.A. Aksent'ev, *Fundamental results on sufficient conditions for the univalence of analytic functions*, Uspekhi Mat. Nauk **30** (1975), 3–60 (in Russian).
- [AvAE] F.G. Avkhadiev, L.A. Aksent'ev and A.M. Elizarov, *Sufficient conditions for finite-valence of analytic functions, and their applications*, Itogi Nauki i Tekhniki Ser. Mat. Analysis, Vol. 25, VINITI, Moscow (1987), 3–121 (in Russian); Transl.: J. Soviet Math. **49** (1990), 715–799.
- [Be] A.F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, New York (1999).
- [Bec1] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **225** (1972), 23–43.
- [Bec2] J. Becker, *Löwnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. **202** (1973), 321–335.
- [Bec3] J. Becker, *Conformal mappings with quasiconformal extensions*, Aspects of Contemporary Complex Analysis, Proc. Confer. Durham, 1979, D.A. Brannan and J.G. Clunie, eds, Academic Press, New York (1980), 37–77.
- [BeP] J. Becker und Ch. Pommerenke, *Über die quasikonforme Fortsetzung schlichter Funktionen*, Math. Z. **161** (1978), 69–80.
- [Ber1] L. Bers, *Holomorphic differentials as functions of moduli*, Bull. Amer. Math. Soc. **67** (1961), 206–210.
- [Ber2] L. Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
- [Ber3] L. Bers, *Uniformization, modules and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [Ber5] L. Bers, *On a theorem of Abikoff*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 83–87.
- [Ber4] L. Bers, *Holomorphic families of isomorphisms of Möbius groups*, J. Math. Kyoto Univ. **26** (1986), 73–76.
- [BerR] L. Bers and H.L. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.
- [Br] L.E.F. Brouwer, *Über die periodischen Transformationen der Kugel*, Math. Ann. **80** (1919), 39–41.
- [BrT] J. Brown and A. Tsao, *On the Zalcman conjecture for starlike and typically real functions*, Math. Z. **191** (1986), 467–474.
- [Bu] J. Burbea, *Grunsky inequalities and Fredholm spectrum in general domains*, J. Anal. Math. **45** (1987), 1–36.
- [CCH] D.M. Campbell, J.G. Clunie and W.K. Hayman, *Research problems in complex analysis*, Aspects of Contemporary Complex Analysis, Proc. Confer. Durham, 1979, D.A. Brannan and J.G. Clunie, eds, Academic Press, New York (1980), 527–571.
- [Ca] L. Carleson, *On mappings conformal at the boundary*, J. Anal. Math. **19** (1967), 1–13.
- [CaG] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York (1993).
- [Car] K. Carne, *The Schwarzian derivative for conformal maps*, J. Reine Angew. Math. **408** (1990), 10–33.
- [Ch1] M. Chuaqui, *On a theorem of Nehari and quasidisks*, Ann. Acad. Sci. Fenn. Ser. A I Math. **18** (1993), 117–124.
- [Ch2] M. Chuaqui, *A unified approach to univalence criteria in the unit disk*, Proc. Amer. Math. Soc. **123** (1995), 441–453.
- [CO1] M. Chuaqui and B. Osgood, *The Schwarzian derivative and conformally natural quasiconformal extensions from one to two to three dimensions*, Math. Ann. **292** (1992), 267–280.
- [CO2] M. Chuaqui and B. Osgood, *Weak Schwarzians, bounded hyperbolic distortion, and smooth quasymmetric functions*, J. Anal. Math. **68** (1996), 209–252.

- [CF] P.E. Conner and E. Floyd, *Differentiable periodic maps*, Ergeb. Math. Grenzgeb., Vol. 33, Springer-Verlag, Berlin (1964).
- [Da] V.A. Danilov, *Estimates of distortion of quasiconformal mappings in space C_α^m* , Siberian Math. J. **14** (1973), 362–369.
- [DB] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.
- [Di] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford (1989).
- [Do] A. Douady, *Prolongement de mouvements holomorphes [d'après Ślodkowski et autres]*, Séminaire N. Bourbaki, Exposé 775 (1993–1994), 1–12.
- [DE] A. Douady and C.J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.
- [Du] P. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [DL] P. Duren and O. Lehto, *Schwarzian derivatives and homeomorphic extensions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **477** (1970), 1–11.
- [Ea1] C.J. Earle, *Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 355–361.
- [Ea2] C.J. Earle, *Some maximal holomorphic motions*, Contemp. Math. **211** (1997), 183–192.
- [EE] C.J. Earle and A.L. Epstein, *Quasiconformal variation of slit domains*, Proc. Amer. Math. Soc. **129** (2001), 3363–3372.
- [EF1] C.J. Earle and R.F. Fowler, *Holomorphic families of open Riemann surfaces*, Math. Ann. **270** (1985), 249–273.
- [EF2] C.J. Earle and R.F. Fowler, *A new characterization of infinite dimensional Teichmüller spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 149–153.
- [EGL] C.J. Earle, F.P. Gardiner and N. Lakic, *Vector fields for holomorphic motions of closed sets*, Contemp. Math. **211** (1997), 193–225.
- [EK] C.J. Earle and I. Kra, *On sections of some holomorphic families of closed Riemann surfaces*, Acta Math. **137** (1976), 49–79.
- [EKK] C.J. Earle, I. Kra and S.L. Krushkal, *Holomorphic motions and Teichmüller spaces*, Trans. Amer. Math. Soc. **343** (1994), 927–948.
- [EM] C.J. Earle and C. McMullen, *Quasiconformal isotopies*, Holomorphic Functions and Moduli I, Math. Sci. Res. Inst. Publ., Vol. 10, D. Drasin et al., eds, Springer-Verlag, New York (1988), 143–154.
- [EMi] C.J. Earle and S. Mitra, *Variation of moduli by holomorphic motions*, The Tradition of Ahlfors and Bers, Stony Brook, NY, 1988, Contemp. Math. **256** (2000), 39–67.
- [EN] C.J. Earle and S. Nag, *Conformally natural reflections in Jordan curves with applications to Teichmüller spaces*, Holomorphic Functions and Moduli II, D. Drasin et al., eds, Math. Sci. Res. Inst. Publ., Vol. 11, Springer-Verlag, New York (1998), 179–194.
- [EL2] C.J. Earle and L. Zhong, *Isometrically embedded polydisks in infinite dimensional Teichmüller spaces*, J. Geom. Anal. **9** (1999), 51–71.
- [El] U. Elias, *Nonoscillation theorems on convex sets*, J. Math. Anal. Appl. **52** (1975), 129–141.
- [Ep1] C. Epstein, *The hyperbolic Gauss map and quasiconformal reflections*, J. Reine Angew. Math. **372** (1986), 96–135.
- [Ep2] C. Epstein, *Univalence criteria in hyperbolic space*, J. Reine Angew. Math. **380** (1987), 196–214.
- [ErH] A. Eremenko and D.H. Hamilton, *On the area distortion by qc mappings*, Proc. Amer. Math. Soc. **123** (1995), 2793–2797.
- [ErL] A. Eremenko and M.Yu. Lyubich, *The dynamics of analytic transformations*, Leningrad Math. J. **1** (1990), 563–634.
- [FKZ] M. Fait, J.G. Krzyż and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. **51** (1976), 279–285.
- [Ga] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer-Verlag, Berlin (1964).
- [Gar] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
- [GaL] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Math. Surveys Monogr., Vol. 76, Amer. Math. Soc., Providence, RI (2000).
- [Ge] F.W. Gehring, *Characteristic Properties of Quasidisks*, Les Presses de l'Université de Montréal (1982).
- [GeH] F.W. Gehring and K. Hag, *Reflections on reflections in quasidisks*, Papers on Analysis: A Volume Dedicated to Olli Martio on the Occasion of His 60th Birthday, Report Univ. Jyväskylä 83 (2001), 81–90.

- [Gol] G.M. Goluzin, *Geometric Theory of Functions of Complex Variables*, Transl. Math. Monogr., Vol. 26, Amer. Math. Soc., Providence, RI (1969).
- [Gri] A.Z. Grinshpan, *Univalent functions and measurable mappings*, Siberian Math. J. **27** (1986), 825–837.
- [Gru] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45** (1939), 29–61.
- [Ha] R.S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. **138** (1969), 399–406.
- [Har1] R. Harmelin, *Injectivity, quasiconformal reflections and the logarithmic derivative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 61–68.
- [Har2] R. Harmelin, *A univalence criterion*, Complex Var. Theory Appl. **10** (1988), 327–331.
- [HiF] E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, Colloq. Publ. Ser., Vol. 31, Amer. Math. Soc., Providence, RI (1957).
- [Hu] O. Hübner, *Die Faktorisierung konformer Abbildungen und Anwendungen*, Math. Z. **92** (1966), 95–109.
- [Hum] J.A. Hummel, *The Grunsky coefficients of a schlicht function*, Proc. Amer. Math. Soc. **15** (1964), 142–150.
- [IS] Y. Iwayoshi and H. Shiga, *A finiteness theorem for holomorphic families of Riemann surfaces*, Holomorphic Functions and Moduli II, Math. Sci. Res. Inst. Publ., Vol. 11, D. Drasin et al., eds, Springer-Verlag, New York (1988), 207–219.
- [IM] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press (2001).
- [Ke] B. Kerekjarto, *Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche*, Math. Ann. **80** (1919), 36–38.
- [Kl] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford (1991).
- [Ko2] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer-Verlag, New York (1997).
- [KS] W. von Koppenfels and F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin (1959).
- [Kr1] I. Kra, *Automorphic Forms and Kleinian Groups*, Benjamin, Reading, MA (1972).
- [Kr2] I. Kra, *Families of univalent functions and Kleinian groups*, Israel J. Math. **60** (1987), 89–127.
- [Kr3] I. Kra, *Quadratic differentials*, Rev. Romain Math. Pure Appl. **39** (1994), 751–787.
- [Kru1] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Wiley, New York (1979).
- [Kru2] S.L. Krushkal, *Differential operators and univalent functions*, Complex Var. Theory Appl. **7** (1986), 107–127.
- [Kru3] S.L. Krushkal, *On the Grunsky coefficient conditions*, Siberian Math. J. **28** (1987), 104–110.
- [Kru4] S.L. Krushkal, *On an interpolating family of a univalent analytic function*, Siberian Math. J. **28** (1987), 88–94.
- [Kru5] S.L. Krushkal, *Extension of conformal mappings and hyperbolic metrics*, Siberian Math. J. **30** (1989), 730–744.
- [Kru6] S.L. Krushkal, *Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings*, Comment. Math. Helv. **64** (1989), 650–660.
- [Kru7] S.L. Krushkal, *On the question of the structure of the universal Teichmüller space*, Soviet Math. Dokl. **38** (1989), 435–437.
- [Kru8] S.L. Krushkal, *New developments in the theory of quasiconformal mappings*, Geometric Function Theory and Applications of Complex Analysis to Mechanics: Studies in Complex Analysis and Its Applications to Partial Differential Equations, Vol. 2, R. Kühnau and W. Tutschke, eds, Pitman Res. Notes Math. Ser., Longman, Harlow, UK (1991), 3–26.
- [Kru9] S.L. Krushkal, *Univalent functions and holomorphic motions*, J. Anal. Math. **66** (1995), 253–275.
- [Kru10] S.L. Krushkal, *On Grunsky conditions, Fredholm eigenvalues and asymptotically conformal curves*, Mitt. Math. Sem. Gissen **228** (1996), 17–23.
- [Kru11] S.L. Krushkal, *On quasireflections and holomorphic functions*, Proc. 1996 Ashkelon Workshop in Complex Function Theory, L. Zalcman, ed., Israel Math. Conf. Proc., Vol. 11, Amer. Math. Soc., Providence, RI (1997), 173–185.
- [Kru12] S.L. Krushkal, *Quasireflections with respect to analytic curves*, Complex Var. Theory Appl. **33** (1997), 145–157.

- [Kru13] S.L. Krushkal, *Quasireflections and holomorphic functions*, Analysis and Topology: A Volume Dedicated to the Memory of S. Stoilow, C. Andreian Cazacu, O. Lehto and T.M. Rassias, eds, World Scientific, Singapore (1998), 467–481.
- [Kru14] S.L. Krushkal, *Quasiconformal mirrors*, Siberian Math. J. **40** (1999), 742–753.
- [Kru15] S.L. Krushkal, *On the best approximation and univalence of holomorphic functions*, Complex Analysis in Contemporary Mathematics. In honor of 80th Birthday of Boris Vladimirovich Shabat, E.M. Chirka, ed., Fasis, Moscow (2001), 153–166 (in Russian).
- [Kru16] S.L. Krushkal, *Quasiconformal reflections across arbitrary planar sets*, Sci. Ser. A Math. Sci. **8** (2002), 57–62.
- [Kru17] S.L. Krushkal, *Polygonal quasiconformal maps and Grunsky inequalities*, J. Anal. Math. **90** (2003), 175–196.
- [Kru18] S.L. Krushkal, *Quasireflections, Fredholm eigenvalues and Finsler metrics*, Dokl. Math. **69** (2004), 221–224.
- [Kru19] S.L. Krushkal, *A bound for reflections across Jordan curves*, Georgian Math. J. **10** (2003).
- [Kru20] S.L. Krushkal, *Variational principles in the theory of quasiconformal maps*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, Amsterdam (2005), 31–98 (this volume).
- [Kru21] S.L. Krushkal, *Univalent holomorphic functions with quasiconformal extensions (variational approach)*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, Amsterdam (2005), 165–241 (this volume).
- [KrK1] S.L. Krushkal und R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner-Texte Math., Bd. 54, Teubner, Leipzig (1983).
- [KrK2] S.L. Krushkal and R. Kühnau, *A quasiconformal dynamic property of the disk*, J. Anal. Math. **72** (1997), 93–103.
- [Krz] J. Krzyz, *Convolution and quasiconformal extension*, Comment. Math. Helv. **51** (1976), 99–104.
- [KrzL] J.G. Krzyz and J. Lawrynowicz, *Quasiconformal mappings of the unit disk with two invariant points*, Michigan Math. J. **14** (1967), 487–492.
- [KrzP] J.G. Krzyz and D. Partyka, *Harmonic extensions of quasisymmetric mappings*, Complex Var. Theory Appl. **33** (1997), 159–176.
- [Ku1] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77–105.
- [Ku2] R. Kühnau, *Eine Verschärfung des Koebeschen Viertelsatzes für quasikonform fortsetzbaren Abbildungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 77–83.
- [Ku3] R. Kühnau, *Zur quasikonforme Fortsetzbarkeit schlichter konformer Abbildungen*, Bull. Soc. Sci. Lett. Łódź **24** (1975), 1–4.
- [Ku4] R. Kühnau, *Über die Werte des Doppelverhältnisses bei quasikonformer Abbildung*, Math. Nachr. **95** (1980), 237–251.
- [Ku5] R. Kühnau, *Funktionalabschätzungen bei quasikonformen Abbildungen mit Fredholmschen Eigenwerten*, Comment. Math. Helv. **56** (1981), 297–206.
- [Ku6] R. Kühnau, *Zu den Grunskyschen Koeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 125–130.
- [Ku7] R. Kühnau, *Quasikonforme Fortsetzbarkeit, Fredholmsche Eigenwerten und Grunskysche Koeffizientenbedingungen*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 383–391.
- [Ku8] R. Kühnau, *Zur Berechnung der Fredholmschen Eigenwerte ebener Kurven*, Z. Angew. Math. Mech. **66** (1986), 193–201.
- [Ku9] R. Kühnau, *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q -quasikonforme Fortsetzbarkeit?* Comment. Math. Helv. **61** (1986), 290–307.
- [Ku10] R. Kühnau, *Möglichst konforme Spiegelung an einem Jordanbogen auf der Zahlenkugel*, Complex Analysis, J. Hersch and A. Huber, eds, Birkhäuser, Basel (1988), 139–156.
- [Ku11] R. Kühnau, *Möglichst konforme Spiegelung an einer Jordankurve*, Jahresber. Deutsch. Math.-Verein. **90** (1988), 90–109.
- [Ku12] R. Kühnau, *Möglichst konforme Spiegelung an Jordanbögen und ein Faktorisierungssatz bei quasikonformen Abbildungen*, Bericht der Mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum–Graz, Berich, Bd. 300 (1988), 1–12.

- [Ku13] R. Kühnau, *Möglichst konforme Spiegelung an einem Jordanbogen auf der Zahlenebene*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 357–367.
- [Ku14] R. Kühnau, *Interpolation by extremal quasiconformal Jordan curves*, Siberian Math. J. **32** (1991), 257–264.
- [Ku15] R. Kühnau, *Möglichst konforme Jordankurven durch vier Punkte*, Rev. Romain Math. Pure Appl. **36** (1991), 383–393.
- [Ku16] R. Kühnau, *Einige neuere Entwicklungen bei quasikonformen Abbildungen*, Jahresber. Deutsch. Math.-Verein. **94** (1992), 141–169.
- [Ku17] R. Kühnau, *Ersetzungssätze bei quasikonformen Abbildungen*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **52** (1998), 65–72.
- [Ku18] R. Kühnau, *Über die Grunskysche Koeffizientenbedingungen*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **54** (2000), 53–60.
- [Ku19] R. Kühnau, *Drei Funktionale eines Quasikreises*, Ann. Acad. Sci. Fenn. Ser. A I Math. **25** (2000), 413–415.
- [Ku20] R. Kühnau, *Zur möglichst konformen Spiegelung*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **55** (2001), 85–94.
- [Ku21] R. Kühnau, *Bemerkung zur quasikonformen Fortsetzung*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **56** (2002), 53–55.
- [KuB] R. Kühnau und H. Blaar, *Kriterien für quasikonforme Fortsetzbarkeit konformer Abbildungen eines Kreisringes bzw. des Inneren oder Äusseren einer Ellipse*, Math. Nachr. **91** (1979), 183–196.
- [Le] N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Nauka, Moscow (1975) (in Russian).
- [Leh] M. Lehtinen, *A real-analytic quasiconformal extension of a quasisymmetric function*, Ann. Acad. Sci. Fenn. Ser. A I Math. **3** (1977), 207–213.
- [Leh1] O. Lehto, *An extension theorem for quasiconformal mappings*, Proc. London Math. Soc. (3) **14** (1965), 187–190.
- [Leh2] O. Lehto, *Quasiconformal mappings and singular integrals*, Istit. Naz. Alta Mat. Symp. Math., Vol. 18, Academic Press, London (1976), 429–453.
- [Leh3] O. Lehto, *Remarks on Nehari's theorem about the Schwarzian derivative and schlicht functions*, J. Anal. Math. **36** (1979), 184–190.
- [Leh4] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987).
- [LV] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin (1965).
- [LyM] M. Lyubich and Y. Minsky, *Laminations in holomorphic dynamics*, J. Differential Geom. **47** (1997), 17–94.
- [Ma] W. Ma, *The Zalcman conjecture for close-to-convex functions*, Proc. Amer. Math. Soc. **104** (1988), 741–744.
- [MSS] R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Super. **16** (1983), 193–217.
- [Mar] G. Martin, *The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions*, Proc. Amer. Math. Soc. **125** (1997), 1093–1103.
- [MMPV] O. Martio, V. Miklukov, S. Ponnusamy and M. Vuorinen, *On some properties of quasiplanes*, Results Math. **42** (2002), 107–113.
- [MSa] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978–1979), 383–401.
- [Mc] C. McMullen, *Complex Dynamics and Renormalization*, Princeton Univ. Press, Princeton (1994).
- [MSu] C. McMullen and D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math. **135** (1988), 351–395.
- [Mik] V.M. Miklukov, *On quasiconformally flat surfaces in Riemannian manifolds*, Izv. Math. **67** (2003), 931–953.
- [Mi] I.M. Milin, *Univalent Functions and Orthogonal Systems*, Nauka, Moscow (1971); English transl.: Amer. Math. Soc., Providence, RI (1977).
- [Mil] J. Milnor, *Dynamics in one Complex Variable. Introductory Lectures*, Vieweg, Braunschweig (1999).
- [Mit] S. Mitra, *Teichmüller spaces and holomorphic motions*, J. Anal. Math. **81** (2000), 1–33.
- [Na] S. Nag, *Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).

- [Ne1] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
- [Ne3] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York (1952).
- [Ne2] Z. Nehari, *Some criteria of univalence*, Proc. Amer. Math. Soc. **5** (1954), 700–704.
- [Nev] R. Nevanlinna, *Eindeutige analytische Funktionen*, 2nd edn, Springer-Verlag, Berlin (1953).
- [Os1] B. Osgood, *Univalence criteria in multiply connected domains*, Trans. Amer. Math. Soc. **260** (1980), 559–473.
- [Os2] B.G. Osgood, *Some properties of f''/f' and the Poincaré metric*, Indiana Univ. Math. J. **31** (1982), 449–461.
- [Os3] B. Osgood, *Old and new in the Schwarzian derivative*, Quasiconformal Mappings and Analysis Ann Arbor, MI, 1975, Springer-Verlag, New York (1998), 275–308.
- [OS1] B. Osgood and D. Stowe, *A generalization of Nehari's univalence criterion*, Comment. Math. Helv. **65** (1990), 234–242.
- [OS2] B. Osgood and D. Stowe, *The Schwarzian derivative and conformal mapping of Riemannian manifolds*, Duke Math. J. **67** (1992), 57–99.
- [Pa1] D. Partyka, *Spectral values and eigenvalues of a quasicircle*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **46** (1992), 81–98.
- [Pa2] D. Partyka, *The smallest positive eigenvalue of a quasisymmetric automorphism of the unit circle*, Topics in Complex Analysis, Banach Center Publications, Vol. 31, Inst. of Math. Polish Acad. of Sci., Warszawa (1995), 303–310.
- [Pa3] D. Partyka, *The Grunsky type inequalities for quasisymmetric automorphisms of the unit circle*, Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. **31** (2000), 135–142.
- [Pf1] J. Pfaltzgraff, *Subordination chains and univalence of holomorphic maps in $\mathbb{C}^n$* , Math. Ann. **210** (1974), 55–68.
- [Pf2] J. Pfaltzgraff, *Subordination chains and quasiconformal extension of holomorphic maps in $\mathbb{C}^n$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), 13–25.
- [Pf3] J. Pfaltzgraff, *k -Quasiconformal extension criteria in the disk*, Complex Var. **21** (1993), 293–301.
- [Pfl] A. Pfluger, *On a coefficient problem for schlicht functions*, Advances in Complex Function Theory, Maryland 1973/74, Lecture Notes in Math., Vol. 505, Springer-Verlag, Berlin (1976), 79–91.
- [Po1] Chr. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
- [Po3] Chr. Pommerenke, *On the Epstein univalence criterion*, Results Math. **10** (1986), 143–146.
- [Po2] Chr. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin (1992).
- [PR] Chr. Pommerenke and B. Rodin, *Holomorphic families of Riemann mapping functions*, J. Math. Kyoto Univ. **26** (1) (1986), 13–22.
- [Re1] E. Reich, *Quasiconformal mappings of the disk with given boundary values*, Advances in Complex Function Theory, W.E. Kirwan and L. Zalcman, eds, Lecture Notes in Math., Vol. 505, Springer-Verlag, Berlin (1976), 101–137.
- [Re2] E. Reich, *On the mapping with complex dilatation $e^{i\theta}$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 261–268.
- [Re3] E. Reich, *A quasiconformal extension using the parametric representations*, J. Anal. Math. **54** (1990), 246–258.
- [Re4] E. Reich, *Extremal quasiconformal mappings of the disk*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, R. Kühnau, ed., Elsevier, Amsterdam (2002), 75–136.
- [RC] E. Reich and J.X. Chen, *Extensions with bounded $\bar{\partial}$ -derivative*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 377–389.
- [RS] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, L.V. Ahlfors et al., eds, Academic Press, New York (1974), 375–391.
- [Rei] H.M. Reimann, *Ordinary differential equations and quasiconformal mappings*, Invent. Math. **33** (1976), 247–270.
- [Ren] H. Renelt, *Elliptic Systems and Quasiconformal Mappings*, Wiley, New York (1988).
- [Res] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monogr., Vol. 73, Amer. Math. Soc., Providence, RI (1989).
- [Ro] B. Rodin, *Behavior of the Riemann mapping function under complex analytic deformations of the domain*, Complex Var. Theory Appl. **5** (1986), 189–195.

- [Roy1] H.L. Royden, *A modification of the Neumann–Poincaré method for multiply connected domains*, Pacific J. Math. **2** (1952), 385–394.
- [Roy2] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud., Vol. 66, Princeton University Press, Princeton (1971), 369–383.
- [RN] C. Ryll-Nardzewski, *Une extension d'un theoreme de Sturm aux fonctions analytiques*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **7** (1950), 5–6.
- [Sa] A. Sadulaev, *Plurisubharmonic functions*, Itogi Nauki i Tekhniki, Seria Sovremennye Problemy Matematiki, Fundamentalnye Napravleniya (Totals of Science and Technics, Series Fundamental Directions), Vol. 8, VINITI, Moscow (1985), 65–113 (in Russian).
- [Schi1] M. Schiffer, *The Fredholm eigenvalues of plane domains*, Pacific J. Math. **7** (1957), 1187–1225.
- [Schi2] M. Schiffer, *Fredholm eigenvalues and Grunsky matrices*, Ann. Polon. Math. **39** (1981), 149–164.
- [ScSc1] M. Schiffer and G. Schober, *Coefficient problems and generalized Grunsky inequalities for schlicht functions with quasiconformal extensions*, Arch. Ration. Mech. Anal. **60** (1976), 205–228.
- [Scho1] G. Schober, *Continuity of curve functionals and techniques involving quasiconformal mappings*, Arch. Ration. Mech. Anal. **29** (1968), 378–389.
- [Scho2] G. Schober, *Semicontinuity of curve functionals*, Arch. Ration. Mech. Anal. **53** (1969).
- [Scho3] G. Schober, *Estimates for Fredholm eigenvalues based on quasiconformal mapping*, Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen. Lecture Notes in Math., Vol. 333, Springer-Verlag, Berlin (1973), 211–217.
- [Scho4] G. Schober, *Univalent Functions – Selected Topics*, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin (1975).
- [Se] V.I. Semenov, *An extension principle for quasiconformal deformations of the plane*, Russian Acad. Sci. Sb. Math. **77** (1994), 127–137.
- [Sh3] Y.L. Shen, *Pull-back operators by quasisymmetric functions and invariant metrics on Teichmüller spaces*, Complex Var. **42** (2000), 289–307.
- [She] V.G. Sheretov, *Analytic Functions with Quasiconformal Extensions*, Tver State Univ., Tver (1991).
- [Shi1] H. Shiga, *Characterization of quasi-disks and Teichmüller spaces*, Tôhoku Math. J. **37** (1985), 541–552.
- [Shi2] H. Shiga, *Remarks on holomorphic families of Riemann surfaces*, Tôhoku Math. J. **38** (1986), 539–549.
- [Shi3] H. Shiga, *On monodromies of holomorphic families of Riemann surfaces and modular transformations*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 541–549.
- [ShT] H. Shiga and H. Tanigawa, *Grunsky's inequality and its application to Teichmüller spaces*, Kodai Math. J. **16** (1993), 361–378.
- [Si] J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. **105** (1962), 322–357.
- [SI1] Z. Ślodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.
- [SI2] Z. Ślodkowski, *Natural extensions of holomorphic motions*, J. Geom. Anal. **7** (1997), 637–651.
- [SI3] Z. Ślodkowski, *Holomorphic motions commuting with semigroups*, Studia Math. **119** (1996), 1–16.
- [Sm1] P.A. Smith, *Transformations of finite period*, Ann. of Math. **39** (1938), 127–164.
- [Sm2] P.A. Smith, *Transformations of finite period, II*, Ann. of Math. **40** (1939), 690–711.
- [Sp] G. Springer, *Fredholm eigenvalues and quasiconformal mapping*, Acta Math. **111** (1963), 121–142.
- [St1] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Anal. Math. **30** (1976), 464–480.
- [St2] K. Strebel, *Extremal quasiconformal mappings*, Results Math. **10** (1986), 168–210.
- [St3] K. Strebel, *On the dilatation of extremal quasiconformal mappings of polygons*, Comment. Math. Helv. **74** (1999), 143–149.
- [Su1] T. Sugawa, *The Bers projection and the λ -lemma*, J. Math. Kyoto Univ. **32** (1992), 701–713.
- [Su2] T. Sugawa, *Holomorphic motions and quasiconformal extensions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 239–252.
- [Su3] T. Sugawa, *A remark on the Ahlfors–Lehto univalence criterion*, Ann. Acad. Sci. Fenn. Ser. A I Math. **27** (2002), 151–161.
- [Sul1] D. Sullivan, *Conformal dynamical systems. Geometric dynamics (Rio de Janeiro, 1981)*, Lecture Notes in Math., Vol. 1007, Springer-Verlag, Berlin (1983), 725–252.

- [Sul2] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. I: Solution of the Fatou–Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418.
- [Sul3] D. Sullivan, *Quasiconformal homeomorphisms and dynamics. II: Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.
- [ST] D. Sullivan and W.P. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.
- [Ta] H. Tanigawa, *Holomorphic families of geodesic discs in infinite dimensional Teichmüller spaces*, Nagoya Math. J. **127** (1992), 117–128.
- [Te1] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Math. Naturw. Kl. **22** (1939), 1–197.
- [Te2] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer analytischen Ringflächenschar von der Parametern*, Deutsche Math. **7** (1944), 309–336.
- [Te3] O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Math. **7** (1944), 336–343.
- [Th] W.P. Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1, Princeton Math. Ser., Vol. 35, Princeton (1997).
- [Tu] P. Tukia, *Quasiconformal extension of quasimetric mappings compatible with a Fuchsian group*, Acta Math. **154** (1985), 153–193.
- [TV] P. Tukia and J. Väisälä, *Quasimetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 97–114.
- [Vai1] J. Väisälä, *Lectures on n -dimensional quasiconformal mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag, Berlin (1971).
- [Vai2] J. Väisälä, *Quasi-symmetric embeddings in euclidean spaces*, Trans. Amer. Math. Soc. **264** (1981), 191–204.
- [Vai3] J. Väisälä, *Quasimetricity and unions*, Manuscripta Math. **68** (1990), 101–111.
- [Ve1] J. Velling, *A geometric interpretation of the Ahlfors–Weill mappings and an induced foliation of $\mathbb{H}^3$* , Holomorphic Functions and Moduli I, Math. Sci. Res. Inst. Publ., Vol. 10, D. Drasin et al., eds, Springer-Verlag, New York (1988), 193–202.
- [Ve2] J. Velling, *Mapping the disk to convex subregions*, Analysis and Topology: A Volume Dedicated to the Memory of S. Stoilow, C. Andreian Cazacu, O. Lehto and T.M. Rassias, eds, World Scientific, Singapore (1998), 719–723.
- [Vu] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin (1988).
- [Wal] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th edn, Amer. Math. Soc., Providence, RI (1965).
- [War] S.E. Warschawski, *On the effective determination of conformal maps*, Contribution to the Theory of Riemann Surfaces, L. Ahlfors, E. Calabi et al., eds, Princeton Univ. Press, Princeton (1953), 177–188.
- [We] S. Werner, *Spiegelungskoeffizient und Fredholmscher Eigenwert für gewisse Polygone*, Ann. Acad. Sci. Fenn. Ser. A I Math. **22** (1997), 165–186.
- [Xi] J. Xiao, *Hadamard convolutions and some sufficient conditions for quasiconformal extensions*, Northeast. Math. J. **5** (1989), 321–329.
- [Ya] S. Yang, *Quasiconformal reflection, uniform and quasiconformal extension domains*, Complex Var. **17** (1992), 277–286.
- [Za] V.P. Zaharyuta, *Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein–Walsh theorem for analytic functions of several complex variables*, Ann. Polon. Math. **33** (1976), 137–148 (in Russian).
- [Zh1] I.V. Zhuravlev, *Univalent functions with quasiconformal extension and Teichmüller spaces*, Inst. Math., Novosibirsk, Preprint (1979) (in Russian).
- [Zh2] I.V. Zhuravlev, *Univalent functions and Teichmüller spaces*, Soviet Math. Dokl. **21** (1980), 252–255.

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CHAPTER 12

Beltrami Equation

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Abstract

We consider the Beltrami equation $\bar{\partial} w = \mu(z) \partial w$ in the classical case where $\|\mu\|_{\infty} < 1$, in the relaxed classical case or in the degenerate case where $|\mu| < 1$ a.e. and $\|\mu\|_{\infty} = 1$, and in the alternating case where $|\mu| < 1$ a.e. in part of the domain and $|\mu| > 1$ a.e. in other part of the domain. We mainly discuss existence, uniqueness, representation and nature of solutions.

1. Introduction and notations

1.1. The Beltrami equation

The *Beltrami equation* in a domain D , $D \subset \mathbb{C}$, has the following form in complex notation:

$$w_{\bar{z}} = \mu(z)w_z, \quad (\text{B})$$

where $\mu : D \rightarrow \mathbb{C}$ is a measurable function, and

$$w_{\bar{z}} = \bar{\partial}w = \frac{1}{2}(w_x + iw_y) \quad \text{and} \quad w_z = \partial w = \frac{1}{2}(w_x - iw_y),$$

are partial derivatives of w in $\bar{z}$ and z .

For $\mu \equiv 0$, (B) reduces to the Cauchy–Riemann equation.

In real variables x, y, u and v , $z = x + iy$, $w = u + iv$, (B) has the form

$$\begin{cases} v_y = \alpha u_x + \beta u_y, \\ -v_x = \beta u_x + \gamma u_y, \end{cases} \quad (\text{B}')$$

where α, β and γ are given measurable functions in x and y .

This survey is devoted to the Beltrami equation (B). In addition to the theory of Beltrami equation, there is a highly developed theory of the *Beltrami equation of the second kind*

$$w_{\bar{z}} = v(z)\bar{w}_z,$$

see for instance [KrushKül,Ren].

The Beltrami equation of the second type plays an important role in the theory of harmonic mappings in the plane, see [Ren] and [BshHe].

1.2. Historical remarks

The system (B') first appears in Gauss [Ga] in connection of finding isothermal coordinates on a surface. Local coordinates u and v on a given surface are called *isothermal* if the curves $u = \text{const}$ are orthogonal to the curves $v = \text{const}$, or equivalently, if the length element ds is given by

$$ds^2 = \Lambda(u, v)(du^2 + dv^2).$$

The transition from given local coordinates x and y to isothermal coordinates u and v is an injective mapping $(x, y) \rightarrow (u, v)$ satisfying

$$a(x, y)dx^2 + 2b(x, y)dxdy + c(x, y)dy^2 = \Lambda \cdot (du^2 + dv^2), \quad \Lambda > 0,$$

where u and v are solutions of the Beltrami system (B') with

$$\alpha = \frac{b}{\sqrt{\Delta}}, \quad \beta = \frac{a}{\sqrt{\Delta}}, \quad \gamma = \frac{c}{\sqrt{\Delta}} \quad \text{and} \quad \Delta = ac - b^2 \geq \Delta_0 > 0.$$

Gauss [Ga] proved the existence and uniqueness of a solution in the case of real analytic α , β and γ , or, equivalently, when μ are real-analytic. The equation then appeared in Beltrami studies [Belt] on surface theory. There is a long list of the names associated with proofs of existence and uniqueness theorem for more general classes of μ 's. Among them we mention Korn [Ko] and Lichtenstein [Li] (Hölder continuous μ 's, 1916) and Lavrent'ev [La1] (continuous μ 's, 1935). Morrey [Mo] in 1938 was the first to prove the existence and uniqueness of a homeomorphic solution for a measurable μ . Morrey's proof is based on PDE methods. In the mid 1950s Ahlfors [Ahl3], Bojarski [Bo1] and Vekua [Ve] proved the existence in the measurable case by singular integral methods. Ahlfors and Bers [AhlBer] established the analytic dependence of solutions on parameters.

1.3. Applications of Beltrami equations

The Beltrami equation was first used in various areas such as differential geometry on surfaces, see Section 1.2, hydrodynamics and elasticity. Most applications of the Beltrami equation are based on the close relation to quasiconformal (qc) mappings. Plane qc mappings appeared already implicitly in the late 1920s in papers by Grötzsch [Grö]. The relation between qc mappings and the Beltrami equation was noticed by Ahlfors [Ahl1] and Lavrent'ev [La1] in the 1930s. The important connection between the theory of Beltrami equation and the theory of plane qc mappings has stimulated an intensive study and enriched both theories. Most notable is the contribution of the qc theory to the modern development of Teichmüller spaces and Kleinian groups.

The Beltrami equation turned out to be instrumental in the study of Riemann surfaces, Teichmüller spaces, Kleinian groups, meromorphic functions, low-dimensional topology, holomorphic motion, complex dynamics, Clifford analysis, control theory and robotics. [Abi, Ahl4, Ahl5, As2, As3, AsMa, Ber1–Ber6, Bo2, Ca, Dr, EaEe, FaKr, Ge3, IwMa1, IwMa2, Krush2, KrushKü1, KrushKü2, Kün, Le3, LaSh, MaSaSu, Str, Sr, Su1, Su2, Vas, Vä1–Vä4, Ve] is only a partial list of references. Part of the list consists of books and expository papers where further references can be found.

1.4. Classification of Beltrami equations

We say that μ is *bounded* in D if $\|\mu\|_\infty < 1$, and that μ is *locally bounded* in D if $\mu|_A$ is bounded whenever A is a relatively compact subdomain of D . The study of Beltrami equation is divided into three cases according to the nature of $\mu(z)$ in D :

- (1) *The classical case*: $\|\mu\|_\infty < 1$.
- (2) *The relaxed classical case*: $|\mu| < 1$ a.e. and $\|\mu\|_\infty = 1$.
- (3) *The alternating case*: $|\mu| < 1$ a.e. in parts of D and $1/|\mu| < 1$ a.e. in other parts of D .

1.5. ACL solutions

By writing $f : D \rightarrow \mathbb{C}$ we will assume that D is a domain in $\mathbb{C}$, that is an open and connected set, and that f is continuous. A function $f : D \rightarrow \mathbb{C}$ is a *solution of (B)*, if f is ACL in D , and its ordinary complex partial derivatives, which exist a.e. in D , satisfy (B) a.e. in D . Here ACL means *absolutely continuous on lines*, i.e., f is absolutely continuous on almost every horizontal and almost every vertical lines in every rectangle R , $\bar{R} \subset D$ with sides parallel to the coordinate axes, see [Ahl5] or [LeVi].

Some authors, cf. [IwMa1, IwMa2] and [GuMaSuVu], include in the definition of a solution the assumption that f belongs to the Sobolev class $W_{\text{loc}}^{1,1}$. Note that $W_{\text{loc}}^{1,1}$ implies the ACL properly, but not vice versa.

A solution $f : D \rightarrow \mathbb{C}$ of (B) which is homeomorphism in D is called a μ -*homeomorphism* or μ -*conformal*. In cases (1) and (2) a solution $f : D \rightarrow \mathbb{C}$ of (B) will be called *elementary* if f is open and discrete, meaning that f maps every open set onto an open set and that the pre-image of every point is a discrete set in D .

It should be noted that if $f : D \rightarrow \mathbb{C}$ is an open ACL mapping, then by a result of Gehring and Lehto, cf. [GeLe], see also [Ahl5] and [LeVi], f is differentiable a.e. in D .

For an open and sense-preserving mapping $f : D \rightarrow \mathbb{C}$, the *complex dilatation* of f is defined by

$$\mu_f(z) = \mu(z) = \bar{\partial} f(z) / \partial f(z) \quad (1.1)$$

when $\partial f(z) \neq 0$, and by $\mu(z) = 0$ when $\partial f(z) = 0$. Then μ is measurable, $|\mu| < 1$ a.e. and the *dilatation* of f which is defined by

$$K(z) = K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \quad (1.2)$$

is finite a.e.

1.6. Ellipticity of the Beltrami equation

In the classical case (1) in Section 1.4, the system (B') is *uniformly*, or in a different terminology, *strongly elliptic*, i.e.,

$$\Delta = \alpha\delta - \beta^2 \geq \Delta_0 > 0. \quad (1.3)$$

In the relaxed classical case (2) in Section 1.4, (B') is *elliptic*, i.e.,

$$\Delta = \alpha\delta - \beta^2 \geq 0. \quad (1.4)$$

In the alternating case (3) in Section 1.4, (B') is elliptic in every subdomain of D , where either μ or $1/\mu$ is locally bounded.

2. The classical case: $\|\mu\|_\infty < 1$

2.1. Quasiconformal mappings

Consider the Beltrami equation (B) in Section 1.1. with $\|\mu\|_\infty < 1$, and let $f : D \rightarrow \mathbb{C}$ be a homeomorphic solution. Since $|\mu| < 1$ a.e., f is sense preserving, and since $\|\mu\|_\infty < 1$, f is a quasiconformal mapping.

There are several equivalent definitions of quasiconformality and some of them have been extended to mappings between metric spaces, see, e.g., [HeiKo] and [Hei]. For an extension to $\mathbb{R}^n$, see [Ge2, Ge3, La2, Vä1, Vä2], and to Banach spaces, see [Vä3, Vä4]. According to the analytic definition, a sense-preserving embedding $f : D \rightarrow \mathbb{C}$ is quasiconformal if f is ACL and there exists $k \in [0, 1)$ such that a.e. in D ,

$$|\bar{\partial} f(z)| \leq k |\partial f(z)|. \quad (2.1)$$

At points $z \in D$ where f is differentiable,

$$K(z) = K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = \lim_{r \rightarrow 0} \frac{\max_{|w-z|=r} |f(z) - f(w)|}{\min_{|w-z|=r} |f(z) - f(w)|} \quad (2.2)$$

is the *dilatation of f at z* . Then the *maximal dilatation of f in D* which is defined by

$$K_f = \operatorname{ess\,sup}_{z \in D} K(z) \quad (2.3)$$

satisfies

$$K_f = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}. \quad (2.4)$$

The other two basic types of definitions of quasiconformality are a geometric one, which uses the quasiinvariance of certain conformal invariants like the modulus of path families, see [Ahl2, LeVi, Pf] and [Vä1], and a metric definition which is based, roughly speaking, on the extent of roundedness of the image of balls; see [Vä1, HeiKo]. If $f : D \rightarrow \mathbb{C}$ is a homeomorphic solution of (B) and h is holomorphic in $f(D)$, then $g = h \circ f$ is a solution. Indeed, g is ACL, and ∂g and $\bar{\partial} g$ satisfy $\bar{\partial} g = \mu(z) \partial g$ a.e. in D . Note that since non-constant holomorphic mappings are open and discrete, so is g . By Stoilow's factorization theorem [St] every open and discrete mapping $g : D \rightarrow \mathbb{C}$ can be written as a composition $h \circ f$, where f is homeomorphism and h is a nonconstant holomorphic mapping in $f(D)$. In view of Stoilow factorization theorem, it is not hard to see that if g is an elementary solution, see Section 1.5, then $g = h \circ f$, where f is a homeomorphic solution of (B) and h is holomorphic.

2.2. Main problems

The main problems in the classical case concerning solutions are:

- existence,
- uniqueness,
- regularity,
- representation of the solution set,
- removability of isolated singularities,
- boundary behavior,
- mapping properties.

2.3. Integrability

Let $f : D \rightarrow \mathbb{C}$ be a homeomorphic solution. Then by using the local integrability of the Jacobian J of f in D and the dilatation inequality

$$(|\partial f(z)| + |\bar{\partial} f(z)|)^2 \leq \frac{1 + |\mu(z)|}{1 - |\mu(z)|} J(z), \quad (2.5)$$

which can be easily verified, one can conclude that f belongs to the Sobolev class $W_{\text{loc}}^{1,2}(D)$, and that the partial derivatives ∂f and $\bar{\partial} f$ are a.e. the generalized $W_{\text{loc}}^{1,2}$ derivatives of f . In particular, every homeomorphic solution belongs to $W_{\text{loc}}^{1,1}(D)$.

Bojarski [Bo1] showed that in fact $f \in W_{\text{loc}}^{1,p}(D)$ for some $p > 2$ which depends on $k = \|\mu\|_\infty$ or equivalently on K . Settling a long standing conjecture by Gehring and Reich [GeRei], Astala [As1] proved that if $f : D \rightarrow \mathbb{C}$ is qc and E is a Borel set in D , then

$$|f(E)| \leq M|E|^{1/K}$$

with a constant $M = M(K) = 1 + O(K - 1)$. As a consequence, Astala settled a long standing conjecture by Gehring and Lehto, proving that $f \in W_{\text{loc}}^{1,p}$ for every $p < p_0$, where

$$p_0 = \frac{2K}{K-1} = 1 + \frac{1}{k} \quad (2.6)$$

and that p_0 is best possible, see [As2].

It should be noted that a $W_{\text{loc}}^{1,2}$ mapping which satisfies the Beltrami equation with $\|\mu\|_\infty \leq k$ is a K -quasiregular mapping with $K = (1+k)/(1-k)$, according to the notation in [Ri] or a mapping of bounded distortion, according to the notation in [Resh]. Note that such solutions are elementary in our notation; see Section 1.5.

Astala, cf. [As2], proved that every $W_{\text{loc}}^{1,q}$ solution with

$$q > q_0 = 1 + k \quad (2.7)$$

belongs to $W_{\text{loc}}^{1,2}$, and hence is elementary. Iwaniec and Martin [IwMa1, IwMa2] showed that q_0 is the best lower bound, i.e., given $k \in [0, 1)$ there is a measurable $\mu : \mathbb{C} \rightarrow \mathbb{C}$

with $\|\mu\|_\infty = k$ such that the equation (B) has a nonelementary solution f which belongs to $W_{\text{loc}}^{1,q}(\mathbb{C})$ for every $q < q_0$, and does not belong to $W_{\text{loc}}^{1,2}(\mathbb{C})$.

One of main theorems in the theory of Beltrami equations in the classical case, and hence in the theory of plane quasiconformal mappings, is the following existence and uniqueness theorem.

THEOREM 2.1. *Let $\mu(z)$ be a complex-valued measurable function in a domain D in the complex plane $\mathbb{C}$ with $\|\mu\|_\infty < 1$. Then:*

- (i) *equation (B) has a sense-preserving homeomorphic solution $f : D \rightarrow \mathbb{C}$;*
- (ii) *f is unique up to a post-composition by a conformal mapping.*

The following corollary can be derived from Theorem 2.1 and Stoilow's factorization theorem. The corollary says that every homeomorphic solution of (B) in the case $\|\mu\|_\infty < 1$ generates the set of all elementary solutions.

COROLLARY 2.2. *Let $h : D \rightarrow \mathbb{C}$ be a homeomorphic solution of (B) with $\|\mu\|_\infty < 1$ and f an elementary solution of (B). Then there exists a unique holomorphic function g in $h(D)$ such that $f = g \circ h$.*

In view of the corollary, the homeomorphic solutions may be viewed as *prime solutions*.

REMARK 2.3. Given μ as in Theorem 2.1, the measurable metric $ds = |dz + \mu d\bar{z}|$ defines a complex structure in D and makes D a Riemann surface, and every homeomorphic solution of (B) becomes a conformal mapping from D with this conformal structure. For this reason Theorem 2.1 is known as the measurable Riemann mapping theorem, see [AhlBer].

2.4. Methods of proof of uniqueness and existence

In this section we survey some of the methods in the proofs of existence and uniqueness of homeomorphic solutions.

2.5. Uniqueness

Let f and g be two homeomorphic solutions in D . Then f and g are in $W_{\text{loc}}^{1,2}$, and hence $h = g \circ f^{-1} \in W_{\text{loc}}^{1,1}$, and since the partial derivatives of h satisfy the Cauchy–Riemann equations, h is conformal by Weyl's lemma, see [Ahl5], and the uniqueness follows.

2.6. Existence

Local solutions. The standard approach in finding a local solution at a point $z_0 \in D$ goes as follows, cf. [LeVi]. One approximates μ in a certain neighborhood U of z_0 by a sequence of “nice” functions μ_n , like polynomials in z and $\bar{z}$. For each μ_n one finds a normalized homeomorphic solution $f_n : U_n \rightarrow \mathbb{C}$, $U_n \subset U$, say by substituting a power series

in z and $\bar{z}$, where one has much freedom in choosing some of the coefficients. Then one verifies that $V = \bigcap U_n$ is a neighborhood of z_0 . Next, with the aid of suitable distortion theorems for quasiconformal mappings, equicontinuity of the sequence f_n is obtained, and one deduces that there is a subsequence which converges locally uniformly to a homeomorphic solution f of (B) in V . A similar method can be applied in the relaxed classical case.

Global solutions via local solutions. In view of Theorem 2.1(ii), see also Remark 2.3, it suffices to find a local solution of (B) at every point $z \in D$. Then the existence of a global homeomorphic solution follows by the uniformization theorem.

Indeed, since every homeomorphic ACL solution is in $W_{\text{loc}}^{1,2}$, it follows that if f and g are two local homeomorphic solutions of (B), say in subdomains U and V , respectively, then $g \circ f^{-1}$ is conformal in $f(U \cap V)$. It thus follows that the set of all local homeomorphic solutions defines a conformal structure in D , which makes D a planar Riemann surface. By the classical Uniformization Theorem, cf. [FaKr], there is a conformal mapping F of D (endowed with the conformal structure which is defined by the local solutions of (B)) into $\mathbb{C}$, meaning if $f: U \rightarrow \mathbb{C}$, $U \subset D$, is a local homeomorphic solution of (B), then $h = F \circ f^{-1}$ is conformal in $f(U)$, and $F|U = h \circ f$. Since a post-composition of a solution by a conformal mapping is again a solution it follows that F is a global solution of (B) in D .

Global solutions directly. There are two basic methods of finding global homeomorphic solutions directly. One of them is based on PDE methods and the other one on singular integral methods.

PDE methods. Following [Mo], consider the system (B') first for smooth coefficients. Define a boundary value problem for $u(z) = \operatorname{Re} f(z)$, get the existence of u and then of v . Finally, by using approximation, one gets a global homeomorphic solution in $W_{\text{loc}}^{1,2}$ for the original μ .

Singular integral methods. Here we follow [Ahl5, Bo1] and [Ve]. Consider (B) with a measurable μ in a domain D in $\mathbb{C}$ with $\|\mu\|_{\infty} = k < 1$. Extend μ on $\mathbb{C}$ by setting $\mu = 0$ on $\mathbb{C} \setminus D$. Then the extended function, denoted again by μ , is measurable in $\mathbb{C}$, and has the same L^{∞} norm k . Obviously, the existence of a homeomorphic solution of (B) in $\mathbb{C}$ yields a homeomorphic solution in D . Therefore it suffices to consider (B) in $\mathbb{C}$.

We first consider the case where $\mu \in C_0^1(D)$, i.e., μ has continuous partial derivatives and a compact support in $\mathbb{C}$. A homeomorphic solution $f: \mathbb{C} \rightarrow \mathbb{C}$ in this case is conformal outside of a certain compact set, and hence extends to a homeomorphism of $\bar{\mathbb{C}}$ onto itself with $f(\infty) = \infty$. Since a composition of a solution with a function of a form $z \rightarrow az + b$, $a, b \in \mathbb{C}$ is again a solution, we may look for a solution $f: \mathbb{C} \rightarrow \mathbb{C}$ which has the Laurent expansion

$$f(z) = z + g(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad (2.8)$$

in a neighborhood of ∞ .

Setting (2.8) in (B) we get

$$g_{\bar{z}} = \mu(z)[1 + g_z]. \quad (2.9)$$

Then $g_z = f_z - 1$ is in the class $L^p(\mathbb{C})$ for every $p \geq 1$. In each of these classes the inverse of the operator $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ is given by the Cauchy transform

$$P : L^p \rightarrow L^p \quad (2.10)$$

which is defined by

$$Ph(z) = -\frac{1}{\pi} \int \frac{h(\zeta)}{\zeta - z} dm(\zeta), \quad (2.11)$$

where the singular integral is defined over $\mathbb{C}$ as a principal value integral. By applying P to (2.8) we obtain

$$g = Pg_{\bar{z}} = P\mu(1 + g_z) \quad (2.12)$$

and hence

$$g_z = T\mu(1 + g_z), \quad (2.13)$$

where

$$T = \frac{\partial}{\partial z} P = -\frac{1}{\pi} \int \frac{h(\zeta)}{(\zeta - z)^2} dm(\zeta). \quad (2.14)$$

Thus $h := g_z$ satisfies the equation

$$h = T\mu(1 + h). \quad (2.15)$$

Since $\|T\|_p \rightarrow 1$ as $p \rightarrow 2$, it follows that there is p_0 such that for $2 < p < p_0$,

$$\|T\mu\|_p = \|T\|_p \|\mu\|_\infty < 1,$$

and hence, by the fixed point theorem for contracting operators, (2.15) has a solution $h \in L^p$ which can be found by successive approximations. Thus

$$f = z + P\mu(1 + h) \quad (2.16)$$

is a solution of (B).

Next one shows that f is a local homeomorphism in $\overline{\mathbb{C}}$, and hence, a homeomorphism of $\overline{\mathbb{C}}$ onto itself, see [Ahl5] and [Bo1]. One can normalize f so that it fixes the points 0, 1 and ∞ . The *normalized solution* is denoted by f^μ .

The case where μ is measurable with compact support is obtained by approximation, see [LeVi].

To obtain a solution in the general case, suppose first that μ is measurable in $\mathbb{C}$ and that $\mu = 0$ in Δ . Then

$$\mu_1(z) = \mu\left(\frac{1}{z}\right) \frac{z^2}{\bar{z}^2}$$

is measurable, $\|\mu_1\|_\infty = \|\mu\|_\infty$ and $\mu_1 = 0$ in $\mathbb{C} \setminus \Delta$. Then (B) with μ_1 instead of μ has a normalized solution g .

Finally, given a measurable function μ in $\mathbb{C}$ with $\|\mu\|_\infty < 1$, we look for a mapping h such that $f = h \circ g$, where g is a homeomorphic solution of (B) with $\mu(z)\chi(z)$ instead of μ with χ being the characteristic function of $\mathbb{C} \setminus \Delta$. Then μ_h can be computed, and since $\mu_h = 0$ in $\mathbb{C} \setminus \Delta$, a homeomorphic solution of (B) with μ_h instead of μ exists. This completes the proof of existence.

2.7. Smoothness of the solutions

Shabat [Sh] showed that for $n > 0$ and $0 < \alpha < 1$ if $\mu \in C^{n,\alpha}(D)$, i.e., the partial derivatives of order n of μ are α -Hölder continuous in D , and f is a solution of (B) then $f \in C^{n+1,\alpha}(D)$, and if $\alpha = 1$, then for all $\varepsilon > 0$, $f \in C^{n+2-\varepsilon}$.

2.8. Analytic dependence on parameters

Ahlfors and Bers [AhlBer] examined the dependence of a normalized solution of (B) on a parameter. They showed that if μ as an element of L^∞ depends analytically on a complex parameter τ then the normalized solution of (B) as an element of $W_{\text{loc}}^{1,p}$ depends analytically on τ .

3. The relaxed classical case: $|\mu| < 1$ a.e. and $\|\mu\|_\infty = 1$

3.1. Approximation

Recall that in our notation, a μ -homeomorphism in a domain D , $D \subset \mathbb{C}$ is an ACL homeomorphic solution of (B) in D , see Section 1.5. For some functions μ with $|\mu(z)| < 1$ a.e. and $\|\mu\|_\infty = 1$ there are no μ -homeomorphisms, i.e., homeomorphic solutions of (B), as illustrated below in Examples 3.2 and 3.3. Even when a μ -homeomorphism exists, it is not known whether it is unique and generates the set of all elementary solutions. As in the classical case, by *elementary solution* we mean an open and discrete solution.

Contrary to the classical case where every ACL solution belongs to $W_{\text{loc}}^{1,1}$, in the relaxed classical case an ACL solution need not be in $W_{\text{loc}}^{1,1}$. Recall that the $W_{\text{loc}}^{1,1}$ property implies the ACL property but not vice versa.

The main problems in the theory of μ -homeomorphisms are as in the classical case, see Section 2.2.

Approximation theorems like the following one are instrumental in many existence proofs.

THEOREM 3.1 [RaSrYa2]. *Let μ_n and μ be measurable in D , and let $|\mu(z)| < 1$ a.e. Suppose $|\mu_n| \leq |\mu|$ and $\mu_n \rightarrow \mu$ a.e., $\|\mu_n\|_\infty < 1$ and let $f_n: D \rightarrow \mathbb{C}$ be μ_n -homeomorphism. If*

- (i) $f_n \rightarrow f$ locally uniformly,
- (ii) $K_\mu(z) \in L^1_{\text{loc}}(D)$,

then

- (i) f is either constant or a μ -homeomorphism,
- (ii) $f^{-1} \in W^{1,2}_{\text{loc}}$,
- (iii) f generates the set of all $W^{1,2}_{\text{loc}}$ solutions, i.e., the set $\{h \circ f: h \text{ analytic in } f(D)\}$ coincides with the set of $W^{1,2}_{\text{loc}}$ solutions.

In applying this theorem one needs at least that $K(z) \in L^1_{\text{loc}}(D)$. Note that in the classical case $K(z) \in L^\infty(D)$.

3.2. Examples

The following two examples show that the assumption $K \in L^p$ for some $p \geq 1$, or even that K belongs to L^p , for all $1 \leq p < \infty$, does not guarantee the existence of μ -homeomorphisms.

EXAMPLE 3.2. Let

$$\mu(re^{i\theta}) = -\frac{e^{2i\theta}}{1+2r}$$

in $\Delta = \{|z| < 1\}$. Then

$$K = 1 + \frac{1}{r} \in L^1(\Delta).$$

We shall show that (B) has no μ -homeomorphic solution in Δ . Indeed, consider the mapping

$$g(re^{i\theta}) = (1+r)e^{i\theta}.$$

It is easy to see that g is a μ -homeomorphism of $\Delta \setminus \{0\}$ onto $1 < |w| < 2$.

Suppose that (B) has a homeomorphic solution $f: \Delta \rightarrow \mathbb{C}$. Then, since f and g are locally quasiconformal in $\Delta \setminus \{0\}$, it follows by the classical uniqueness theorem that $h = g \circ f^{-1}$ is conformal in $f(\Delta) \setminus f(0)$ and can be extended to $f(\Delta)$ that is impossible.

EXAMPLE 3.3. Let

$$\mu(re^{i\theta}) = -e^{2i\theta} \frac{1 + 1/(\log 1/r) - 1/(\log^2 1/r)}{1 + 1/(\log 1/r) + 1/(\log^2 1/r)}$$

in $\Delta(1/e) = \{z: |z| < 1/e\}$. Then

$$K = \log^2 \frac{1}{r} + \log \frac{1}{r} \in \bigcap_{1 \leq p < \infty} L^p(\Delta(1/e)).$$

We shall show that (B) has no μ -homeomorphic solution in $\Delta(1/e)$.

Indeed, consider the mapping

$$g(re^{i\theta}) = \left(1 + \frac{1}{\log \frac{1}{r}}\right) e^{i\theta}.$$

It is easy to see that g is a μ -homeomorphism of $\Delta(1/e) \setminus \{0\}$ onto $1 < |w| < 2$. Then the assertion follows as in the previous example.

REMARK 3.4. Contrary to the classical case, in the relaxed classical case as well as in the alternating case, see Section 4, the existence and nature of a solution may depend not only on the behavior of $|\mu|$ but also on $\arg \mu$. For instance, let μ be as in Example 3.2. Then (B) has no homeomorphic solutions in Δ . However, if μ is replaced by $\tilde{\mu} = -\mu$ then (B) with $\tilde{\mu}$ instead of μ has a solution $f(re^{i\theta}) = re^{1-1/r}e^{i\theta}$ which maps Δ homeomorphically onto itself.

Furthermore, as a consequence of Proposition 3.25 in [GuMaSuVu], Gutlyanskii, Martio, Sugawa and Vuorinen show that given μ which is locally bounded in $\Delta \setminus \{0\}$, there is an ACL homeomorphic solution of (B) in Δ , where μ is replaced by $\tilde{\mu}(z) = |\mu(z)|\bar{z}/z$.

The effect of $\arg \mu$ is noticed in [Vo1, Vo2, And] and [ReWa] in the classical case, in [Le1, Le2, Ba, SrYa2, SrYa5, SrYa6], and [GuMaSuVu] in the relaxed classical case, and in [Ya, SrYa1, SrYa4] in the alternating case.

In the following sections we present various conditions on $\mu(z)$ or on $K(z)$ which yield the existence of homeomorphic solutions and some of their properties.

3.3. The singular set

Given a measurable function μ in D with $\|\mu\|_\infty = 1$ the *singular set* E of μ is the set of all points $z \in \bar{D}$ such that $\|\mu\|D \cap U\|_\infty = 1$ for every neighborhood $U = U(z)$ of z . Note that

$$z_0 \in E \iff \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{|z-z_0| < \varepsilon} K(z) = \infty.$$

The study of μ -homeomorphisms in the relaxed classical case can be divided into the following cases:

- (i) E is specified and $E \subset \partial D$;
- (ii) E is specified and $E \subset D$;
- (iii) E is not specified.

3.4. Case (i): The singular set E is specified and $E \subset \partial D$

3.4.1. Existence and uniqueness. Note that $E \subset \partial D$ means that μ is locally bounded. Bers [Ber1, 3.1], see also [Bo1] and [Beli], showed that in this case a μ -homeomorphism exists and it is unique up to a post composition by a conformal mapping. The proof follows by a normal families argument or by application of the Uniformization Theorem.

3.4.2. Boundary behavior. We consider the case that $D = \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $E \subset \mathbb{R} \subset \partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. Let f be a μ -homeomorphism of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ into $\mathbb{C}$. Note that f can be viewed as a conformal embedding of $\mathbb{H}$ endowed with the measurable conformal structure $ds = |dz + \mu(z)d\bar{z}|$ into $\mathbb{C}$. It is well known that if f is conformal in $\mathbb{H}$ (endowed with the Euclidean metric), then $f(\mathbb{H})$ is a proper subset of $\mathbb{C}$, and if, in addition, $f(\mathbb{H})$ is a Jordan domain, f has a homeomorphic extension on $\overline{\mathbb{H}}$, the closure of $\mathbb{H}$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The same is true for a μ -homeomorphism in $\mathbb{H}$ when $\|\mu\|_\infty < 1$.

If μ and f are as above, $f(\mathbb{H})$ is either $\mathbb{C}$, see Example 3.7 in Section 3.4.5, or a proper subset of $\mathbb{C}$. Thus, one may ask under what further conditions on μ every μ -homeomorphism f satisfies

- (i) $f(\mathbb{H}) \neq \mathbb{C}$, and
- (ii) if, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\overline{\mathbb{H}}$.

3.4.3. Let $r > 0$ and let ρ be measurable a.e. positive function in $(0, r)$. We say that ρ is *locally bounded away from 0*, if $\rho|_A \geq \rho_0(A) > 0$ a.e. for all relatively compact subsets A of $(0, r)$. We say that $\rho(t) \rightarrow 0$ a.e. as $t \rightarrow 0^+$, if $\rho \rightarrow 0$ as $t \rightarrow 0^+$, $t \in (0, r) \setminus A$ for some set A of linear measure zero.

Let $U, U \subset \mathbb{C}$ be an open set which meets $\mathbb{R}$. We say that $|\mu(x, y)| \rightarrow 1$ a.e. uniformly in U as $y \rightarrow 0^+$, $x \in U \cap \mathbb{R}$, if U has a subset A with $m_2(A) = 0$ such that for all $x \in U \cap \mathbb{R}$,

$$\lim_{y \rightarrow 0^+} |\mu(x, y)| = 1, \quad (x, y) \in U \setminus A,$$

and there exists $C > 1$ such that, for all (x_1, y) and (x_2, y) in $U \setminus A$,

$$\frac{1}{C} \leq \frac{1 - |\mu(x_1, y)|}{1 - |\mu(x_2, y)|} \leq C. \quad (3.1)$$

Note that (3.1) is equivalent to the following condition

$$\frac{1}{2C} \leq \frac{K(x_1, y)}{K(x_2, y)} \leq 2C. \quad (3.2)$$

It is shown in [SrYa5] that $|\mu(x, y)| \rightarrow 1$ a.e. uniformly in U if and only if there are positive measurable functions $\rho: (0, r) \rightarrow \mathbb{R}_+$ and $M: U \rightarrow \mathbb{R}_+$ with

$$\rho \rightarrow 0 \quad \text{a.e. as } t \rightarrow 0^+$$

and

$$\frac{1}{C} \leq M(x, y) \leq C$$

such that

$$|\mu(x, y)| = 1 - \rho(y)M(x, y).$$

THEOREM 3.5 [SrYa5]. *Let f be an embedding of $\mathbb{H}$ into $\mathbb{C}$ and $a \in \mathbb{R}$. Suppose that a has a neighborhood U in $\mathbb{C}$ such that f is ACL in $U^+ = U \cap \mathbb{H}$ with a locally bounded complex dilatation μ in U^+ . If*

- (a) $|\mu(x, y)| \rightarrow 1$ a.e. uniformly in U^+ as $y \rightarrow 0^+$, $x \in U \cap \mathbb{R}$,
- (b) $\arg \mu$ is continuously differentiable and $\partial(\arg \mu)/\partial y$ is bounded in U^+ ,
- (c) $|\arg \mu| \leq 2\theta_0 < \pi$ in U^+ ,

then

- (i) $f(\mathbb{H}) \neq \mathbb{C}$,
- (ii) if, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\mathbb{H} \cup (U \cap \mathbb{R})$.

COROLLARY 3.6. *Suppose that the conditions of Theorem 3.5 hold for all $a \in \mathbb{R}$, and that $f(H)$ is a Jordan domain, then f has a homeomorphic extension on $\mathbb{H} \cup \mathbb{R}$.*

The main tool in the proof of Theorem 3.5 is the method of deformation of the complex dilatation which is defined below.

3.4.4. Deformation of the complex dilatation. See [SrYa1]. Let μ be a locally bounded complex-valued measurable function in a domain U , and let g be an embedding of U into $\mathbb{C}$, and suppose that g is locally quasiconformal. The deformation $g * \mu$ of μ which is induced by g is defined in $g(U)$ by

$$g * \mu(w) = \frac{A\mu - B}{A - B\mu} \circ g^{-1}(w), \quad w \in g(U),$$

where $A = \partial g$ and $B = \bar{\partial} g$.

It is easy to see that $g * \mu$ is measurable and locally bounded in $g(U)$. Moreover, if h is a locally quasiconformal embedding of $g(U)$ into $\mathbb{C}$ with complex dilatation $\mu_h = g * \mu$ a.e. in $g(U)$, then $f = h \circ g$ is locally quasiconformal with complex dilatation $\mu_f = \mu$ a.e. in U .

For the proof of Theorem 3.5 two deformations of the complex dilatations are used:

$$\mu_2 = g_1 * \mu \quad \text{and} \quad \mu_3 = g_2 * \mu_1.$$

The mappings g_1 and g_2 are chosen so that they are identity on $U \cap \mathbb{R}$ and on $g_1(U \cap \mathbb{R})$, respectively, so that $\arg \mu_2 = 0$ and such that μ_3 is bounded in $g_2 \circ g_1(V \cap \mathbb{H})$ for some neighborhood V of a , $V \subset U$. Since μ_3 is bounded there is a μ_3 -homeomorphism h in V^+ such that $f = h \circ g_2 \circ g_1$ in V^+ . Then f and h have the same boundary extension properties and the theorem follows.

The mapping g_1 is chosen to be a diffeomorphism in some square $S(a) = \{|x - a|, |y| < \delta\}$, such that its inverse

$$x + iy = G(\xi, \eta) = g_1^{-1}(\xi, \eta)$$

is given by

$$x = \varphi(\xi, \eta), \quad y = \eta,$$

where φ is a solution of the ordinary differential equation

$$\frac{dx}{dy} = -\tan \theta(x, y)$$

with the initial condition

$$\varphi(\xi, 0) = \xi.$$

The mapping g_2 is chosen as

$$g_2(\xi, \eta) = \left(\xi, \int_0^\eta \rho(t) dt \right).$$

Then g_2 is locally quasiconformal in $g_1(S)$ and it is identity on $S \cap \mathbb{R}$.

Note, that g_1 and g_2 map horizontal lines into horizontal lines, and vertical lines into vertical lines, respectively.

3.4.5. Examples

EXAMPLE 3.7. In the following example, f and μ satisfy the conditions of Corollary 3.6 with $E = \mathbb{R}$. Here f maps $\mathbb{H}$ homeomorphically onto itself, and it has a homeomorphic extension onto $\mathbb{H} \cup \mathbb{R}$, as asserted in the theorem, but not on $\overline{\mathbb{H}}$. More precisely, here $f(\mathbb{H} \cup \mathbb{R}) = \mathbb{H} \cup \mathbb{R}_+ \cup \{\infty\}$, and the cluster set of f at ∞ is the closed ray $[-\infty, 0]$.

The mapping f in this example is defined by

$$f(x, y) = e^{x+2i \arctan y^2}, \quad y > 0.$$

Then the complex dilatation of f is given by

$$\mu(x, y) = \frac{y^2 - 4y + 1}{y^2 + 4y + 1}$$

and

$$1 - |\mu| = 1 - \mu = \frac{4y}{y^2 + 4y + 1}.$$

Therefore $1 - |\mu(x, y)| = \rho(y)M(x, y)$, where

$$\rho(y) = 4y \quad \text{and} \quad M(x, y) = \frac{1}{y^2 + 4y + 1},$$

and thus the conditions of Corollary 3.6 hold.

EXAMPLE 3.8. In the following example f is a μ -homeomorphism in $\mathbb{H}$ with μ satisfying conditions (a) and (b) in Theorem 3.5, but not condition (c). In this example f maps $\mathbb{H}$ onto $\mathbb{C}$.

The mapping f in this example is defined by

$$f(x, y) = x + i \log y, \quad y > 0.$$

Then clearly $f(\mathbb{H}) = \mathbb{C}$. The complex dilatation of f is given by

$$\mu(x, y) = \frac{y - 1}{y + 1}.$$

Then

$$1 - |\mu| = \frac{2y}{y + 1} = \rho(y)M(x, y),$$

where

$$\rho(y) = 2y, \quad M(x, y) = \frac{1}{y + 1} \quad \text{and} \quad \arg \mu(x, y) \rightarrow \pi \quad \text{as } y \rightarrow 0^+,$$

and thus (a) and (b) hold, and (c) fails.

3.4.6. The last Example 3.8 shows that condition (c) is indispensable in Theorem 3.5. It is shown in [SrYa5] that the same is true for conditions (a) and (b).

3.5. Case (ii): The singular set E is specified and $E \subset D$

Srebro and Yakubov [SrYa2] proved the following existence and uniqueness theorem in the case when $D = \mathbb{C}$ and $E = \mathbb{R}$.

THEOREM 3.9. *Let μ be a complex-valued measurable function such that μ is locally bounded in $\mathbb{C} \setminus \mathbb{R}$. Suppose that, for every point $x_0 \in \mathbb{R}$, there are positive numbers $r = r(x_0)$ and $R = R(x_0)$ such that, for a.e. $z = x + iy$ in the disk $D(x, r_0) = \{z: |z - x_0| < r\}$,*

$$\mu(z) = e^{2i\theta(z)} [1 - \rho(y)M(z)], \quad (\text{M})$$

for some continuously differentiable function $\theta: D(x_0, r) \rightarrow (-\pi/2, \pi/2)$, continuous function $\rho: (-r, r) \rightarrow \mathbb{R}$, with $\rho(0) = 0$ and $\rho(y) > 0$ for all $y \neq 0$, and measurable function $M: D(x_0, r) \rightarrow \mathbb{C}$ such that the following condition (R) holds a.e. in $D(x_0, r)$:

$$\frac{1}{R} \leq \operatorname{Re} M(z) \leq R \quad \text{and} \quad |\operatorname{Im} M(z)| < R. \quad (\text{R})$$

Then:

- (i) *a μ -homeomorphism $f: \mathbb{C} \rightarrow \mathbb{C}$ exists,*
- (ii) *f is unique up to a post composition of f by a conformal map,*
- (iii) *f generates the set of all $W_{\text{loc}}^{1,2}$ solutions.*

3.5.1. In [SrYa6] the existence and uniqueness of a μ -homeomorphism is proved in the case where E is a union of finite number of smooth closed arcs or cross cuts in a domain D under the following assumptions on D and E :

ASSUMPTIONS. Let D be a domain in $\mathbb{C}$, and let E be a relatively closed subset of D . Suppose that each point z_0 in E has a neighborhood $U(z_0)$ such that $E \cap U(z_0)$ is either a C^3 arc or a *finite star* with a *vertex* at z_0 and end points in $\partial U(z_0)$, meaning that $E \cap U(z_0)$ is a finite union of arcs, each connecting the point z_0 with $\partial U(z_0)$ and which are disjoint except for their common end point z_0 .

In the later case, we assume also that each of the arms is C^3 and that the arms meet at nonzero angles.

According to the assumptions, E is a countable union of arcs E_n with end points in the union of the boundary of D in $\overline{\mathbb{C}}$ and the set of all star vertices of E . Furthermore, since the arms are C^3 , each E_n excluding its end points has a neighborhood U_n , $U_n \subset D$ such that the U_n 's are disjoint and for every inner point z_0 in E_n , there are $r = r(z_0) > 0$ and a C^3 path $\gamma: [-r, r] \rightarrow E_n$ with $\gamma(0) = z_0$ and $|\gamma'(s)| = 1$, $|s| \leq r$ such that the normals at points of γ do not intersect in U_n .

THEOREM 3.10 [SrYa6]. *Let D , E , E_n and U_n be as in Section 3.5.1. Let $\mu(z)$ be a complex-valued measurable function in D which is locally bounded in $D \setminus E$. Suppose that for each $n = 1, 2, \dots$, there are:*

- (a) a C^1 function $\theta_n : U_n \rightarrow \mathbb{R}$,
- (b) a continuous function $\rho_n : \mathbb{R} \rightarrow [0, \infty)$ with $\rho_n(0) = 0$,
- (c) a measurable function $M_n : U_n \rightarrow \mathbb{C}$, and
- (d) a positive number R_n such that for a.e. $z \in U_n$

$$1/R_n \leq \operatorname{Re} M_n(z) \leq R_n \quad \text{and} \quad |\operatorname{Im} M_n(z)| \leq R_n, \quad (\mathbf{R}_n)$$

$$\mu(z) = e^{2i\theta_n(z)} [1 - \rho_n(\operatorname{dist}(z, E_n)) M_n(z)], \quad (\mathbf{M}_n)$$

and such that

$$dz + \mu(z) d\bar{z} = dz + e^{2i\theta(z)} d\bar{z} \neq 0$$

when restricted to E_n . Then:

- (i) a μ -homeomorphism $f : D \rightarrow \mathbb{C}$ exists,
- (ii) f is unique up to a post composition of f by a conformal map,
- (iii) f generates the set of all $W_{\text{loc}}^{1,2}$ solutions.

THEOREM 3.11 [SrYa6]. *Let D, E, E_n, U_n be as in Theorem 3.10. Let $U_{n,1}$ and $U_{n,2}$ be the connected components of $U_n \setminus E_n$, and let $U = \bigcup U_n$. Let $\mu(z)$ be a complex-valued measurable function in D which is locally bounded in $D \setminus E$. Suppose that μ is as in Theorem 3.10, except that for some of the sets U_n where either $\|\mu|_{U_{n,1}}\|_\infty < 1$ or $\|\mu|_{U_{n,2}}\|_\infty < 1$. Then the assertions (i)–(iii) of Theorem 3.10 hold.*

The main tool in proving these existence and uniqueness theorems is the method of the deformation of the complex dilatation; see Section 3.4.4.

3.6. Case (iii): The singular set E is not specified

In this case the control on μ which gives existence, uniqueness and allows a study of properties may be expressed either by integral estimates like the convergence or divergence of

$$\int_U F(\mu) dm \quad \text{or} \quad \int_U \Phi(K_\mu) dm, \quad U \subset D$$

for some functions F or Φ or by measure estimates like

$$|\{z \in D : |\mu(z)| > 1 - \varepsilon\}| \leq \varphi(\varepsilon) \quad \text{or} \quad |\{z \in D : K_\mu(z) > t\}| \leq \psi(t)$$

for some functions φ or ψ . These two basic forms of control are related. Indeed, suppose that

$$\Phi : [1, \infty) \rightarrow \mathbb{R}_+$$

is a nondecreasing function such that for a Borel $B \subset D$,

$$\int_B \Phi(K(z)) dm \leq M_B < \infty.$$

For $t > 1$, set

$$E = E_t = \{z \in B: K(z) > t\}.$$

Then

$$\begin{aligned} \Phi(t)|E| &\leq \int_E \Phi(K(z)) dm \\ &= \int_E \Phi(K(z)) dm + \int_{B \setminus E} \Phi(K(z)) dm \\ &= \int_B \Phi(K(z)) dm \leq M_B. \end{aligned}$$

Hence,

$$|E_t| \leq \frac{M_B}{\Phi(t)} := \Psi(t).$$

We first survey results involving integral estimates and then results involving measure estimates.

3.6.1. Pesin [Pe] established the existence of a μ -homeomorphism f in the unit disk Δ when K is p exponentially integrable, i.e.,

$$\int_{\Delta} e^{[K(z)]^p} dx dy < \infty$$

for some $p > 1$. He also proved that under this condition

$$f \in \bigcup_{p < 2} W_{\text{loc}}^{1,p}(\Delta) \quad \text{and} \quad f^{-1} \in W_{\text{loc}}^{1,2}(\Delta).$$

3.6.2. Miklyukov and Suvorov [MiSu] obtained the existence of a μ -homeomorphism $f: \Delta \rightarrow \mathbb{C}$ when

$$K(z) \leq K_0 + Q(z)$$

for some positive constant K_0 and a function $Q(z) \in W_0^{1,2}$. They also showed that f is unique and $f^{-1} \in W_{\text{loc}}^{1,2}(f(\Delta))$. Recently Martio and Miklyukov [MaMi] generalized this result to the case where $K(z)$ is dominated by some function $Q(z)$ which belongs to $W_{\text{loc}}^{1,2}(\Delta)$.

3.6.3. Lehto [Le1,Le2] considered the case where μ is locally bounded in $\mathbb{C} \setminus E$ for some compact set E in D with $m_2(E) = 0$, and defined for $z \in \mathbb{C}$ and $r > 0$

$$\psi_\mu(z, r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|1 - e^{-2i\theta} \mu(z + re^{i\theta})|^2}{1 - |\mu(z + re^{i\theta})|^2} d\theta.$$

By using results of Andreian Cazacu [And] and Reich and Walczak [ReWa], Lehto showed in [Le1] that if, for all z in D and $0 < r_1 < r_2$,

$$I(r_1, r_2, r) = \int_{r_1}^{r_2} \frac{dr}{r \psi_\mu(z, r)} \rightarrow \infty \quad \text{as } r_1 \rightarrow 0 \text{ or as } r_2 \rightarrow \infty, \quad (\text{L})$$

then there exists a homeomorphism $f: \mathbb{C} \rightarrow \mathbb{C}$ such that for every relatively compact domain U in $\mathbb{C} \setminus E$, $f|_U$ is a solution of B .

Cima and Derrick [CiDe] showed that (L) is not a necessary condition for existence.

3.6.4. Brakalova and Jenkins [BrJe] modified Pesin's condition and proved that given a measurable function μ in $\mathbb{C}$ with $\|\mu\|_\infty \leq 1$, a μ -homeomorphism f of $\mathbb{C}$ onto itself with $f(0) = 0$ and $f(1) = 1$ exists if the following two conditions hold,

$$\iint_B \exp \frac{1}{1 - |\mu|} dA < \Phi_B \quad (\text{BJ1})$$

for every bounded measurable set B in D , where Φ_B is a constant which depends on B , and

$$\iint_{\{|z| < R\} \cap D} \frac{1}{1 - |\mu|} dA = O(R^2), \quad R \rightarrow \infty. \quad (\text{BJ2})$$

They also showed that $f \in \bigcup_{1 \leq p < 2} W_{\text{loc}}^{1,p}(\mathbb{C})$ and that f generates the set of all $W_{\text{loc}}^{1,2}$ homeomorphic solutions. The proof is based on estimates by Reich and Walczak on the change of the modulus of an annulus under a quasiconformal mapping and on an approximation theorem.

Iwaniec and Martin [IwMa1,IwMa2] studied wider classes of μ 's with $K(z)$ satisfying exponentially and subexponentially integrability condition. They introduced the term *principal solution*. A homeomorphism $h: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is called a *principal solution* if there is a discrete set E such that $h \in W_{\text{loc}}^{1,1}(\mathbb{C} \setminus E)$, h satisfies (B) a.e., and at ∞ , h has the normalization

$$h(z) = z + o(1).$$

THEOREM 3.12 [IwMa1,IwMa2]. *There exists a number $p_0 > 1$ such that, if*

$$|\mu(z)| \leq \frac{K(z) - 1}{K(z) + 1} \chi_\Delta$$

and

$$e^K \in L^p(\Delta)$$

for $p \geq p_0$, then (B) has a unique principal solution $h \in z + W_{\text{loc}}^{1,2}(\mathbb{C})$.

THEOREM 3.13 [IwMa1, IwMa2]. *There exists a number $p_* \geq 1$ such that, if*

$$\frac{K(z)}{1 + \log K(z)} \in L^p(\Delta)$$

for $p \geq p_*$, then (B) has a unique principal solution $h \in z + W_{\text{loc}}^{1,Q}(\mathbb{C})$ with Orlicz function $Q(t) = t^2 \log^{-1}(e + t)$.

The proofs of these results are based on a monotonicity principle in the Orlicz–Sobolev classes and deep results in harmonic analysis.

For uniqueness of a principal solution in $z + W_{\text{loc}}^{1,Q}(\mathbb{C})$, Iwaniec and Martin [IwMa1, IwMa2] look at $f = h_1 - h_2$, where h_1 and h_2 are assumed to be principal solutions of (B) in $W_{\text{loc}}^{1,Q}(\mathbb{C})$. Then f is of bounded distortion, satisfies (B) and its differential Df is in L^p . By a Liouville theorem, which they establish for this class of mappings, $f = \text{const}$, and in view of the normalization of h_1 and h_2 , $f = 0$.

3.6.5. By using [ReWa], Gutlyanskii, Martio, Sugawa and Vuorinen [GuMaSuVu] provided conditions for the existence of a $W_{\text{loc}}^{1,1}$ μ -homeomorphism where both $|\mu|$ and $\arg \mu$ are involved.

They also studied the properties of μ -homeomorphisms of $\overline{\mathbb{C}}$ in terms of what they call *angular dilatation* D_{μ, z_0} and *dominating factor* which are defined as follows.

Given $z_0 \in D$, the *angular dilatation* of μ is defined by

$$D_{\mu, z_0} = \frac{|1 - \mu(z) \frac{\bar{z} - \bar{z}_0}{z - z_0}|^2}{1 - |\mu(z)|^2} \quad \text{if } z \in \mathbb{C} \quad \text{and} \quad D_{\mu, z_0}(z) = D_{\mu, 0}(z) \quad \text{if } z_0 = \infty.$$

A function $H : [0, +\infty) \rightarrow \mathbb{R}$ is called a *dominating factor* if the following conditions are satisfied:

- (i) $H(x)$ is continuous and strictly increasing in $[x_0, +\infty)$ and $H(x) = H(x_0)$, for $x \in [0, x_0]$ for some $x_0 \geq 0$, and
- (ii) the function $e^{H(x)}$ is convex in $x \in [0, +\infty)$.

A dominating factor H is said to be of *divergence type* if

$$\int_1^\infty \frac{H(x)}{x^2} dx = +\infty.$$

THEOREM 3.14 [GuMaSuVu]. *Suppose that μ is a measurable function in $\mathbb{C}$ with $|\mu(z)| < 1$ a.e. such that*

- (1) $K \in L^p_{\text{loc}}(\mathbb{C})$ for some $p > 1$; and
 (2) for each point $z_0 \in \mathbb{C}$ either $D_{\mu, z_0} \leq M$ a.e. in the disk $B(z, r_0) = \{|z - z_0| < r_0\}$ or

$$\int_{B(z_0, r_0)} e^{H(D_{\mu, z_0}(z))} dm(z) \leq M \quad \text{if } z_0 \in \mathbb{C}$$

and

$$\int_{|z| > r_0} e^{H(D_{\mu, z_0}(z))} \frac{dm(z)}{|z|^4} \leq M \quad \text{if } z_0 = \infty$$

for some dominating factor $H = H_{z_0}$ of divergence type and positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$.

Then there exists a normalized homeomorphic solution $f: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ of (B) such that $f \in W^{1,q}_{\text{loc}}(\mathbb{C})$, $q = 2p/(1+p)$, and $f^{-1} \in W^{1,2}_{\text{loc}}$.

The following theorem may be viewed as an extended version of the theorems of [BrJe] and [IwMa1, IwMa2].

THEOREM 3.15 [GuMaSuVu]. *Let H be a dominating factor of divergence type. Suppose that μ is a measurable function in $\mathbb{C}$ with $\|\mu\|_{\infty} \leq 1$ which satisfies*

$$\int_{\mathbb{C}} e^{H(K(z))} d\sigma < +\infty,$$

where $d\sigma = (1 + |z|^2)^{-2} dA(z)$ denotes the spherical area element.

Then there exists a normalized solution f of (B) such that $f \in \bigcap_{1 \leq q < 2} W^{1,q}_{\text{loc}}(\mathbb{C})$ and that $f^{-1} \in W^{1,2}_{\text{loc}}(\mathbb{C})$.

3.6.6. David [Da] considered ACL embeddings $f: D \rightarrow \mathbb{C}$ with a complex dilatation μ which satisfies the exponential condition

$$|\{z \in D: |\mu(z)| > 1 - \varepsilon\}| \leq C e^{-d/\varepsilon} \quad (\text{Da})$$

for all $\varepsilon \in (0, \varepsilon_0]$, for some $\varepsilon_0 \in (0, 1]$, $C > 0$ and $d > 0$.

One of the main results in [Da, p. 27], says that if μ is measurable in $\mathbb{C}$, and satisfies the exponential condition (Da), then the Beltrami equation has a homeomorphic solution $f^{\mu}: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, which fixes the points 0, 1 and ∞ . Furthermore, the mapping f^{μ} belongs to $W^{1,s}_{\text{loc}}$ for all $s < 2$. He also proves that if f is another homeomorphic solution and if $f \in W^{1,1}_{\text{loc}}(\mathbb{C})$, then $f = h \circ f^{\mu}$ for some conformal mapping h . David's proof of the existence goes along the lines of Ahlfors [Ahl5] and Bojarski [Bo1] involving a very fine analysis of singular integrals.

3.6.7. In Tukia [Tu], (Da) is replaced by the spherical exponential condition

$$\sigma\{z \in D: K(z) > t\} \leq C e^{-\alpha t} \quad (\text{Tu})$$

for all $t \geq T$, for some $T \geq 0$, $C > 0$ and $\alpha > 0$ and domains D in $\overline{\mathbb{C}}$. Here $\sigma\{\dots\}$ denotes the spherical area of $\{\dots\}$. Tukia proves the existence of a global solution by using David's local solutions and the Uniformization Theorem. The latter is possible by David's uniqueness theorem. Tukia studies also convergence and compactness properties.

3.6.8. Ryazanov, Srebro and Yakubov showed in [RaSrYa1] that if

$$K(z) \leq Q(z) \quad (\text{RSY})$$

for some function $Q \in BMO_{\text{loc}}(D)$, then a μ -homeomorphism $f: D \rightarrow \mathbb{C}$ exists.

Following [RaSrYa1], an ACL sense-preserving homeomorphism $f: D \rightarrow \mathbb{C}$ is called a *BMO- qc mapping* if $K(z) \leq Q(z)$ a.e. in D for some function Q in $BMO(D)$. *BMO $_{\text{loc}}$ - qc mappings* are defined in a similar way. Besides existence, [RaSrYa1] contains a study of *BMO- qc mappings* like removable singularities, reflection principle, boundary behavior and mapping properties. Sastry [Sa] provided necessary and sufficient conditions for a *BMO- qc extension* from $\mathbb{R}$ to the half plane $\mathbb{H}$. Her conditions are of a form similar to Beurling–Ahlfors' where the upper and lower bounds depend on the BMO norm and certain averages of a given *BMO* boundary function, respectively, of a BMO majorant Q of a given *BMO- qc map* of $\mathbb{H}$.

3.7. BMO functions

Recall that a function $Q: D \rightarrow [1, \infty] \in BMO(D)$ if $Q \in L^1_{\text{loc}}(D)$ and

$$\|Q\|_* := \sup_B \frac{1}{|B|} \int_B |Q(z) - Q_B| dx dy < \infty,$$

where sup is taken on all disks B in D , and

$$Q_B = \frac{1}{|B|} \int_B Q(z) dx dy.$$

A function $Q: D \rightarrow [1, \infty] \in BMO_{\text{loc}}(D)$ if $Q \in BMO(U)$ for all relatively compact subdomain U of D .

It is known that

$$L^\infty(D) \subset BMO_{\text{loc}}(D) \subset \bigcap_{1 \leq p < \infty} L^p_{\text{loc}}(D).$$

BMO functions were introduced by John and Nirenberg [JoNi] and are related in various ways to quasiconformal maps, see, e.g., [As4, AsGe, BaeMa, Jon, Rei] and [ReiRy]. In particular, the class of BMO functions is quasiconformally, and, hence, conformally invariant.

This property allows to define *BMO- qc* mappings between Riemann surfaces; see [And] and [AndSt].

The following lemma by John and Nirenberg [JoNi] plays an important role in analysis and its applications. Note that by this lemma condition (RSY) in Section 3.6.8 is locally equivalent to condition (Da) and (Tu) in Sections 3.6.6 and 3.6.7, respectively.

LEMMA 3.16 [JoNi]. *If u is a nonconstant function in $BMO(D)$, then*

$$\left| \{z \in B: |u(z) - u_B| > t\} \right| \leq a e^{-\frac{b}{\|u\|_*} t} |B|$$

for every disk B in D and all $t > 0$, where a and b are absolute positive constants which do not depend on B and u . Conversely, if $u \in L^1_{\text{loc}}$ and if for every disk B in D and for all $t > 0$

$$\left| \{z \in B: |u(z) - u_B| > t\} \right| < a e^{-bt} |B|$$

for some positive constants a and b , then $u \in BMO(D)$.

The existence of a homeomorphic solution in [RaSrYa1] is proved by the aid of the Approximation Theorem 3.2 and two distortion theorems. The main tool in the proof of the distortion theorems is the following modulus inequality. This inequality serves also as the main tool in proving various properties of *BMO- qc* mappings.

3.8. Modulus inequality

Let Γ be a path family in D . A Borel function $\rho: \mathbb{C} \rightarrow [0, \infty]$ is *admissible* for Γ , $\rho \in \text{adm } \Gamma$, if

$$\int_{\gamma} \rho \, ds \geq 1$$

for every locally rectifiable path γ in Γ .

The *conformal modulus* $M(\Gamma)$ of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \iint_{\mathbb{C}} \rho^2 \, dx \, dy.$$

The (conformal) modulus is a conformal invariant and plays an important role in the study of conformal, *qc* and *qr* mappings; see [Je,Vas,LeVi,Ahl5,Vä1,Vu,Ri].

Let $f: D \rightarrow \overline{\mathbb{C}}$ be a *qc* mapping which is $Q(z)$ -*qc* for a given function Q in L^1_{loc} , Γ is a path family in D and $\rho \in \text{adm } \Gamma$. Then, cf. [LeVi, V(6.6)],

$$M(f\Gamma) \leq \iint_{\mathbb{C}} Q(z) \rho^2(z) \, dx \, dy. \quad (3.3)$$

The modulus inequality (3.3) was extended and applied in [MaRySrYa1] to sense preserving homeomorphisms $f: D \rightarrow \mathbb{R}^n$, $n \geq 2$, in $W_{\text{loc}}^{1,n}(D)$ with $K \in L_{\text{loc}}^{n-1}$; see also [MaRySrYa2]. In [MaRySrYa3] the modulus inequality was generalized to noninjective mappings. This inequality corresponds to the K_I modulus inequality, see [Ri], which plays an important role in the quasiregular theory. A modulus inequality which corresponds to the K_O modulus inequality, see [Ri], is also presented in [MaRySrYa3]. It is shown in [MaRySrYa3] that these two types of inequalities hold in the class of all mappings of finite length distortion. This class contains as a subclass all quasiconformal mappings, [MaRySrYa3]. Later these and similar inequalities were extended in [KoOn] to mappings of bounded distortion in $\mathbb{R}^n$ with K satisfying certain integrability conditions. These inequalities are used in [KoOn] and in a series of papers in establishing the basic properties of mappings of bounded distortion in all dimensions, as openness, discreteness, Lusin's condition (N) and more.

The following lemma in the BMO theory, which is established in [RaSrYa1], plays a crucial role in proving equicontinuity. The lemma including its proof extends to BMO functions in higher dimensions, and is used in the study of BMO - qc mappings in higher dimensions, cf. [MaRySrYa1] and [MaRySrYa2].

LEMMA 3.17 [RaSrYa1]. *Let Q be a nonnegative BMO function in Δ . For $0 < t < e^{-2}$, let $A(t) = \{z: t < |z| < e^{-1}\}$. Then*

$$\iint_{A(t)} \frac{Q(z) dx dy}{|z|^2 (\log |z|)^2} \leq c \log \log 1/t,$$

where c is a constant which depends only on the average Q_1 of Q over the disk $|z| < e^{-1}$ and on the BMO norm Q_* of Q in Δ .

3.9. The subclasses **Da**, **Tu**, **BJ** and **RSY**

For a domain D in $\mathbb{C}$, let **Da**(D), **Tu**(D), **BJ**(D) and **RSY**(D) denote the classes of all ACL sense-preserving homeomorphisms $f: D \rightarrow \overline{\mathbb{C}}$ which satisfy a.e. in D the conditions (Da), (Tu), (BJ1)–(BJ2) and (RSY), respectively. Clearly, **Da**(D) $\subset$ **Tu**(D) with equality when D is bounded. By [BrJe, p. 89], **Da**(D) $\subset$ **BJ**(D).

In case that D is a quasidisk then **Tu**(D) = **RSY**(D), otherwise **Tu**(D), and hence, **Da**(D), are proper subclasses of **RSY**(D). Each of the classes **BJ**(D) and **RSY**(D) contains functions which do not belong to the other class. It should be noted that classes **Da**(D), **Tu**(D) and **RSY**(D) are qc invariant, while **BJ**(D) is not.

3.10. FMO functions

Recently Ignatiev and Ryazanov [IgRy] introduced the following class of functions called functions of finite mean oscillation (FMO).

Let $Q : D \rightarrow [1, \infty] \in L^1_{\text{loc}}(D)$, $z_0 \in D$. We say that Q is of *finite mean oscillation at z_0* , $Q \in FMO(z_0)$, if

$$d_Q(z_0) := \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|D(z_0, \varepsilon)|} \int_{D(z_0, \varepsilon)} |Q(z) - Q_{D(z_0, \varepsilon)}| dx dy < \infty,$$

where $D(z_0, \varepsilon) = \{z : |z - z_0| < \varepsilon\}$, and that $Q \in FMO(D)$ if $Q \in FMO(z)$ for all $z \in D$.

1. As in the BMO theory, the following condition

$$\overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{|D(z_0, \varepsilon)|} \int_{D(z_0, \varepsilon)} Q(z) dx dy < \infty, \quad (3.4)$$

implies that $d_Q(z_0) < \infty$, and consequently that $Q \in FMO(z_0)$.

2. Obviously, $BMO(D) \subset BMO_{\text{loc}}(D) \subset FMO(D)$ and $BMO(D) \neq BMO_{\text{loc}}(D)$. Also $BMO_{\text{loc}}(D) \neq FMO(D)$ as shown in the following example.

3.11. Example of a function $Q \in FMO \setminus BMO_{\text{loc}}$

Fix $p > 1$. Set $z_n = 2^{-n}$, $r_n = 2^{-pn^2}$, $c_n = 2^{2n^2}$ and $D_n = \{z : |z - z_n| < r_n\}$, and let

$$Q(z) = \sum_{n=1}^{\infty} c_n \chi(D_n).$$

Obviously, by (3.4), $Q \in BMO_{\text{loc}}(\mathbb{C} \setminus \{0\})$, and hence, $Q \in FMO(\mathbb{C} \setminus \{0\})$.

For $Q \in FMO(0)$, fix N such that $(p-1)N > 1$, and let $\varepsilon = \varepsilon_N = z_N + r_N$. Then $\bigcup_{n \geq N} D_n \subset \Delta(\varepsilon) := \{|z| < \varepsilon\}$ and

$$\begin{aligned} \int_{\Delta(\varepsilon)} Q &= \sum_{n \geq N} \int_{D_n} Q = \pi \sum_{n \geq N} c_n r_n^2 \\ &= \sum_{n \geq N} 2^{2(1-p)n^2} < \sum_{n \geq N} 2^{2(1-p)n} \\ &< C[2^{2(1-p)N}]^2 < 2C\varepsilon^2. \end{aligned}$$

Hence $Q \in FMO(0)$, and consequently $Q \in FMO(\mathbb{C})$.

On the other hand,

$$\int_{\Delta(\varepsilon)} Q^p = \pi \sum_{n > N} c_n^p r_n^2 = \sum_{n > N} 1 = \infty.$$

Hence, $Q \notin L^p(\Delta(\varepsilon))$, $\varepsilon > 0$, and therefore $Q \notin BMO_{\text{loc}}$.

THEOREM 3.18 [RaSrYa2]. *If, for some $Q \in FMO(D)$,*

$$K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \leq Q(z),$$

then (B) has a homeomorphic solution $f : D \rightarrow \mathbb{C}$ such that

- (i) $f^{-1} \in W_{\text{loc}}^{1,2}(D)$;
- (ii) f generates the set of all $W_{\text{loc}}^{1,2}$ solutions.

As in the BMO case the proof is based on the modulus inequality (3.3) and the following lemma which generalizes Lemma 3.17.

LEMMA 3.19. *Suppose that $Q : \Delta \rightarrow [1, \infty] \in FMO(0)$ and that Q is integrable in $\Delta(e^{-1}) \subset D$. Then, for $\varepsilon \in (0, e^{-e})$,*

$$\int_{\varepsilon < |z| < e^{-1}} \frac{Q(z) dx dy}{(|z| \log |z|)^2} \leq C \log \log \frac{1}{\varepsilon},$$

where C is a constant which depends only on the mean value of Q over $\Delta(e^{-1})$ and on $d_Q(0)$; see Section 3.10.

4. Alternating Beltrami equation

4.1. Introduction

Roughly speaking, this term refers to equation (B) in the case where μ is a measurable complex valued function in a domain D in $\mathbb{C}$ with $|\mu| < 1$ a.e. in parts of D and $|\mu| > 1$ a.e. in other parts of D . With further assumptions on μ , every nonconstant solution $f : D \rightarrow \mathbb{C}$ of (B), i.e. an ACL mappings which satisfies (B) a.e., is locally quasiregular (and in particular open, discrete and sense-preserving) in the regions where $|\mu| < 1$, and anti-quasiregular (and in particular open, discrete and sense-reversing) in the regions where $|\mu| > 1$. These mappings may branch at some isolated points in D or may have folds and cusps. This leads to the notation and study of, what we call, *folded quasiregular maps* (FQR-maps) and *branched folded maps* (BF-maps).

A first systematic study of the alternating Beltrami equations which started in [SrYa1], and continued in [SrYa3, SrYa4], revealed new phenomena which may have applications in other areas. Earlier results in this subject can be found in [Dz, Ya] and [Abd].

4.2. The geometric configuration and the conditions on μ

Let E be a one-dimensional set in D with $m_2(E) = 0$ such that (D, E) is a locally finite two-color map painted black and white. The local assumption on E says that every point z of E has a neighborhood U such that $E \cap U$ is an arc or a (finite) *star* with a *vertex* at z , i.e.,

a union of finitely many arcs, each connecting the point z with ∂U and which are disjoint except for their common end point z . In sequel we will assume that $\mu : D \rightarrow \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a measurable function such that $|\mu| < 1$ a.e. in the white regions of $D \setminus E$ and $1/|\mu| < 1$ a.e. in the black regions of $D \setminus E$.

4.3. A symmetric form of the alternating Beltrami equation

In order to include the case $w_z = 0$, we rewrite equation (B), see Section 1.1, in a symmetric form:

$$A(z)w_z + B(z)w_{\bar{z}} = 0, \quad (\tilde{B})$$

where $A(z)$ and $B(z)$ are measurable functions in D and $\mu = -A/B$ at points where $B \neq 0$. Then $|A(z)| < |B(z)|$ a.e. in the white regions and $|A(z)| > |B(z)|$ a.e. in the black regions. With no loss of generality, we may use (B) instead of $(\tilde{B})$, agreeing that $\mu = \infty$ corresponds to $B = 0$ in $(\tilde{B})$. We will assume that at every point of $D \setminus E$ either $|A/B|$ or $|B/A|$ is locally bounded, implying that $(\tilde{B})$ is locally strongly elliptic in $D \setminus E$. Note, that A , B and μ may be continuous on E . We will be mainly interested in solutions which are branched folded mappings in D . These mappings are described in the following sections.

4.4. Branched folded maps

The notion of a *branched folded covering* (BF-covering) was introduced by Tucker, [Tuc]. However, the class of BF-maps and related classes of maps were studied earlier by Hopf [Ho] and others, and in the last thirty years by Church [Ch], Church and Timourian [ChTi], Väisälä, [Vä2] and others. More recently, these mappings have appeared in connection with problems in Graph Theory, cf. Bouchet [Bou], in the study of light harmonic functions, Lyzzaik [Ly], in the study of nonlinear operators in Banach spaces and elliptic boundary value problems, Berger, Church, Dancer and Timourian [ChDaTi] and [BeChTi], and in some aspects of Singularity Theory, Arnold [Ar]. Here BF-maps are defined in a wider sense which is more appropriate for the study of alternating Beltrami equation where smoothness is not assumed.

4.5. Definition of BF-maps

Let X and Y be oriented surfaces. A map $f : X \rightarrow Y$ is called a *branched folded map* (BF-map) if f is *discrete*, i.e., the points of $f^{-1}(y)$ are isolated in X for any y in Y , and if there exists a one-dimensional set E in X such that (X, E) is a locally finite two-color map painted black and white such that f is open and sense-preserving in the white regions and open and sense-reversing in the black regions.

BF-maps according to the classical definition, simplicial maps, open discrete maps, nonconstant quasiregular maps and nonconstant holomorphic maps form subclasses of

BF-maps. A BF-map in the classical definition is locally simplicial, and its critical set looks like the critical set of a BF-map in our definition. The critical set of a classical BF-map and its image look alike. This need not be the case for a BF-map in our definition.

4.6. Discrete maps, canonical maps and classification of critical points

Let $f : X \rightarrow Y$ be a discrete map, where X and Y are oriented surfaces, and let Σ_f denote the *critical set* of f , i.e., the set of points where f is not a local homeomorphism. The critical points of f , i.e., points of Σ_f , are classified into three categories: *branch points*, *folding points* and *cuspidal points*.

A critical point is called a *branch point* if it is isolated in Σ_f . A critical point p will be called a *folding point* if p has a neighborhood U such that $f|U$ is topologically equivalent to the *standard folding map* $(x, y) \rightarrow (x, |y|)$ of R^2 into itself, where p corresponds to the point $(0, 0)$. A critical point which is neither a branch point nor a folding point will be called a *cuspidal point*. The behavior of a map near a cuspidal point is not well understood. Only some special cuspidal points will be classified and described here.

4.7. Branch points, power maps and winding maps

If p is a branch point for a discrete map $f : X \rightarrow Y$, where X and Y are as above, then p has a neighborhood in which f is topologically equivalent to a power map $z \rightarrow z^d$ of $\mathbb{C}$ for some integer $d = d(p) > 1$ where the point p corresponds to the point $z = 0$ in the map z^d . This result follows from the fact that f must be open in some neighborhood of p , cf. [MaSr, Theorem 2.10], and by a theorem of Stoilow, cf. [St] or [Why]. Note, that the power map $z \rightarrow z^d$ is topologically equivalent to a *winding map* g_d of order d , which is defined by

$$g_d(re^{it}) = re^{idt}, \quad d \in \mathbb{Z}_+.$$

4.8. Folding maps

A map $f : X \rightarrow Y$, X and Y as above, is a *proper folding* if f is topologically equivalent to the standard folding $(x, y) \rightarrow (x, |y|)$ of R^2 . In this case X and Y must be simply connected, the critical set Σ_f is a line, called a *folding line*, and every point of Σ_f is a folding point. A *folding map* is defined as a restriction of a proper folding map g to a domain, say U , such that $U \cap \Sigma_g$ is connected.

One may consider folding maps in the following wider sense: $f : X \rightarrow Y$ is a *folding map* along a *folding line* l , if l is a Jordan arc or a Jordan curve in X , $X \setminus l$ has two components X_1 and X_2 , $f|X_1 \cup l$ is a sense-preserving embedding and $f|X_2 \cup l$ is a sense-reversing embedding. If $X \subset \mathbb{C}$ and $Y = \mathbb{C}$, the definitions agree.

4.9. (p, q) -cusp maps and (p, q) -cusp points

The simplest *cusp point* appears as the point $(0, 0)$ in *Whitney's canonical cusp mapping*

$$(x, y) \rightarrow (x, xy - y^3).$$

This map is topologically equivalent to the so-called *straight $(3, 1)$ -cusp map* $f_{3,1}$ of $\mathbb{C}$ onto itself which is defined by $f_{3,1}(z) = z^3/|z|^2$ for $\text{Im } z > 0$ and $f_{3,1}(z) = \bar{z}$ for $\text{Im } z < 0$. Similarly, the *straight (p, q) -cusp map* $f_{p,q}$ of $\mathbb{C}$ is defined for any positive integers p and q with $p \equiv q \pmod{2}$ by $z^p/|z|^{p-1}$ if $\text{Im } z > 0$, and $\bar{z}^q/|z|^{q-1}$ if $\text{Im } z < 0$. Then the critical set of $f_{p,q}$ is the real axis R , the point $z = 0$ is a (p, q) -cusp point and every other point on the real axis is a folding point. Also, $f_{p,q}(x) = x$ for $x \in R$ if p and q are odd and $f_{p,q}(x) = |x|$ for $x \in R$ if p and q are even. A map $f: D \rightarrow \mathbb{C}$ is called a *proper (p, q) -cusp map* if it is topologically equivalent to the straight (p, q) -cusp map, and the point p in D which corresponds to $z = 0$ is a (p, q) -cusp point. The restriction of f to a sub-domain of D which contains the cusp point is called a (p, q) -cusp map. Some of these (p, q) -cusp maps are considered in Lyzzaik [Ly] in connection with the local behavior of light harmonic maps.

4.10. Umbrella and simple umbrella maps

Next, we describe a wider class of cusp points. These are associated with what are called *umbrella maps* and *simple umbrella maps*. Mappings of this type are considered in [KhYa] and in [Ly]. A map $f: \mathbb{C} \rightarrow \mathbb{C}$ is called a *simple straight umbrella map* if $f(0) = 0$ and $f(z) = |z|\varphi(z/|z|)$ for $z \neq 0$, for some continuous piece-wise injective function $\varphi: S^1 \rightarrow S^1$. A mapping $f: X \rightarrow Y$ is called a *proper simple umbrella map*, if f is topologically equivalent to a simple straight umbrella map. The restriction of a proper simple umbrella map $f: X \rightarrow Y$ to a domain D in X , is called a *simple umbrella map*.

A mapping $f: \mathbb{C} \rightarrow \mathbb{C}$ is called a *straight umbrella map* if $f(0) = 0$ and $f(z) = t\gamma_t(z/|z|)$ for $|z| = t > 0$ for some path $\gamma: [0, \infty) \rightarrow F_d$ in the space F_d of all continuous and piecewise injective maps of S^1 into itself of a given degree d , which has the property that all maps $\gamma_t = \gamma(t): S^1 \rightarrow S^1$, $t \in [0, \infty)$, have the same critical points on S^1 . A mapping $f: X \rightarrow Y$ is called a *proper umbrella map*, if f is topologically equivalent to a straight umbrella map. The restriction of f to a domain in X is called an *umbrella map*.

Note that the critical set Σ_f of a proper umbrella map f is either empty, consists of a single point, or is a star. In the latter case the vertex of the star will be called also the *vertex of the umbrella map*, and it is then either a folding point or a cusp point. Also, note that Σ_f may be a star while its image need not be a star. If, however, f is a proper simple umbrella map, and if its critical set is a star, so is its image. One can show that proper umbrella maps and proper (p, q) -cusp maps are proper in the usual sense, and that the restriction of umbrella (resp. simple umbrella) maps to certain subdomains are proper umbrella (resp. simple umbrella) maps.

Locally, a discrete map or even a BF-map $f: D \rightarrow \mathbb{C}$ need not be equivalent to any of the canonical maps mentioned above. However, one can see that if E is a free arc in the

critical set Σ_f of a BF-map $f: D \rightarrow \mathbb{C}$, and if $f(E)$ is contained in an arc, then every point z of E has a neighborhood U such that $U \subset D$ and such that $f|U$ is either a proper folding map or a proper (p, q) -cusp map for some odd integers p and q which depend on the point z . As for the local behavior of a BF-map $f: D \rightarrow \mathbb{C}$ at a vertex z_0 of a star E in the critical set Σ_f of f , one can show that if $f(E)$ is included in a star and the image of each arm of E is included in an arm of $f(E)$ and if $f|E$ is piecewise injective, then the point z_0 as a neighborhood U such that $f|U$ is a proper simple umbrella map.

4.11. Alternating Beltrami equations and FQR-maps

Let D , E and μ be as in Section 4.2. A BF-map $f: D \rightarrow \mathbb{C}$ will be called a *folded quasiregular* map (FQR-map) if f is locally quasiregular or anti-quasiregular at every non-critical point and at every isolated critical point. It follows from the definition of FQR-maps that every FQR-map satisfies an alternating Beltrami equation (B) with μ which is locally bounded in the white regions and such that $1/\mu$ is locally bounded in the black regions. And conversely, a solution of an alternating Beltrami equation with a μ satisfying these assumption is an FQR-map if it is discrete.

One property is clearly shared by classical Beltrami equations and alternating ones, namely, if f is a nonconstant solution of (B) in a domain D and if h is analytic in $f(D)$, then $h \circ f$ is a solution too.

In the classical case, every homeomorphic solution is unique up to a conformal mapping and generates the set of all elementary solutions, and thus may be considered as a prime solution. It is not known, however, whether every alternating Beltrami equation has what are called *prime solutions*, i.e., solutions f which cannot be written as a composition $h \circ g$ where h is a noninjective analytic function, and whether the set of all BF solutions is generated by prime solutions.

4.12. Proper folding, cusp and umbrella solutions

A function $f: D \rightarrow \mathbb{C}$ is called a *proper folding solution* (resp. a *proper (p, q) -cusp* or a *proper umbrella*) if f is a solution to (B) in D and f is a proper folding mapping (resp. a proper (p, q) -cusp or a proper umbrella map). One of the main results in [SrYa1] asserts that every solution which belongs to one of the following families of solutions is prime and generates the set of all BF solutions:

- (a) proper folding solutions,
- (b) proper (p, q) -cusp solutions with $|p - q| = 2$,
- (c) proper umbrella solutions of degree 1 or -1 .

These results may be viewed as uniqueness and representation theorem, and are the counter parts of the classical theory, Theorem 2.1 and Corollary 2.2, and imply a strong rigidity property, namely, if f is a solution of a Beltrami equation (B) of any type mentioned above in (a)–(c), then every other solution of (B) with the same μ , preserves the equivalence relation $z_1 \sim z_2$ if $f(z_1) = f(z_2)$, which is determined by f . This phenomenon does not occur in the classical case where the prime solutions are homeomorphisms which determine only

the trivial equivalence. The rigidity property implies that an alternating Beltrami equation cannot have solutions of different types, as for instance, prime (p, q) -cusp solutions with different pairs (p, q) . Furthermore, one can show that if f is a proper (p, q) -cusp solution with $|p - q| = 2d > 2$ or a proper umbrella solution with $|\deg f| = d > 1$, then B has a prime solution g which is a proper (r, s) -cusp solution with $|r - s| = 2$, or, respectively, a prime umbrella solution of degree 1 or -1 , such that $f(z) = (g(z))^d$.

THEOREM 4.1. *Let D be a simply connected domain in $\mathbb{C}$, E a cross-cut in D with $m_2(E) = 0$, D_1 and D_2 the two connected components of $D \setminus E$ and $\mu: D \rightarrow \mathbb{C} \cup \{\infty\}$ a measurable function such μ is locally bounded in D_1 and $1/\mu$ is locally bounded in D_2 , and let $w = f(z)$ be a proper folding solution of the Beltrami equation (B).*

If $w = g(z)$ is a solution of (B) in D , then there exists a function h which is continuous in $f(D)$ and analytic in $f(D_1) = f(D_2)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.

REMARK 4.2. As a consequence of Theorem 4.1 one obtains the following result: *If (B) has a folding solution f , then any other solution of (B) will preserve the equivalence relation $z_1 \sim z_2$ if $f(z_1) = f(z_2)$.*

THEOREM 4.3. *Let D , E , D_1 , D_2 and μ be as in Theorem 4.1, and let $w = f(z)$ be a proper $(3, 1)$ -cusp solution of the Beltrami equation (B) in D . If $w = g(z)$ is a solution of (B) in D , then there exists a function h which is analytic in $f(D)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.*

COROLLARY 4.4. *Let D , E , D_1 , D_2 and μ be as in Theorem 4.1. Then (B) cannot have both folding solutions and $(3, 1)$ -cusp solutions.*

REMARK 4.5. By Theorems 4.1 and 4.3 it follows that, if a Beltrami equation (B) has a solution f which is either a homeomorphism or a proper folding or a proper $(3, 1)$ -cusp, then f generates the of all BF solution. However, not every (p, q) -cusp solution of (B) has this property as can be seen in the following example. Bellow, we will characterize all (p, q) -cusp solutions and all umbrella solutions which have the generating property.

EXAMPLE 4.6. We present here a function $\mu(z)$ for which equation (B) has a proper $(5, 1)$ -cusp solution f and a proper $(3, 1)$ -cusp solution g . Then, clearly $g \neq h \circ f$ for any analytic function h , since $\deg f = 2$ and $\deg g = 1$. This shows that f does not generate the set of all BF solutions.

Let Φ and Ψ be automorphisms of R^2 which are given in polar coordinates by $\Phi(r, \theta) = (r, \phi(\theta))$ and $\Psi(r, \theta) = (r, \psi(\theta))$, where ϕ and ψ are piecewise linear increasing self-maps of $[0, 2\pi]$ such that: $\phi([0, \pi/5]) = [0, \pi/3]$, $\phi([\pi/5, 4\pi/5]) = [\pi/3, 2\pi/3]$, $\phi([\pi/5, \pi]) = [2\pi/3, \pi]$, $\phi([\pi, 2\pi]) = [\pi, 2\pi]$ and $\psi([0, \pi]) = [0, \pi/2]$, $\psi([\pi, 4\pi/3]) = [\pi/2, \pi]$, $\psi([4\pi/3, 5\pi/3]) = [\pi, 3\pi/2]$, $\psi([5\pi/3, 2\pi]) = [3\pi/2, 2\pi]$. Recall that $f_{3,1}(z) = z^3/|z|^2$ if $\operatorname{Im} z > 0$ and $f_{3,1}(z) = \bar{z}$ if $\operatorname{Im} z \leq 0$. Then $g = \Psi \circ f_{3,1} \circ \Phi$ is a proper $(3, 1)$ -cusp solution of (B) with $\mu = g_{\bar{z}}/g_z$. Next, let $f(z) = (g(z))^2$, $z \in \mathbb{C}$.

Then f is a solution of (B) with the same μ , and f is a topologically equivalent to the straight $(5, 1)$ -cusp map $f_{5,1}$. Consequently, f is a proper $(5, 1)$ -cusp solution.

THEOREM 4.7. *Let D, E, D_1, D_2 and μ be as in Theorem 4.1 and let $w = f(z)$ be a proper $(p + 2, p)$ -cusp solution of Beltrami equation (B) in D . If $w = g(z)$ is a BF solution of (B) in D , then there exists a function h which is analytic in $f(D)$ such that $g = h \circ f$. Furthermore, $g(z_1) = g(z_2)$ whenever $f(z_1) = f(z_2)$.*

COROLLARY 4.8. *Let f be a proper (p, q) -cusp solution of (B) in a domain D . If $|p - q| = 2$, then f generates the solution set.*

THEOREM 4.9. *Let $w = f(z)$ be a proper (p, q) -cusp solution of (B) in a domain D . If $|p - q| = 2$, then f is prime.*

The following theorem shows that a proper (p, q) -cusp solution with $|p - q| > 2$ is not prime and can be factored as $h \circ g$, where g is a proper prime cusp solution and h is an analytic power map, and in particular, f is not prime.

THEOREM 4.10. *Let f be a proper (p, q) -cusp solution of (B) in a domain D with $|p - q| = 2d > 2$ such that its cusp point is at $z = 0$ and $f(0) = 0$. If $d > 1$, then exists a proper prime (n, m) -cusp solution g of (B) in D such that $f(z) = (g(z))^d, z \in D$.*

We now turn to proper umbrella solutions of degree 1 or -1 . It is shown in [SrYa1] that such a solution is prime and generates the set of all BF solutions.

THEOREM 4.11. *Let $f: D \rightarrow \mathbb{C}$ be a proper umbrella solution of equation (B) with $|\deg f| = d \geq 1$ and with a vertex at the point z_0 .*

- (i) *If $d = 1$, then f is a prime solution.*
- (ii) *If $d > 1$, then (B) has a proper prime solution $w = g(z)$ in D such that $f(z) = (g(z))^d + f(z_0)$.*

4.13. Existence of local folding solutions

Let $\mu: D \rightarrow \overline{\mathbb{C}}$ be a measurable function in the unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$ such that μ is locally bounded in $D_1 = \{z \in D: \operatorname{Im} z > 0\}$, and such that $1/\mu$ is locally bounded in $D_2 = \{z \in D: \operatorname{Im} z < 0\}$. Let $E = D \cap \mathbb{R}$.

THEOREM 4.12 ([SrYa4], existence of local folding solutions). *Let D, D_1, D_2, E and μ be as in Section 4.13. If there exist $r \in (0, 1)$, a nonnegative integer m , and a complex valued real analytic function $Q(x, y)$ in $|z| < r, z = x + iy$, with*

$$\operatorname{Re} Q(0, 0) > 0$$

such that for $|z| < r$,

$$\mu(z) = 1 - y^{2m+1} Q(x, y),$$

then (B) has a folding solution in some neighborhood of 0.

4.14. Uniformization and folds

Let μ be as in Section 4.13. Then local solutions in each of the domains D_1 and D_2 are related by conformal mappings, and by the uniqueness theorems for folding solutions, this is the case for local folding solutions. One would expect that if (B) has a local folding solution at every point $z \in D \cap \mathbb{R}$, then (B) has a global folding solution in D . It turns out that this is not the case as shown in the following example.

EXAMPLE 4.13 [SrYa3]. We will construct complex-valued measurable functions $A(z)$ and $B(z)$ in D , such that (B) has a local folding solution at every point in $D \cap \mathbb{R}$, and such that (B) has no global folding solution in D .

Let φ be a diffeomorphic map of D onto itself which maps $D \cap \mathbb{R}$ onto the arc $E = \{(x, y) \in \mathbb{R}^2: y = \lambda x^2\} \cap D$ for some constant $\lambda > 0$. Since E is an analytic arc, there is a neighborhood U of E , $U \subset D$, and a conformal map $\psi: U \rightarrow \mathbb{C}$, such that $\psi(E) \subset \mathbb{R}$. Let $k: \psi(U) \rightarrow \mathbb{R}^2$ denotes the standard folding map $k(x, y) = (x, |y|)$.

Now consider (B) with

$$A(z) = \begin{cases} \mu_\varphi(z), & z \in D_1, \\ 1, & z \in D \setminus D_1, \end{cases}$$

and

$$B(z) = \begin{cases} -1, & z \in D \setminus D_2, \\ -\bar{\mu}_\varphi(z), & z \in D_2, \end{cases}$$

where

$$\mu_\varphi(z) = \varphi_z^- / \varphi_z, \quad z \in D,$$

is the complex dilatation of φ in D .

Clearly, $A(z)$ and $B(z)$ are measurable in D , and since φ is a diffeomorphism $\mu(z) = -A(z)/B(z) = \mu_\varphi(z)$ is locally bounded in D_1 and $1/\mu(z) = -B(z)/A(z) = \bar{\mu}_\varphi(z)$ is locally bounded in D_2 . Furthermore, $k \circ \psi \circ \varphi: \varphi^{-1}(U) \rightarrow \mathbb{C}$ is a local folding solution of (B). Thus (B) satisfies the assumptions in Theorem 4.1.

We will show that for any choice of $\lambda > 2$, (B) has no global folding solution in D . Fix $\lambda > 2$ and suppose that (B) has a global folding solution $g: D \rightarrow \mathbb{C}$. With no loss of generality we may assume that $g(D \cap \mathbb{R}) \subset \mathbb{R}$ and $g(0) = 0$, since otherwise g can be

replaced by another solution $h \circ g$ for some map h which is a solution in $g(D)$, conformal in $\text{int}g(D)$, and which maps $g(D \cap \mathbb{R})$ into $\mathbb{R}$ and $g(0)$ onto 0.

Now, by “unfolding” g one obtains the homeomorphism $G : D \rightarrow \mathbb{C}$,

$$G(z) = \begin{cases} g(z), & z \in D \setminus D_2, \\ \bar{g}(z), & z \in D_2. \end{cases}$$

Then, G and φ have the same complex dilatation $\mu(z) = \mu_\varphi(z)$ in D , and hence, $f = G \circ \varphi^{-1}$ is conformal in D and $f(0) = 0$. With no loss of generality, we may assume that $f'(0) = 1$. This can be obtained by replacing g by cg for some $c > 0$. Consequently the map

$$f(z) = z + a_2 z^2 + \dots$$

belongs to the class S of all normalized univalent maps in D , and hence $|a_2| \leq 2$.

On the other hand, $f(E) \subset \mathbb{R}$, which implies

$$\text{Im } f(t + i\lambda t^2) = 0$$

for all $t \in \mathbb{R}$, with $|t|$ sufficiently small, and straightforward computations give $|a_2| > \lambda$. This contradiction shows that (B) has no global folding solutions.

The following theorem shows that for some μ 's as in Theorem 4.11, there is a local folding solution at every point of $D \cap \mathbb{R}$, yet any global solution must branch somewhere off $D \cap \mathbb{R}$.

THEOREM 4.14 [SrYa4]. *Let D , D_1 , D_2 , E be as in Section 4.11. Given a point $z_0 \in D \setminus E$, there exist μ as in Section 4.13 for which (B) has a local folding solution at every point of E , such that every global solution of (B) in D branches at z_0 , and in particular, (B) has no (global) folding solution in D .*

In view of the last theorem and the fact that local folding solution can be glued conformally, one may ask whether the existence of a global solution in D which folds along E and is locally injective off E implies that (B) has a global folding solution in D . The following theorem answers this question negatively.

THEOREM 4.15 [SrYa4]. *Let D , D_1 , D_2 , E be as in Theorem 4.1. There exists μ satisfying the condition of Section 4.13 such that (B) has a solution which folds along E and is locally injective off E , and (B) has no global folding solution in D .*

5. Open problems

5.1. Problem 1

Let $\mathcal{F}$ denote a given class of μ -homeomorphisms. We say that a closed Jordan curve J is $\mathcal{F}$ -tame if there is a μ -homeomorphism $f \in \mathcal{F}$ which maps the annulus $1/2 < |z| < 2$

into $\mathbb{C}$ such that $f(\partial\Delta) = J$. Note that J is analytically tame iff J is an analytic curve and that J is qc tame, iff J is a quasicircle. The monographs of Gehring and Hag [GeHa] have a long list of characterizations of quasicircles. It is shown in [RaSrYa1] that the class of quasicircles is a proper subclass of BMO - qc circles.

Problems:

- (i) Characterize BMO - qc circles.
- (ii) Is there a class $\mathcal{F}$ of μ -homeomorphisms containing the class of BMO - qc mappings and a Jordan curve J , such that J is $\mathcal{F}$ -tame and not a BMO - qc circle.

5.2. Problem 2

Let $\mathcal{F}$ be a given class of μ -homeomorphisms. We say that a domain D_2 is an $\mathcal{F}$ -image of a domain D_1 if there is $f \in \mathcal{F}$ which maps D_1 onto D_2 .

Problem: Find classes $\mathcal{F}_1$ and $\mathcal{F}_2$ of μ -homeomorphisms and domains D and D' such that D' is $\mathcal{F}_1$ -image of D and not an $\mathcal{F}_2$ -image.

5.3. Problem 3

Find necessary condition on μ in the unit disk Δ with $|\mu| < 1$ a.e. in $\Delta^+ = \{z \in \Delta: \operatorname{Im} z > 0\}$ and $|\mu| > 1$ a.e. in $\Delta^- = \{z \in \Delta: \operatorname{Im} z < 0\}$ such that the Beltrami equation (B) has (a) a folding solution, (b) a (p, q) -cusp solution.

5.4. Problem 4

Let μ be as in Section 5.3. Find sufficient conditions on μ in Δ such that (B) has (a) a folding solution, (b) a (p, q) -cusp solution.

5.5. Problem 5

For which μ 's with $|\mu| < 1$ a.e. and $\|\mu\|_\infty = 1$, existence of a μ -homeomorphism implies uniqueness. In other words, under what condition on μ , $g \circ f^{-1}$ is conformal whenever f and g are μ -homeomorphisms. Note that in our notation neither f nor g are assumed to be in $W_{\text{loc}}^{1,1}(D)$.

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References

- [Abd] A. Abdushukurov, *A system of Beltrami equation and corresponding inhomogeneous system*, Dokl. Akad. Nauk Tadzhik. SSR **28** (4) (1985), 187–189 (in Russian).
- [Abi] W. Abikoff, *The Real Analytic Theory of Teichmüller Spaces*, Springer-Verlag, Berlin (1980).
- [Ahl1] L. Ahlfors, *Zur Theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.
- [Ahl2] L. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. **3** (1953–1954), 1–58.
- [Ahl3] L. Ahlfors, *Conformality with respect to Riemann metrics*, Ann. Acad. Sci. Fenn. A I **205** (1955), 1–22.
- [Ahl4] L. Ahlfors, *Finite generated Kleinian groups*, Amer. J. Math. **86** (1964), 413–429.
- [Ahl5] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Math. Studies, Vol. 10, Van Nostrand, Princeton, NJ (1966).
- [Ahl6] L. Ahlfors, *Quasiconformal mappings, Teichmüller spaces, and Kleinian groups*, Proc. Internat. Congress of Mathematicians, Helsinki 1978, Acad. Sci. Fenn. (1980), 71–84.
- [Ahl7] L. Ahlfors, *Commentary on Zur Theorie der Überlagerungsflächen*, Fields Medalist's lectures World Sci. Ser. 20th Century Math., World Scientific, River Edge, NJ (1997), 8–9.
- [AhlBer] L. Ahlfors and L. Bers, *Riemann mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 385–404.
- [And] C. Andreian Cazacu, *Influence of the orientation of the characteristic ellipses on the properties of the quasiconformal mappings*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Riemann Surfaces, Bucharest (1971), 65–85.
- [AndSt] C. Andreian Cazacu and V. Stanciu, *Normal and compact families of BMO and BMO_{loc}-qc mappings*, Math. Rep. **52** (4) (2000), 407–419.
- [Ar] V.I. Arnold, *Singularities of Caustics and Wave Fronts*, Kluwer, Dordrecht (1990).
- [As4] K. Astala, *A remark on Quasi-conformal mappings and BMO-functions*, Michigan Math. J. **30** (1983), 209–212.
- [As1] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [As2] K. Astala, *Analytic aspects of quasiconformality*, Documenta Mathematica, Extra Volume ICM (1998), II, 617–626.
- [As3] K. Astala, *Recent connections and applications of planar quasiconformal mappings*, European Congress of Mathematics, Budapest 1996, Vol. 1, Progr. Math., Vol. 168, Birkhäuser, Basel (1998), 36–51.
- [AsGe] K. Astala and F.W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Anal. Math. **46** (1986), 16–57.
- [AsIwKoMa] K. Astala, T. Iwaniec, P. Koskela and G. Martin, *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), 703–726.
- [AsIwSa] K. Astala, T. Iwaniec and E. Saksman, *Beltrami operator in the plane*, Duke Math. J. **107** (1) (2001), 27–56.
- [AsMa] K. Astala and G. Martin, *Holomorphic motions*, Papers on Analysis, Rep. Dept. Math. Stat. Univ. Jyväskylä (2002), 27–40.
- [BaeMa] A. Baernstein II and J. Manfredi, *Topics in quasiconformal mappings*, Topics in Modern Harmonic Analysis, Istit. Naz. Alta Math., Roma (1983), 819–862.
- [Ba] T. Bandman, *The behavior of a quasiconformal mapping near singular points of a characteristic*, Soviet Math. Dokl. **15** (6) (1974), 1766–1769.
- [Beli] P.P. Belinskii, *General Properties of Quasiconformal Mappings*, Nauka, Novosibirsk (1974) (in Russian).
- [Belt] E. Beltrami, *Saggio di interpretazione della geometria non Euclidea*, Giorn. Math. **6** (1867), 284–312.
- [BeChTi] M.S. Berger, P.T. Church and J.G. Timourian, *Folds and cusps in Banach spaces with application to nonlinear partial differential equations, II*, Trans. Amer. Math. Soc. **307** (1988), 225–244.
- [Ber1] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Wiley, New York/Chapman & Hall, London, (1958).
- [Ber2] L. Bers, *Uniformizations by Beltrami equations*, Comm. Pure Appl. Math. **14** (1961), 213–235.

- [Ber3] L. Bers, *Uniformization, moduli and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [Ber4] L. Bers, *Quasiconformal mappings with applications to differential equations, function theory and topology*, Bull. Amer. Math. Soc. **83** (1977), 1083–1100.
- [Ber5] L. Bers, *An extremal problem for quasiconformal mappings and a problem of Thurston*, Acta Math. **141** (1978), 73–98.
- [Ber6] L. Bers, *Finite dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. **5** (1981), 131–172.
- [Bo1] B.V. Bojarski, *Generalized solutions of systems of differential equations of first order and elliptic type with discontinuous coefficients*, Mat. Sb. **43** (85) (1957), 451–503.
- [Bo2] B.V. Bojarski, *Subsonic flow of compressible fluid*, Arch. Mech. Stos. **18** (1966), 497–520.
- [Bo3] B.V. Bojarski, *Quasiconformal mappings and general structure properties of systems of non-linear equations elliptic in the sense of Lavrent'ev*, Convegno sulle Trasformazioni Quasiconformi e Questioni Connesse, Rome 1974, Sympos. Math., Vol. 18, Academic Press, London (1976), 485–499.
- [BoGu] B.V. Bojarski and V.Ya. Gutlyanskii, *On the Beltrami equation*, Proc. Internat. Conf. Complex Analysis at the Nankai Institute of Mathematics, 1992; Conf. Proc. Lecture Notes Anal., Vol. I, Internat. Press, Cambridge, MA (1994), 8–33.
- [BoIw] B.V. Bojarski and T. Iwaniec, *Quasiconformal mappings and non-linear elliptic equations in two variables, I, II*, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. **22** (1974), 473–478.
- [Bou] A. Bouchet, *Constructions of covering triangulations with folds*, J. Graph Theory **6** (1982), 57–74.
- [BrJe] M.A. Brakalova and J.A. Jenkins, *On solutions of the Beltrami equation*, J. Anal. Math. **76** (1998), 67–92.
- [BshHe] D. Bshouty and W. Hengartner, *Harmonic mappings in the plane*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), 479–506 (this Volume).
- [Ca] P. Caraman, *n-Dimensional Quasiconformal Mappings*, Ed. Acad. Romane and Abacus Press, Tunbridge Wells, Kent, England (1974).
- [Che] Zh. Chen, *Compactness for (z)-homeomorphisms*, Chinese Ann. Math. Ser. A **21** (6) (2000), 723–726.
- [ChChHe] J. Chen, Zh. Chen and Ch.Q. He, *Boundary correspondence under (z)-homeomorphisms*, Michigan Math. J. **43** (2) (1996), 211–220.
- [Ch] P.T. Church, *Discrete maps on manifolds*, Michigan Math. J. **25** (1978), 251–358.
- [ChDaTi] P.T. Church, E.N. Dancer and J.G. Timourian, *The structure of a nonlinear elliptic operator*, Trans. Amer. Math. Soc. **338** (1993), 1–48.
- [ChTi] P.T. Church and J.G. Timourian, *Deficient points of maps on manifolds*, Michigan Math. J. **27** (1980), 321–338.
- [CiDe] J.A. Cima and W.R. Derrick, *Some solutions of the Beltrami equations with $\|\mu\|_\infty = 1$* , Proc. Sympos. Pure Math. **52** (1991), Part I, 41–44.
- [Da] G. David, *Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1) (1988), 25–70.
- [Dr] D. Drasin, *The inverse problem of the Nevanlinna theory*, Acta Math. **138** (1977), 83–151.
- [Dz] A. Dzuraev, *On a Beltrami system of equations degenerating on a line*, Soviet Math. Dokl. **10** (2) (1969), 449–452.
- [Ea] C. Earle, *Some remarks on the Beltrami equation*, Math. Scand. **36** (1975), 44–48.
- [EaEe] C.J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** (1969), 19–43.
- [FaKr] H. Farkas and I. Kra, *Riemann Surfaces*, Graduate Text in Mathematics, Vol. 71, Springer-Verlag, New York (1992).
- [Ga] C.F. Gauss, *Astronomische Abhandlungen*, Vol. 3, H.C. Schumacher, ed., Altona (1825); or “Carl Friedrich Gauss Werke”, Dieterichsche Universitäts-Druckerei, Göttingen, Vol. IV (1880), 189–216.
- [Ge2] F.W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

- [Ge3] F.W. Gehring, *Topics in quasiconformal mappings*, Proc. Internat. Congress of Mathematicians, Berkeley, CA (1986), 62–80.
- [GeHa] F. Gehring and K. Hag, *Reflections on reflections in quasidisks*, Papers on Analysis, Rep. Dept. Math. Stat., Univ. Jyväskylä **83** (2001), 81–90.
- [GeLe] F.W. Gehring and O. Lehto, *On total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I **272** (1959), 1–9.
- [GeRei] F.W. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I **388** (1966), 1–15.
- [GoVo] V. Goldshtein and S. Vodop'yanov, *Quasiconformal mappings and spaces of functions with generalized first derivatives*, Sibirsk. Mat. Zh. **17** (1976), 515–531.
- [Grö] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des picardschen Satzes*, Ber. Verh. Sächs Akad. Wiss. **80** (1928), 503–507.
- [GuMaSuVu] V. Gutlyanskii, O. Martio, T. Sugawa and M. Vuorinen, *On the degenerate Beltrami equation*, Preprint 282, Reports Dept. Math., University of Helsinki (2001); Michigan J. Math., to appear.
- [Hei] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York (2001).
- [HeiKiMa] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monogr., Oxford Univ. Press, New York (1993).
- [HeiKo] J. Heinonen and P. Koskela, *Sobolev mappings with integrable dilatation*, Arch. Ration. Mech. Anal. **125** (1993), 81–97.
- [Ho] H. Hopf, *Zur Topologie der Abbildungen von Mannigfaltigkeiten, II*, Math. Ann. **102** (1929–1930), 562–623.
- [Ig] A. Igl'nikov, *A certain non-uniformly elliptic first order system*, Mathematical Physics, No. 5, Naukova Dumka, Kiev (1968), 76–78 (in Russian).
- [IgRy] A. Ignat'ev and V. Ryazanov, *Finite mean oscillations in the mappings theory*, Preprint 332, Reports Dept. Math., University of Helsinki (2002).
- [IwMa2] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-Linear Analysis*, Oxford Science Publications (2001).
- [IwMa1] T. Iwaniec and G. Martin, *The Beltrami equation*, Mem. Amer. Math. Soc., to appear.
- [IwŠv] T. Iwaniec and V. Šverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. **118** (1993), 181–188.
- [Je] J. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, New York (1958).
- [JoNi] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.
- [Jon] P.W. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. **29** (1) (1980), 41–66.
- [KhYa] Sh.D. Khushvaktov and E.H. Yakubov, *Folded Riemann surfaces*, Tashkent Gos. Univ. Sb. Nauch. Trudov **689** (1982), 77–88 (in Russian).
- [Ko] A. Korn, *Zwei Anwendungen der Methode der sukzessiven Annäherungen*, Mathematische Abhandlungen H.A. Schwarz zu seinem fünfzig-jährigen Doktorjubiläum, Springer-Verlag, Berlin (1914), 215–229.
- [KoOn] P. Koskela and J. Onninen, *Mappings of finite distortion: Capacity and modulus inequality*, Preprint 257, Dept. Math. Stat., University of Jyväskylä (2002).
- [Kr] V.I. Kruglikov, *The existence and uniqueness of mappings that are quasiconformal in the mean*, Metric Questions of the Theory of Functions and Mappings, Naukova Dumka, Kiev (1973), 123–147.
- [Krush1] S.L. Krushkal, *On mean quasiconformal mappings*, Soviet. Math. Dokl. **5** (1964), 966–969.
- [Krush2] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Winston, Washington/Wiley, New York (1979).
- [Krush3] S.L. Krushkal, *Variational principle in the theory of quasiconformal maps*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam, (2005), this volume.
- [KrushKü1] S.L. Krushkal and R. Kühnau, *Quasiconformal Mappings – New Methods and Applications*, Nauka, Novosibirsk (1984) (in Russian); *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner-Text Math., Vol. 54, Teubner, Leipzig (1983) (in German).

- [KrushKü2] S. Krushkal and R. Kühnau, *A quasiconformal dynamic property of the disk*, J. Anal. Math. **72** (1997), 93–103.
- [Küh] R. Kühnau, *Identitäten für schlichte Lösungen der Beltramischen Gleichung*, Identities for Univalent Solutions of the Beltrami Equation, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **31** (1) (1982), 123–128 (in German).
- [Kün] H.P. Küenzi, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
- [La1] M.A. Lavrentieff (Lavrent'ev), *Sur une classe de représentations continues*, Rec. Math. **42** (1935), 407–423.
- [La2] M.A. Lavrent'ev, *Sur une critere differentiel des transformations homeomorphes des domaines a trois dimensions*, Dokl. Akad. Nauk SSSR **20** (1938), 241–242.
- [LaSh] M.A. Lavrent'ev and B.V. Shabat, *Problems of Hydrodynamics and their Mathematical Models*, Nauka, Moscow (1977) (in Russian).
- [Le1] O. Lehto, *Homeomorphisms with given dilatations*, Proc. Fifteen Scandinavian Congress, Oslo 1968, Springer-Verlag, Berlin (1970), 58–73.
- [Le2] O. Lehto, *Remarks on generalized Beltrami equation and conformal mappings*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Riemann Surfaces, Brasov 1969, Publ. House Acad. S. R. Romania, Bucharest (1971), 203–214.
- [Le3] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1986).
- [LeVi] O. Lehto and K. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edn, Springer-Verlag, New York (1973).
- [Li] L. Lichtenstein, *Zur Theorie der konformen Abbildungen; Konforme Abbildungen nicht-analytischer singularitätenfreier Flächenstücke auf ebene Gebiete*, Bull. Acad. Sci. Cracovie (1916), 192–217.
- [Ly] A. Lyzzaik, *Local properties of light harmonic mappings*, Canadian J. Math. **44** (I) (1992), 135–153.
- [MaSaSu] R. Mané, P. Sad and D. Sullivan, *On the dynamics of rational mappings*, Ann. Sci. École Norm Sup. (4) **16** (2) (1983), 193–217.
- [Ma] O. Martio, *Boundary values and injectiveness of the solutions of Beltrami equations*, Ann. Acad. Sci. Fenn. Ser. A I **402** (1967), 1–27.
- [MaMi] O. Martio and V.M. Miklyukov, *On existence and uniqueness of degenerate Beltrami equation*, Preprint 347, Reports Dept. Math., University of Helsinki (2003), 1–12.
- [MaRiVä] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **448** (1969), 1–40.
- [MaRySrYa1] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *On the theory of $Q(x)$ -homeomorphisms*, Dokl. Math. **381** (1) (2001), 20–22.
- [MaRySrYa3] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *Mappings of finite length distortion*, Preprint 322, Reports Dept. Math., University of Helsinki (2002).
- [MaRySrYa2] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *On boundary behavior of Q -homeomorphisms*, Israel Math. Conf. Proc. (2004), 93–103.
- [MaSr] O. Martio and U. Srebro, *On the local behavior of quasiregular maps and branched covering maps*, J. Anal. Math. **36** (1979), 198–212.
- [MiSu] V.M. Miklyukov and G.D. Suvorov, *The existence and uniqueness of quasiconformal mappings with unbounded characteristics*, Studies in the Theory of Functions of Complex Variable and Its Application, Inst. Math. Akad. Nauk. Ukrain., Kiev (1972), 43–53 (in Russian).
- [Mo] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
- [MüQiYa] S. Müller, T. Qi and B.S. Yan, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), 217–243.
- [Ov] I.S. Ovchinnikov, *The existence of mappings on the plane for degenerate first order elliptic systems*, Dokl. Akad. Nauk SSSR **191** (1970), 526–529 (in Russian).
- [Pe] I.N. Pesin, *Mappings that are quasiconformal in the mean*, Soviet. Math. Dokl. **10** (4) (1969), 939–941.
- [Pf] A. Pfluger, *Quasikonforme Abbildungen und logarithmische Kapazität*, Ann. Inst. Fourier (Grenoble) **2** (1951), 69–80.

- [PoRy] V.L. Potemkin and V.I. Ryazanov, *On the noncompactness of classes of mappings with measure restrictions on dilatation*, Ukrain. Mat. Zh. **50** (11) (1998), 1522–1531 (in Russian).
- [RaRe] T. Rado and P.V. Reichelderfer, *Continuous Transformations in Analysis*, Springer-Verlag, Berlin (1955).
- [Rei] E. Reich, *Extremal quasiconformal mappings of the disk*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, North-Holland, Amsterdam (2002), 75–136.
- [ReWa] E. Reich and H. Walczak, *On the behavior of quasiconformal mappings at a point*, Trans. Amer. Math. Soc. **117** (1965), 338–351.
- [Reiman] H.M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. **49** (1974), 260–276.
- [ReiRy] H.M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation*, Lecture Notes in Math., Vol. 487, Springer-Verlag, Berlin (1975).
- [Ren] H. Renelt, *Quasikonforme Abbildungen und elliptische Systeme erster Ordnung in der Ebene*, Teubner, Leipzig (1982); English transl.: *Elliptic Systems and Quasiconformal Mappings*, Pure Appl. Math., Wiley, Chichester (1988).
- [Resh] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monographs, Vol. 73, Amer. Math. Soc., Providence, RI (1989).
- [Ri] S. Rickman, *Quasiregular Mappings*, A Series of Modern Surveys in Mathematics Springer-Verlag, New York (1993).
- [Ry1] V. Ryazanov, *On convergence theorems for homeomorphisms of the Sobolev classes*, Ukrain. Mat. Zh. **47** (1995), 249–259.
- [Ry2] V. Ryazanov, *On convergence and compactness theorems for ACL homeomorphisms*, Rev. Roumaine Math. Pures Appl. **41** (1996), 133–139.
- [RaSrYa1] V. Ryazanov, U. Srebro and E. Yakubov, *BMO-quasiconformal mappings*, J. Anal. Math. **83** (2001), 1–20.
- [RaSrYa2] V. Ryazanov, U. Srebro and E. Yakubov, *The Beltrami equation and FMO functions*, Contemp. Math. **364** (2004).
- [Sa] S. Sastry, *Boundary behavior of BMO-QC automorphisms*, Preprint (2001).
- [Sch1] E.A. Scherbakov, *Homeomorphic solutions of elliptic systems with degeneration*, Sborn. Trud. Phys.-Techn. Inst. Nizk. Temp., Vyp. 1, Charkov (1969), 100–116 (in Russian).
- [Sch2] E.A. Scherbakov, *On methods of proofs of existence of homeomorphic solutions of elliptic systems with degeneration*, Nauch. Trud. Kubask. Gos. Univ., Vyp. 217 (1976), 83–88 (in Russian).
- [Sh] B.V. Shabat, *On a generalized solution to a system of equations in partial derivatives*, Math. Sb., **59** (1945), 193–210 (in Russian).
- [Sr] U. Srebro, *Extremal length and quasiconformal maps*, Israel Math. Conf. Proc. **14** (2000), 135–158.
- [SrYa1] U. Srebro and E. Yakubov, *Alternating Beltrami equation and branched folded mappings*, J. Anal. Math. **70** (1996), 65–89.
- [SrYa2] U. Srebro and E. Yakubov, μ -Homeomorphisms, Contemp. Math. (Lipa's Legacy) **211** (1997), 473–479.
- [SrYa3] U. Srebro and E. Yakubov, *Uniformization of maps with folds*, Israel Math. Conf. Proc. **11** (1997), 229–232.
- [SrYa4] U. Srebro and E. Yakubov, *On Folding solutions of the Beltrami equation*, Ann. Acad. Sci. Fenn. Ser. A I Math. **26** (2001), 225–231.
- [SrYa5] U. Srebro and E. Yakubov, *On boundary behavior of μ -homeomorphisms*, Israel J. Math. **141** (2004), 249–261.
- [SrYa6] U. Srebro and E. Yakubov, in preparation.
- [St] S. Stoilow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Deuxième édition, augmentée de notes sur les fonctions analytiques et leurs surfaces de Riemann, Gauthier-Villars, Paris (1956) (in French).
- [Str] K. Strebel, *Quadratic Differentials*, Ergeb. Math. Grenzgeb. (3) [Results in Mathematics and Related Areas (3)], Bd. 5, Springer-Verlag, Berlin (1984).
- [Su1] D. Sullivan, *Quasiconformal mappings and dynamics I. Solution of the Fatou–Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418.

- [Su2] D. Sullivan, *Quasiconformal mappings in dynamics, topology and geometry*, Proc. Internat. Congress of Mathematicians, Berkeley 1986, Amer. Math. Soc., Providence, RI (1987), 1216–1228.
- [Tuc] A.W. Tucker, *Branched and folded coverings*, Bull. Amer. Math. Soc. (1936), 859–862.
- [Tu] P. Tukia, *Compactness properties of μ -homeomorphisms*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 47–69.
- [Vä1] J. Väisälä, *Lectures in n -Dimensional Quasiconformal Mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag (1971).
- [Vä2] J. Väisälä, *Local topological properties of countable mappings*, Duke Math. J. **41** (1974), 541–546.
- [Vä3] J. Väisälä, *Free quasiconformality in Banach spaces, I*, Ann. Acad. Sci. Fenn. Ser. A I Math. **15** (1990), 355–379.
- [Vä4] J. Väisälä, *Free quasiconformality in Banach spaces, II*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 255–310.
- [Vas] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Lecture Notes in Math., Vol. 1788, Springer-Verlag, Berlin (2002).
- [Ve] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford (1962).
- [Vo1] L.I. Volkovyskii, *Investigations on the type problem for a simply connected Riemann surface*, Trudy Math. Inst. Steklov **34** (1950) (in Russian).
- [Vo2] L.I. Volkovyskii, *Quasiconformal Mappings*, Izdat. L'vovsk. Univ., L'vov (1954) (in Russian).
- [VoLuAr] L.I. Volkovyskii, G.L. Lunts and I.G. Aramanovich, *Problems in the Theory of Functions of Complex Variables*, Pergamon Press, Oxford–New York–Paris (1965) (translated from Russian).
- [Vu] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag (1988).
- [Why] G.T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Coll. Publ., New York (1942).
- [Ya] E.H. Yakubov, *On solutions of Beltrami equation with degeneration*, Soviet Math. Dokl. **19** (6) (1978), 1515–1516.
- [Zhong] Zhong Li, *A remark on homeomorphism solutions of the Beltrami equation*, Beijing Daxue Xuebao Ziran Kexue Ban **25** (1989), 8–17.
- [Zo] V.A. Zorich, *Quasiconformal mappings and asymptotic geometry of manifolds*, Uspekhi Math. Nauk **345** (3) (2002), 3–28 (in Russian); English transl.: Russian Math. Surveys **57** (2002), 437–462.

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CHAPTER 13

The Application of Conformal Maps in Electrostatics

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1. Introduction

The importance of conformal maps in mathematical physics arises mainly because of the possibility of solving boundary value problems in the plane with the help of complex analytic functions. The applications considered in this chapter are in electrostatics as an example; cf., e.g., [10], and further references in [9].

One is led to existence theorems, analytical expressions for the solutions and numerical methods. Beside these aspects *a priori* estimates are possible by using distortion theorems for conformal maps; cf. [2] as a historically first example in the case of the hydrodynamical analogue. (It seems that the *a priori* estimates arise mainly from extremal problems in conformal mapping theory with a corresponding quadratic differential which is a complete square.)

But in real physics the problems are in space, and the rich theory of conformal maps unfortunately exists only in the plane, in view of the classical theorem of Liouville. Therefore such applications of conformal maps are possible only in the case of “planar problems”. This means problems in space which can be prescribed in a plane completely. But the question is: What does this mean exactly? Let us consider the following concrete example.

Let the doubly-connected domain $\mathfrak{G}$ with the two boundary components $\mathfrak{C}_1$ and $\mathfrak{C}_2$, where $\mathfrak{C}_2$ is the outer component, be given in the finite complex plane $z = x + iy$. For our purpose of “practice” it is enough to assume $\mathfrak{C}_1$ and $\mathfrak{C}_2$ as piecewise analytic (although our proofs do not always depend on this). In “physical reality” not only $\mathfrak{C}_1$ but also $\mathfrak{C}_2$ is bounded. But for “mathematical reasons” we will also consider the case, where $\mathfrak{C}_2$ is not bounded.

Now we consider the electrostatic condenser (see Figure 1) in the space (x, y, U) with the two plates C_1, C_2 defined by

$$C_k: \text{all points } (x, y, U) \text{ with } z = x + iy \in \mathfrak{C}_k, -h \leq U \leq 0, k = 1, 2. \quad (1.1)$$

Here $h > 0$ is the “height” of the condenser.

Our aim is to determine the capacity *cap* of this condenser, mainly in dependence of the height h . Beside a universal constant, depending on the dielectric constant, we can usually find the following formula in the literature

$$\text{cap} \approx \frac{2\pi}{\log R} h, \quad (1.2)$$

where $\frac{1}{2\pi} \log R$ denotes the conformal module of the doubly-connected domain $\mathfrak{G}$ as a “plane” condenser. The quantity R is defined as the quotient > 1 of the radii of an annulus which is conformally equivalent to $\mathfrak{G}$. For this definition of the conformal module of $\mathfrak{G}$, compare [8].

But the crucial question is: What does the sign “ $\approx$ ” in (1.2) mean? We can find only very vague answers to this question in the literature, even in the simplest case of an annulus $\mathfrak{G}$. In this latter case we have the classical “Leyden jar” (in German also “Kleistsche Flasche”).

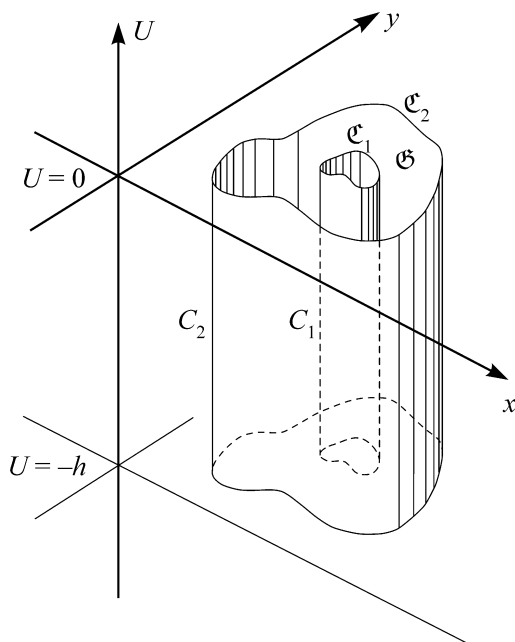


Fig. 1.

Already for this example in [13, pp. 334–345], we can find:

Um diesen Fall genähert zu realisieren, muss man sich ein System unendlich langer cylindrischer Flächen mit parallelen Erzeugenden denken, die so mit Elektrizität geladen sind, dass die Dichtigkeit längs jeder Erzeugenden constant ist. Diese Anordnung ist natürlich in der Wirklichkeit unmöglich; sie wird aber eine gute Annäherung an die Wahrheit darstellen, auch wenn die cylindrischen Flächen begrenzt sind, wenn nur die Querdimensionen und gegenseitigen Entfernungen der Flächen klein sind im Vergleich zu der Längenerstreckung der Cylinder, und wenn nur nach dem Zustande in den mittleren Theilen der Cylinder gefragt wird, so dass der Einfluss der Endflächen vernachlässigt werden kann.

In [14] we can read in the article of Schottky on p. 78:

Die Bedingung der unendlichen Ausdehnung [of the Leyden jar] kann für ein mittleres Gebiet des Leiters ersetzt werden durch die Bedingung, daß die Randeffecte sich in dem untersuchten Gebiet nicht mehr bemerkbar machen. Diese Frage ist natürlich mit Hilfe dreidimensionaler Annäherungsüberlegungen von Fall zu Fall zu prüfen.

But again the execution is missing. Also an exact explanation for the approximation of the capacity corresponding to (1.2) is not given (in [14] on p. 79: “...soweit die Randeffecte vernachlässigt werden können”).

And Koosis [5] writes on p. 129: “If we imagine a very long metallic cylinder perpendicular to the x – y plane, cutting the latter, near its own middle, precisely in $\partial D, \dots$ ”. Again a precise explanation of the phrase “very long” is absent. Cf. also [16].

The aim of this chapter is to arrive at a precise result in case of the formula (1.2) for the capacity.

We mention that our theme is closely connected to the general question of boundary effects of an electrostatic condenser. This question arises in several connections; cf. the references in [12, Chapter 4], [15, p. 230], [7]. It should be remarked that Hermann von Helmholtz (1868) was possibly the first who attacked these questions [3]. His idea appears today in many texts; cf., for example, [10, p. 300].

It is clear that our problem is not a problem of Geometric Function Theory in the plane as such, but a problem of potential theory in space. But surprisingly in the following considerations several aspects of conformal mapping theory (in the plane) will arise.

2. Results

Our main result is the following.

THEOREM 1. *The capacity cap of our condenser (1.1) satisfies*

$$\frac{2\pi h}{\log R} < \text{cap} < \frac{2\pi h}{\log R} \left(1 + \sqrt{\frac{8}{\pi} F^* \log R \cdot \frac{1}{h}} \right), \quad (2.1)$$

if in the plane z the area F^* inside of $\mathfrak{C}_2$ is finite.

Of course (2.1) can also be interpreted as an approximation formula for cap if $F^* \log R$ is small while h is fixed.

Although not of interest in “real” physics, it is also of mathematical interest to have an estimation for the capacity in the case $F^* = \infty$. In this case we will derive the following result, weaker in the order of magnitude.

THEOREM 2. *In any case (also for $F^* = \infty$) the capacity satisfies*

$$\frac{2\pi h}{\log R} < \text{cap} < \frac{2\pi h}{\log R} (1 + E h^{-2/5}), \quad (2.2)$$

where the constant E depends only on $\mathfrak{G}$.

In the case of a convex $\mathfrak{C}_2$ we will derive the following sharper result.

THEOREM 3. *In any case (also for $F^* = \infty$) for a convex outer boundary component $\mathfrak{C}_2$, the capacity satisfies*

$$\frac{2\pi h}{\log R} < \text{cap} < \frac{2\pi h}{\log R} (1 + E' h^{-2/3}), \quad (2.3)$$

where the constant E' depends only on $\mathfrak{G}$.

Beside the trivial left-hand sides of (2.1)–(2.3) there is also a sharper inequality in this other direction.

THEOREM 4. *There is a constant $c > 0$, depending only on $\mathfrak{G}$ (not on h), such that*

$$\frac{2\pi h}{\log R} + c < \text{cap}. \quad (2.4)$$

Especially in the case of a finite area F^* inside of $\mathfrak{C}_2$ (cf. Theorem 1) this means

$$\text{cap} = \frac{2\pi h}{\log R} \left(1 + O\left(\frac{1}{h}\right) \right). \quad (2.5)$$

Here it is impossible to improve the magnitude of the error term. This answers the question about the precision of (1.2).

Finally we consider as an simple example the classical Leyden jar.

3. The underlying main inequalities

The following inequalities for the capacity of our condenser are basically:

$$\text{cap} \geq \iint \left(\int_{\mathfrak{C}(t, \tau)} \frac{ds}{a} \right)^{-1} dt d\tau \quad (ds - \text{arclength element}), \quad (3.1)$$

$$\frac{1}{\text{cap}} \geq \int \frac{dt}{\iint_{\mathfrak{F}(t)} \frac{d\sigma}{a}} \quad (d\sigma - \text{area element}). \quad (3.2)$$

Here $\mathfrak{C}(t, \tau)$ denotes the members of a family of curves joining the two electrodes in the space between the electrodes. These curves depend on two (real) parameters t and τ . For different pairs t, τ the corresponding curves are disjoint. It is not necessary that these curves fill the space between the two electrodes completely. The curves are piecewise smooth; also the dependence on the parameters. (In fact in our case all things will be very simple and piecewise real-analytic.) The function a in (3.1) is defined in such a way that in the corresponding point $a dt d\tau$ always is the infinitesimal area element orthogonal to the corresponding curve $\mathfrak{C}(t, \tau)$ through this point.

In (3.2) $\mathfrak{F}(t)$ denotes the members of a family of (piecewise smooth) closed surfaces which separate the two electrodes. These surfaces depend on a (real) parameter t . For different values of the parameter t the corresponding surfaces are disjoint. We can think of the system of skins of an onion. It is not necessary that these surfaces fill the space between the two electrodes completely. The function a in (3.2) is now defined in such a way that in the corresponding point $a dt$ is always the infinitesimal distance between the surfaces $\mathfrak{F}(t)$ through this point and the surfaces $\mathfrak{F}(t + dt)$. (Of course this a can be calculated with some gradient.) Because of this function a geometry in three-space plays an essential role in our considerations.

The formulas (3.1), (3.2) were derived in [6] from formulas of Hersch [4] for the connection between the capacity and the 2-module (reciprocal extremal length) of some families of curves (resp. surfaces).¹

Although we always tacitly suppose that the space is homogeneous, that means we have a constant value for the dielectric, it is in principle also possible to study the inhomogeneous case with the formulas in [6] which are more general than (3.1) and (3.2).

In this connection we mention also that it is possible to justify the formula (1.2) in the case of *fixed* electrodes, that means for a condenser with a fixed height h . But now we define a constant value for the dielectric constant in the space between the electrodes (that means for $z \in \mathfrak{G}$, $-h \leq U \leq 0$), which is “much greater” than the value in the rest of the space.

In [6] it is also mentioned that there is a close connection between the technique with the formulas (3.1), (3.2) and the technique in [12] following the Dirichlet principle and the Thomson principle. For other applications of this technique cf. [7]. There are further other situations which we can attack with this method. We think for example of the condenser which arises if we rotate a doubly-connected domain in a plane about an axis which lies also in this plane and has a “great” distance from that domain.

4. A lemma for conformal mapping of an annulus

Let be $\mathfrak{F}$ the class of all schlicht conformal mappings $\mathfrak{z}(\zeta)$ of the annulus $1 < |\zeta| < R$ ($< +\infty$) which transform $|\zeta| = R$ onto $|\mathfrak{z}| = 1$ and the annulus onto a part of $|\mathfrak{z}| < 1$ which does not contain the point $\mathfrak{z} = 0$. Then the image of $|\zeta| = 1$ is contained in $|\mathfrak{z}| \leq r$ with an r defined by

$$\mu(r) = \log R; \quad (4.1)$$

cf. [11] ($\mu(r)$ is the module of the Grötzsch extremal domain). With (4.1), r is also defined as a function $r(R)$ of R . We have $\mu(r) < \log \frac{4}{r}$ [11], and therefore

$$r(R) < \frac{4}{R}. \quad (4.2)$$

(This makes, of course, sense only for $R > 4$; but there are also simple estimates possible for $R \leq 4$; cf. [11].)

Our aim is now to estimate the values of $\mathfrak{z}(\zeta)$ for fixed $|\zeta| = \rho$ with $1 < \rho < R$. Again it is possible to obtain the sharp inequality with the corresponding result of Grötzsch. But because this needs elliptic integrals we prefer the following way with a nonsharp but simpler result.

¹These formulas of J. Hersch are not so well known. In the literature these formulas were often mistaken for the connection between the conformal capacity and the three-modules.

LEMMA 1. For all mappings $z(\zeta)$ of the class $\mathfrak{F}$ the image of the circle $|\zeta| = \rho$ with $1 < \rho < R$ is contained in the disk

$$|z| \leq \tau(\rho) =: 1 - \frac{1}{4} \frac{1 - r(R)}{R + R^2} (R - \rho). \quad (4.3)$$

For the *proof* we can assume $z(R) = 1$ without loss of generality. After reflection we have the mapping in the greater annulus $1 < |\zeta| < R^2$ with an image containing the disk $|z - 1| < 1 - r(R)$. Therefore the inverse mapping transforms this disk onto a part of the disk $|\zeta - R| < R + R^2$, and an application of Schwarz's lemma leaves us with

$$\left| \frac{dz}{d\zeta} \right|_{\zeta=1} \leq \frac{R + R^2}{1 - r(R)}. \quad (4.4)$$

Now we apply with a fixed ρ ($1 < \rho < R$) the Koebe $\frac{1}{4}$ -theorem (after a similarity): The image of the disk $|\zeta - R| < R - \rho$ covers the disk

$$|z - 1| < \frac{1}{4} (R - \rho) \left| \frac{dz}{d\zeta} \right|_{\zeta=R},$$

because of (4.4) also the disk

$$|z - 1| < \frac{1}{4} (R - \rho) \frac{1 - r(R)}{R + R^2}. \quad (4.5)$$

Because in similar way also all disks

$$|z - e^{i\alpha}| < \frac{1}{4} (R - \rho) \frac{1 - r(R)}{R + R^2}, \quad \alpha \text{ is real,}$$

are covered from the image of the annulus $\rho < |\zeta| < 2R - \rho$, we get (4.3).

5. A lemma for area distortion in the complex plane

LEMMA 2. For all schlicht conformal mappings $w = z + a_2 z^2 + a_3 z^3 + \dots \in \mathcal{S}$ the following holds for the area I_τ of the image of $|z| \leq \tau < 1$:

$$I_\tau \leq \pi \frac{\tau^2 + 4\tau^4 + \tau^6}{(1 - \tau^2)^4}, \quad (5.1)$$

with equality only for the Koebe functions.

If the mappings are additionally convex, we have

$$I_\tau \leq \pi \frac{\tau^2}{(1 - \tau^2)^2} \quad (5.2)$$

with equality only for the rotations of $\mathfrak{w} = \mathfrak{z}/(1 - \mathfrak{z})$.

PROOF. We have with $\mathfrak{z} = pe^{i\varphi}$ after de Branges

$$I_\tau = \int_0^\tau \int_0^{2\pi} \left| \frac{d\mathfrak{w}}{d\mathfrak{z}} \right|^2 p \, dp \, d\varphi = \pi \sum_{n=1}^{\infty} n |a_n|^2 \tau^{2n} \leq \pi \sum_{n=1}^{\infty} n^3 \tau^{2n}.$$

This yields (5.1).

In the case of convex mappings we arrive at (5.2) by using the stronger inequality $|a_n| \leq 1$; cf., for example, [1, p. 45]. $\square$

LEMMA 3. Let $z = z(\zeta)$ be a schlicht conformal mapping of the annulus $1 < |\zeta| < R$ ($< +\infty$) onto a ring domain such that $|\zeta| = R$ transforms onto the outer boundary component and $z = 0$ is not contained in the outside of the inner boundary component. Then the area A_ρ inside the image of $|\zeta| = \rho$ satisfies

$$A_\rho \leq \frac{B}{(R - \rho)^4}, \quad (5.3)$$

where the constant

$$B = 1536\pi R^{*2} \left[\frac{R + R^2}{1 - r(R)} \right]^4, \quad \text{with } r(R) \text{ after (3.1),} \quad (5.4)$$

depends only on the ring domain. Here R^* denotes the conformal radius of the simply-connected domain inside the image of $|\zeta| = R$, with respect to $z = 0$.

If the image of $|\zeta| = R$ is convex, we have the stronger inequality

$$A_\rho \leq \frac{B'}{(R - \rho)^2}, \quad (5.5)$$

where the constant

$$B' = 4\pi R^{*2} \left[\frac{R + R^2}{1 - r(R)} \right]^2, \quad \text{with } r(R) \text{ after (4.1),} \quad (5.6)$$

again depends only on the ring domain; the definition of R^* is the same as before.

PROOF. We have to combine Lemmas 1 and 2; cf. Figure 2.

With the corresponding Riemann mapping function $\mathfrak{z} = \mathfrak{z}(z)$ we transform the simply-connected domain inside the outer boundary component in the plane z onto the unit disk $|\mathfrak{z}| < 1$ with $\mathfrak{z}(0) = 0$, $\mathfrak{z}'(0) > 0$. If we now consider the resulting mapping $\zeta \rightarrow \mathfrak{z}$ we obtain by Lemma 1, the image of $|\zeta| = \rho$ inside of $|\mathfrak{z}| \leq \tau(\rho)$. Therefore in view of Lemma 2 the area inside of the image of $|\zeta| = \rho$ in the plane z satisfies (with setting $z = R^* \cdot \mathfrak{w}$)

$$A_\rho \leq \pi R^{*2} \frac{\tau^2 + 4\tau^4 + \tau^6}{(1 - \tau^2)^4}, \quad \tau = \tau(\rho) \text{ with (4.3).} \quad (5.7)$$

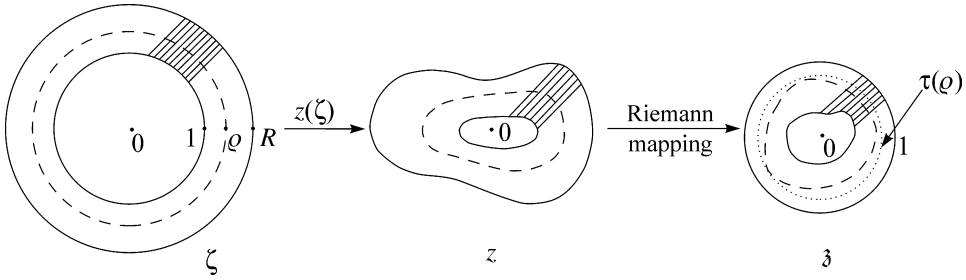


Fig. 2.

In the case of a convex outer boundary component of the ring domain we have to use (5.2), again with $A_\rho = R^{*2} I_{\tau(\rho)}$.

From (5.7) we get

$$\begin{aligned}
 A_\rho &\leq \pi R^{*2} \frac{\tau^2 + 4\tau^4 + \tau^6}{(1 + \tau)^4} \frac{1}{(1 - \tau)^4} \\
 &\leq \pi R^{*2} \cdot 6 \cdot \frac{1}{(1 - \tau)^4} \\
 &= \pi R^{*2} \cdot 6 \left[\frac{4(R + R^2)}{1 - r(R)} \right]^4 \frac{1}{(R - \rho)^4}.
 \end{aligned} \tag{5.8}$$

(Of course it is possible to replace the constant “6” by a smaller one; but we avoid this discussion.) This leaves us with (5.3) and (5.4).

If we have the convex case, instead of (5.8) we obtain

$$\begin{aligned}
 A_\rho &\leq \pi R^{*2} \frac{\tau^2}{(1 + \tau)^2} \frac{1}{(1 - \tau)^2} \leq \pi R^{*2} \cdot \frac{1}{4} \frac{1}{(1 - \tau)^2}, \\
 A_\rho &\leq \pi R^{*2} \cdot \frac{1}{4} \left[\frac{4(R + R^2)}{1 - r(R)} \right]^2 \frac{1}{(R - \rho)^2}.
 \end{aligned} \tag{5.9}$$

□

6. Proof of Theorem 1

For the proof of (2.1) we choose the following family of surfaces $\mathfrak{F}(t)$ in (3.2) for $0 < t < \log R$. Our surfaces $\mathfrak{F}(t)$ consists of five parts:

- $\mathfrak{F}_1(t)$ is the cylinder defined in the space (x, y, U) by $|\zeta(z)| = e^t$, $-h \leq U \leq 0$. Here $\zeta(z)$ is a conformal mapping of our doubly-connected domain $\mathfrak{G}$ in the $(z = x + iy)$ -plane onto the annulus $1 < |\zeta| < R$; cf. Figure 3.
- $\mathfrak{F}_2(t)$ is the surface in the plane $U = \lambda t$ whose orthogonal projection in the plane z (with $U = 0$) is the complement of the domain outside of $\mathfrak{C}_1$. The positive constant λ is defined later. Of course it is also possible that $\mathfrak{F}_2(t)$ is degenerated, e.g., to a curve.

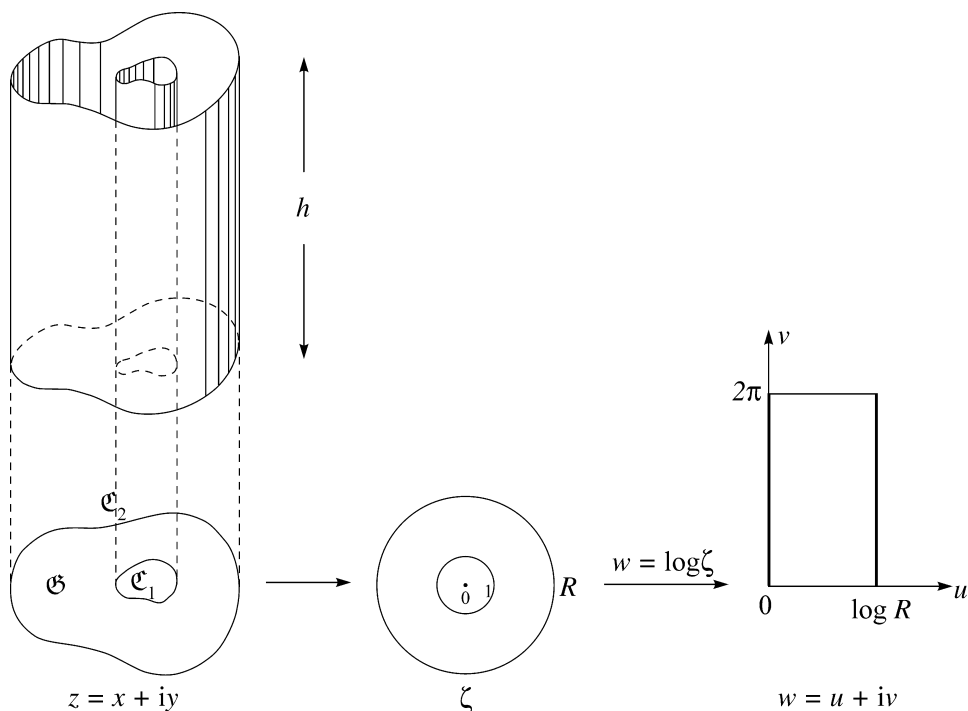


Fig. 3.

(c) $\mathfrak{F}_2^*(t)$ is defined symmetrically to $\mathfrak{F}_2(t)$ with respect to the plane $U = -h/2$.

(d) $\mathfrak{F}_3(t)$ joins $\mathfrak{F}_1(t)$ and $\mathfrak{F}_2(t)$ and is defined by $U = \lambda(t - u(z))$, whereby the orthogonal projection on the plane z (with $U = 0$) is defined by $1 \leq |\zeta(z)| \leq e^t$ or $0 \leq u \leq t$.

(e) $\mathfrak{F}_3^*(t)$ is defined symmetrically to $\mathfrak{F}_3(t)$ with respect to the plane $U = -h/2$.

Now we use in (3.2) the closed surface

$$\mathfrak{F}(t) = \mathfrak{F}_1(t) + \mathfrak{F}_2(t) + \mathfrak{F}_2^*(t) + \mathfrak{F}_3(t) + \mathfrak{F}_3^*(t). \quad (6.1)$$

We start with $\iint_{\mathfrak{F}_1(t)} \frac{d\sigma}{a}$. For this part we obviously have $a = |\frac{dz}{dw}|$, independent of U . This leaves us with

$$\iint_{\mathfrak{F}_1(t)} \frac{d\sigma}{a} = \int_{-h}^0 \left(\oint \frac{|dz|}{a} \right) dU = \int_{-h}^0 \left(\int_0^{2\pi} dv \right) dU = 2\pi h, \quad (6.2)$$

where the integration in $\oint$ is over the closed curve $|\zeta(z)| = e^t$.

For the part $\mathfrak{F}_2(t)$ we simply have $a \equiv \lambda$, therefore

$$\iint_{\mathfrak{F}_2(t)} \frac{d\sigma}{a} = \frac{1}{\lambda} F_i, \quad (6.3)$$

where F_i is the area of $\mathfrak{F}_2(t)$ (independent of t).

For $\mathfrak{F}_3(t)$ we have $\frac{a}{\lambda} = \cos \alpha$ with

$$\operatorname{tg} \alpha = |\operatorname{grad}(\lambda u(z))| = \lambda |\operatorname{grad} u(z)| = \lambda \left| \frac{dw}{dz} \right|,$$

where in every point α is the angle between the surface $\mathfrak{F}_3(t)$ and the plane $U = \text{const}$ through the same point. For every surface element $d\sigma$ of $\mathfrak{F}_3(t)$ we have a orthogonal projection $d\sigma' = \cos \alpha d\sigma$ in $\mathfrak{G}$. This leaves us with

$$\begin{aligned} \iint_{\mathfrak{F}_3(t)} \frac{d\sigma}{a} &= \iint \frac{d\sigma'}{\lambda \cos^2 \alpha} = \frac{1}{\lambda} \iint (1 + \operatorname{tg}^2 \alpha) d\sigma' \\ &= \frac{1}{\lambda} \left(F_t + \lambda^2 \iint \left| \frac{dw}{dz} \right|^2 d\sigma' \right), \end{aligned}$$

where F_t is the area of the projection of $\mathfrak{F}_3(t)$, that is the doubly-connected domain $1 \leq |\zeta(z)| \leq e^t$. Here the last integral is the corresponding area in the w -plane, that is $2\pi t$. So we finally get

$$\iint_{\mathfrak{F}_3(t)} = \frac{1}{\lambda} F_t + \lambda 2\pi t. \quad (6.4)$$

If we set $F_t^* = F_i + F_t$ for the area in the plane z inside the closed curve $|\zeta(z)| = e^t$, we obtain by summarizing (6.2)–(6.4) and the similar values for $\mathfrak{F}_2^*(t)$ and $\mathfrak{F}_3^*(t)$:

$$\iint_{\mathfrak{F}(t)} \frac{d\sigma}{a} = 2\pi h + \frac{2}{\lambda} F_i + \frac{2}{\lambda} F_t + \lambda 4\pi t = 2\pi h + \frac{2}{\lambda} F_t^* + \lambda 4\pi t,$$

therefore with (3.2)

$$\frac{1}{\operatorname{cap}} \geq \int_0^{\log R} \frac{dt}{2\pi h + \frac{2}{\lambda} F_t^* + \lambda 4\pi t}, \quad (6.5)$$

$$\operatorname{cap} \leq \frac{2\pi h + \frac{2}{\lambda} F^* + \lambda 4\pi \log R}{\log R}. \quad (6.6)$$

Here we assume that the area F^* inside of $\mathfrak{C}_2$ is finite.

If we choose $\lambda^2 = F^*/(2\pi \log R)$ in the right-hand side of (6.6) we obtain with this best value (2.1); the left-hand side estimate $\frac{2\pi h}{\log R} < \operatorname{cap}$ is of course trivial.

REMARK. This estimate with the result (6.5) and (6.6), (2.1) is of course also valid in the case when we replace the inner plate $\mathfrak{C}_1$ in (1.1), for example, by the complete solid

$$C'_1: \text{ all points } (x, y, U) \text{ with } z = x + iy \text{ not outside of } \mathfrak{C}_1, -h \leq U \leq 0. \quad (6.7)$$

7. Proof of Theorem 2 (including also the case $F^* = \infty$)

If the area F^* inside of $\mathfrak{C}_2$ is not necessarily finite, we have to consider in (6.5) the involved area F_t^* in more detail. We define $A_\rho = F_{\log \rho}^*$ for $1 < \rho < R$. This quantity A_ρ is then the area inside the image of $|\zeta| = \rho$ in the plane z (cf. again Figure 2). If we substitute $t = \log \rho$ in (6.5) we obtain with Lemma 3

$$\begin{aligned} \frac{1}{cap} &\geq \int_1^R \frac{d\rho/\rho}{2\pi h + \frac{2}{\lambda} B(R-\rho)^{-4} + \lambda 4\pi \log R} \\ &= \frac{1}{2\pi h + \lambda 4\pi \log R} \\ &\quad \times \int_1^R \left[1 - \frac{2B/\lambda}{(2\pi h + \lambda 4\pi \log R)(R-\rho)^4 + \frac{2}{\lambda} B} \right] \frac{d\rho}{\rho}. \end{aligned} \quad (7.1)$$

Here we can replace $d\rho/\rho$ by $d\rho$. If we further put

$$D^4 = \frac{\lambda}{2B}(2\pi h + \lambda 4\pi \log R)$$

and substitute $X = D(R - \rho)$, we get

$$\frac{1}{cap} \geq \frac{1}{2\pi h + \lambda 4\pi \log R} \left[\log R - \frac{1}{D} \int_0^{D(R-1)} \frac{dX}{X^4 + 1} \right].$$

Because of

$$\int_0^\infty \frac{dX}{X^4 + 1} = \frac{\pi}{2\sqrt{2}}$$

this leaves us after setting $\lambda = h^{3/5}$ with

$$\begin{aligned} \frac{1}{cap} &\geq \frac{1}{2\pi h + h^{3/5} 4\pi \log R} \\ &\quad \times \left[\log R - \frac{\pi}{2\sqrt{2}} \sqrt[4]{2B} \cdot h^{-3/20} (2\pi h + h^{3/5} 4\pi \log R)^{-1/4} \right] \end{aligned} \quad (7.2)$$

and therefore (2.2).

8. Proof of Theorem 3 ($\mathfrak{C}_2$ is convex)

If the area F^* inside of $\mathfrak{C}_2$ is not necessarily finite but $\mathfrak{C}_2$ is convex, we can work with the stronger estimate (5.5) instead of (5.3).

Analogously to (7.1) we obtain now

$$\begin{aligned} \frac{1}{cap} &\geq \int_1^R \frac{d\rho/\rho}{2\pi h + \frac{2}{\lambda} B'(R-\rho)^{-2} + \lambda 4\pi \log R} \\ &= \frac{1}{2\pi h + \lambda 4\pi \log R} \int_1^R \left[1 - \frac{2B'/\lambda}{(2\pi h + \lambda 4\pi \log R)(R-\rho)^2 + \frac{2}{\lambda} B'} \right] \frac{d\rho}{\rho}. \end{aligned}$$

Again we replace $d\rho/\rho$ by ρ . Further we put

$$D'^2 = \frac{\lambda}{2B'}(2\pi h + \lambda 4\pi \log R)$$

and now substitute $X' = D'(R - \rho)$. The result is now

$$\frac{1}{cap} \geq \frac{1}{2\pi h + \lambda 4\pi \log R} \left[\log R - \frac{1}{D'} \int_0^{D'(R-1)} \frac{dX'}{X'^2 + 1} \right].$$

Here we can estimate with

$$\int_0^\infty \frac{dX'}{X'^2 + 1} = \frac{\pi}{2}.$$

This leaves us after setting $\lambda = h^{1/3}$ with

$$\begin{aligned} \frac{1}{cap} &\geq \frac{1}{2\pi h + h^{1/3} 4\pi \log R} \\ &\quad \times \left[\log R - \frac{\pi}{2} \sqrt{2B'} h^{-1/6} (2\pi h + h^{1/3} 4\pi \log R)^{-1/2} \right] \end{aligned}$$

and therefore (2.3).

9. Estimation in the other direction

If we use in (3.1) the curve family $\mathfrak{C}(t, \tau)$ defined by

$$\begin{aligned} z &= z(e^{u+i\tau}) \quad \text{with } 0 \leq u \leq \log R, \tau \text{ fixed, } 0 \leq \tau \leq 2\pi, \\ U &= t = \text{fixed,} \quad -h \leq t \leq 0, \end{aligned} \tag{9.1}$$

with the two parameters t and τ , we have

$$a = \left| \frac{dz}{dw} \right|, \quad \int_{\mathfrak{C}(t, \tau)} \frac{ds}{a} = \int |dw| = \log R \tag{9.2}$$

(for the mapping $z(\zeta)$, cf. Figure 2).

So we obtain from (3.1) only the trivial inequality

$$cap \geq \frac{2\pi h}{\log R}. \quad (9.3)$$

Our aim is now to obtain (2.4) from here by using a deformation of this curve family in the lower, respectively, in the upper part. This means for U near to $-h$ or near to 0 , respectively. Obviously it is in both cases sufficient for this purpose to change only a small “bundle” of the curves $\mathfrak{C}(t, \tau)$ in (9.1) such that for the new curves $\mathfrak{C}^*(t, \tau)$ in this bundle

$$\int_{\mathfrak{C}^*(t, \tau)} \frac{ds}{a} \leq \log R - \varepsilon. \quad (9.4)$$

Here ε denotes a positive and sufficiently small constant. The parameters t and τ correspond to this bundle and vary in a domain which contains an inner point.

Of course it is enough to prescribe the change from these $\mathfrak{C}(t, \tau)$ to the new $\mathfrak{C}^*(t, \tau)$ in the upper part. Then we choose a concrete such “bundle” by restricting the corresponding t, τ by

$$-\delta_1 \leq t \leq 0, \quad \tau_0 - \delta_2 \leq \tau \leq \tau_0 + \delta_2,$$

with $\delta_1 > 0$ and $\delta_2 > 0$ sufficiently small. Here τ_0 denotes a value of the parameter τ for which the image of the segment $v = \tau_0, 0 \leq u \leq \log R$ in the plane z ($=$ projection of $\mathfrak{C}(t, \tau)$ into the plane z) has two finite endpoints on an open analytic part of $\mathfrak{C}_1$, resp. $\mathfrak{C}_2$.

The deformation from $\mathfrak{C}(t, \tau)$ to $\mathfrak{C}^*(t, \tau)$ in this bundle is such that we always have the same endpoints on the plates of our condenser, and such that $\mathfrak{C}^*(t, \tau)$ always lies again in the same cylinder surface $F(\tau)$ as $\mathfrak{C}(t, \tau)$, defined by the points

$$F(\tau): z = z(e^{u+i\tau}), \quad -\infty < u < +\infty. \quad (9.5)$$

Of course we try to construct this deformation by enlarging the essential function a in (3.1), that means by lifting $\mathfrak{C}(t, \tau)$ to a new $\mathfrak{C}^*(t, \tau)$. For the following, cf. Figure 4.

After this lifting (in this bundle) we have to replace $a = \left| \frac{dz}{dw} \right|$ in (3.1) by the new function

$$a = a' \left| \frac{dz}{dw} \right|, \quad (9.6)$$

where $a'dt$ is the infinitesimal distance between the curves $\mathfrak{C}^*(t, \tau)$ and $\mathfrak{C}^*(t + dt, \tau)$:

$$\int_{\mathfrak{C}^*(t, \tau)} \frac{ds}{a} = \int \left| \frac{dw}{dz} \right| \frac{1}{a'} \frac{ds}{dS} dS.$$

Here on the right-hand side the integration is over the orthogonal projection of $\mathfrak{C}^*(t, \tau)$ with the new line element $dS = \cos \alpha ds$ ($=$ projection of ds); α is the angle between

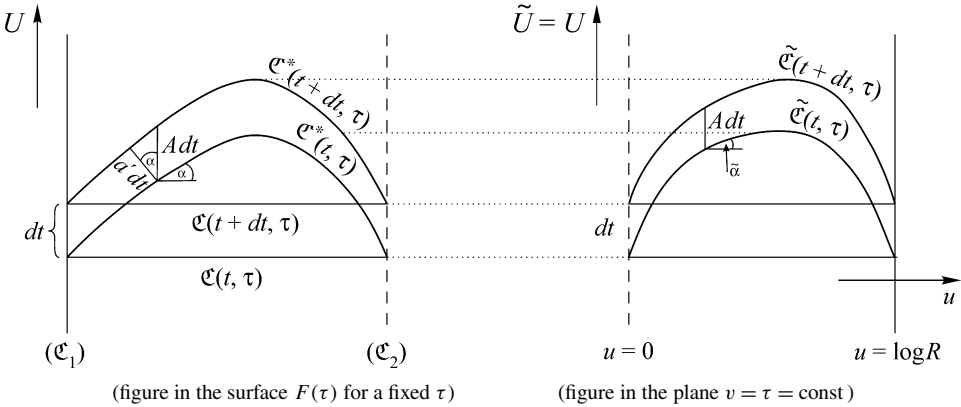


Fig. 4. Transformation of the cylinder surface $F(\tau)$ with $z \rightarrow w = \log \zeta(z)$, $U = U$ for a fixed $v = \tau$.

$\mathfrak{C}^*(t, \tau)$ and the plane $U = \text{const}$ through the corresponding point. If we introduce additionally the infinitesimal distance between $\mathfrak{C}^*(t, \tau)$ and $\mathfrak{C}^*(t + dt, \tau)$ in the orthogonal direction

$$A = a' \cos \alpha, \quad (9.7)$$

we obtain

$$\int_{\mathfrak{C}^*(t, \tau)} \frac{ds}{a} = \int \left| \frac{dw}{dz} \right| \frac{dS}{A \cos^2 \alpha} = \int \left| \frac{dw}{dz} \right| \frac{1 + \operatorname{tg}^2 \alpha}{A} dS.$$

Now we transform the whole situation in the cylinder surface $F(\tau)$ into the plane $v = \tau = \text{const}$ of the space $(u, v, \tilde{U})$ with the mapping $w = \log \zeta(z)$, $\tilde{U} = U$; cf. again Figure 4. From the curves $\mathfrak{C}(t, \tau)$ we get new curves $\tilde{\mathfrak{C}}(t, \tau)$ in this plane. Instead of the angle α we have now a new angle $\tilde{\alpha}$, with

$$\operatorname{tg} \alpha^* = \left| \frac{dz}{dw} \right| \operatorname{tg} \alpha \quad (9.8)$$

because of $du = \left| \frac{dw}{dz} \right| dS$.

So we arrive at

$$\int_{\mathfrak{C}^*(t, \tau)} \frac{ds}{a} = \int_0^{\log R} \frac{1}{A} \left(1 + \left| \frac{dw}{dz} \right|^2 \operatorname{tg}^2 \tilde{\alpha} \right) du. \quad (9.9)$$

We remark that of course A and $\tilde{\alpha}$ depend on u and t , $\left| \frac{dw}{dz} \right|$ depends on u and τ . The choice of $\mathfrak{C}^*(t, \tau)$ means now the choice of $\tilde{\mathfrak{C}}(t, \tau)$. This influences only A and $\tilde{\alpha}$, not $\left| \frac{dw}{dz} \right|$. So the latter function is fixed in the following consideration.

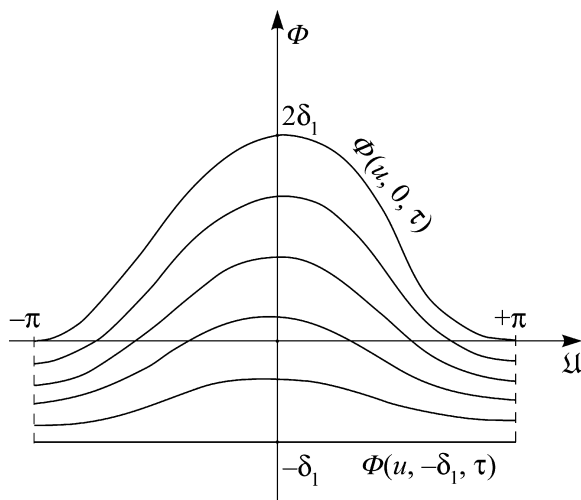


Fig. 5.

Now we define for example the following concrete deformation

$$\tilde{\mathfrak{C}}(t, \tau): \Phi(u; t, \tau) = (t + \delta_1)(1 + \cos u) + t, \quad -\delta_1 \leq t \leq 0, \quad (9.10)$$

with the abbreviation $u = \frac{2\pi}{\log R}u - \pi$.

This definition of a curve family $\tilde{\mathfrak{C}}(t, \tau)$ is really even independent of τ in the fixed interval $\tau_0 - \delta_2 \leq \tau \leq \tau_0 + \delta_2$ under consideration.

We have

$$A = \frac{\partial \Phi}{\partial t} = 2 + \cos u,$$

$$\operatorname{tg} \tilde{\alpha} = \left| \frac{\partial \Phi}{\partial u} \right| = \frac{2\pi}{\log R} (t + \delta_1) |\sin u|.$$

So the integrand on the right-hand side of (9.9) becomes

$$\begin{aligned} \frac{1}{A} \left(1 + \left| \frac{dw}{dz} \right|^2 \operatorname{tg}^2 \tilde{\alpha} \right) &= \frac{1 + \left| \frac{dw}{dz} \right|^2 \left(\frac{2\pi}{\log R} \right)^2 (t + \delta_1)^2 \sin^2 u}{2 + \cos u} \\ &= 1 - \frac{2 \cos^2 \frac{u}{2} [1 - \lambda(t + \delta_1)^2]}{1 + 2 \cos^2 \frac{u}{2}} \\ &< 1 - \frac{2}{3} \cos^2 \frac{u}{2} [1 - \lambda(t + \delta_1)^2], \end{aligned} \quad (9.11)$$

where

$$\lambda = \left| \frac{dw}{dz} \right|^2 \left(\frac{2\pi}{\log R} \right)^2 \sin^2 u$$

is a bounded function of u and $v = \tau$. Because of $(t + \delta_1)^2 \leq \delta_1^2$ it is from here obviously that for sufficiently small δ_1 the integral in (9.9) satisfies (9.4). Theorem 4 is proved.

10. Special case: Leyden jar

In special cases we can obtain better estimates for the capacity with our method. We consider here as a typical example the Leyden jar (in German also “Kleistsche Flasche”) which indeed represents the simplest example. Then we have an annulus in the z -plane. This annulus can be chosen without loss of generality in form of $1 < |z| < R$. This means $\zeta(z) \equiv z$. Already in this simplest case the question of an inequality for the corresponding capacity cap of the three-dimensional condenser (= Leyden jar; cf. Figure 6) is not trivial.

(a) We start with the inequality of the form $1/cap \geq \dots$ corresponding to (2.1). But now we use the earlier estimate (6.5) because we can obtain here immediately $F_t^* = \pi e^{2t}$. To avoid a too complicated integral (6.5) we also use the simple estimate

$$2t < e^{2t} - 1.$$

Then (6.5) leaves us with

$$\frac{1}{cap} \geq \frac{1}{2\pi} \int_0^{\log R} \frac{dt}{h + \frac{1}{\lambda} e^{2t} + 2\lambda t} \geq \frac{1}{2\pi} \int_0^{\log R} \frac{dt}{h - \lambda + (\lambda + \frac{1}{\lambda}) e^{2t}}.$$

After substitution $\tau = e^{2t}$ we obtain

$$\begin{aligned} \frac{1}{cap} &\geq \frac{1}{4\pi} \int_1^{R^2} \frac{d\tau}{\tau(h - \lambda + (\lambda + \frac{1}{\lambda})\tau)}, \\ \frac{1}{cap} &\geq \frac{1}{4\pi(h - \lambda)} \log \frac{R^2(h + \frac{1}{\lambda})}{h - \lambda + (\lambda + \frac{1}{\lambda})R^2}. \end{aligned} \quad (10.1)$$

It is difficult to give the best choice of the free parameter $\lambda > 0$ in dependence of h and R . But if we consider (for fixed R) the asymptotic expression, for great values of h ,

$$\frac{1}{cap} \geq \frac{\log R}{2\pi h} - \frac{1}{4\pi h^2} \left[(R^2 - \log R^2)\lambda + (R^2 - 1)\frac{1}{\lambda} \right] + O(h^{-3}), \quad (10.2)$$

the best choice for λ to obtain the smallest value of the expression in the brackets [] is

$$\lambda = \sqrt{\frac{R^2 - 1}{R^2 - \log R^2}}. \quad (10.3)$$

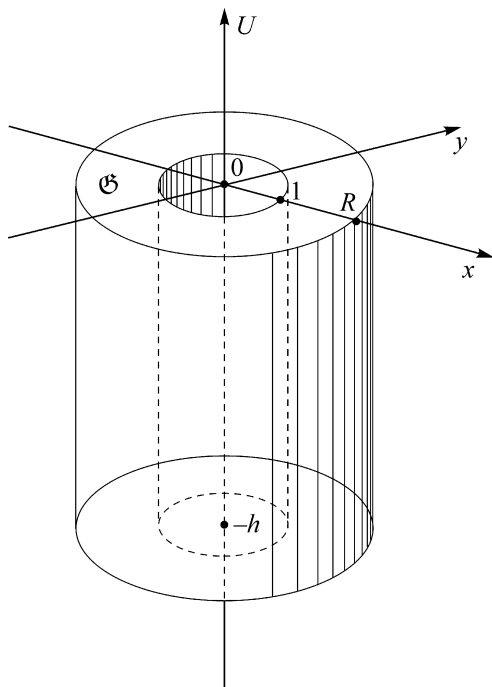


Fig. 6.

Then (10.2) writes as

$$cap \leq \frac{2\pi h}{\log R} + \frac{2\pi}{(\log R)^2} \sqrt{(R^2 - 1)(R^2 - \log R^2)} + O(h^{-1}). \quad (10.4)$$

Of course it is also possible to deduce from (10.1) an asymptotic formula for the other case of small values for $R - 1$ with a fixed h .

(b) Now we seek an estimate in the other direction for the capacity cap of our Leyden jar. For this reason we use another family of curves in (3.1) than in the general case of Section 9. This family is simpler and more effective in this case. This or a similar family is recommendable generally in those cases in which, e.g., $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are Jordan curves, such that the domain inside of $\mathfrak{C}_1$ and the domain outside of $\mathfrak{C}_2$ contains inner points. (If, for example, $\mathfrak{C}_1$ is a slit this simple method does not work.)

In our case of the Leyden jar let $\mathfrak{C}(t, \tau)$ in (3.1) in the first (main) part simply be defined as the segment $U = t$ ($-h \leq t \leq 0$), $\arg z = \tau$ ($0 \leq \tau < 2\pi$), $1 < |z| < R$, with

$$a = a(x, y, U) = |z|. \quad (10.5)$$

(In more general cases the field lines are recommended.) This leaves us for the arcs $\mathfrak{C}(t, \tau)$ with

$$\int_{\mathfrak{C}(t, \tau)} \frac{ds}{a} = \log R. \quad (10.6)$$

We define the remaining curves $\mathfrak{C}(t, \tau)$ in the following way; cf. Figure 7. Again $\mathfrak{C}(t, \tau)$ is situated in the half plane $\arg z = \tau$ ($0 \leq \tau < 2\pi$) and consists of the segment $U = t$ ($0 < t < \frac{1}{2}$ resp. $-h - \frac{1}{2} < t < -h$) prolonged with two half circles centered at $(\cos \tau, \sin \tau, 0)$ and $(R \cos \tau, R \sin \tau, 0)$ (resp. $(\cos \tau, \sin \tau, -h)$ and $(R \cos \tau, R \sin \tau, -h)$). Then we obtain, putting all together, a family of curves $\mathfrak{C}(t, \tau)$ which is admissible in (3.1). For the crucial function a in (3.1) we have in the part with $|z| \leq 1$ obviously $a \geq \frac{1}{2}$, whereas $a \geq R$ in the part with $|z| \geq R$. Therefore (3.1) leaves us with

$$\begin{aligned} cap &\geq \frac{2\pi h}{\log R} + 2 \cdot 2\pi \int_0^{1/2} \frac{d\rho}{\log R + \pi\rho(2 + \frac{1}{R})}, \\ cap &\geq \frac{2\pi h}{\log R} + \frac{4}{2 + \frac{1}{R}} \log \frac{\log R + \frac{1}{2}\pi(2 + \frac{1}{R})}{\log R}. \end{aligned} \quad (10.7)$$

We summarize our result.

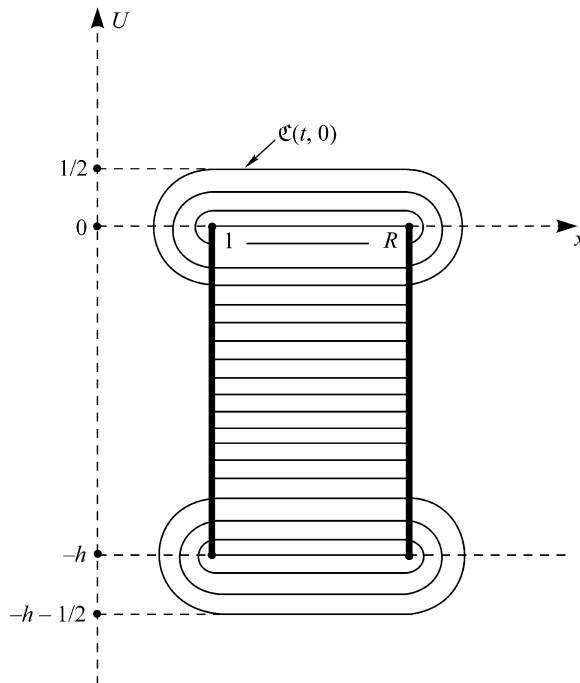


Fig. 7. Intersection with the half plane $\arg z = 0$.

THEOREM 5. *The capacity cap of the Leyden jar in the space (x, y, U) with the plates $x^2 + y^2 = 1$, $-h \leq U \leq 0$, and $x^2 + y^2 = R^2$, $-h \leq U \leq 0$, satisfies (10.1) (with λ from, e.g., (10.3)) and (10.7).*

References

- [1] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [2] P. Frank und K. Löwner, *Eine Anwendung des Koebeschen Verzerrungssatzes auf ein Problem der Hydrodynamik*, Math. Z. **3** (1919), 78–86.
- [3] H. Helmholtz, *Wissenschaftliche Abhandlungen*, Bd. 1, Johann Ambrosius Barth, Leipzig (1882).
- [4] J. Hersch, (a) *Sur une forme générale du théorème de Phragmén–Lindelöf*, C. R. Acad. Sci. Paris **237** (1953), 641–643; (b) *“Longueurs extrémales” dans l’espace, résistance électrique et capacité*, ibid. **238** (1954), 1639–1641.
- [5] P. Koosis, *The Logarithmic Integral, II*, Cambridge Univ. Press, Cambridge, NY (1992).
- [6] R. Kühnau, *Der Modul von Kurven- und Flächenscharen und räumliche Felder in inhomogenen Medien*, J. Reine Angew. Math. **243** (1970), 184–191.
- [7] R. Kühnau, *Boundary effects for an electrostatic condenser*, J. Math. Sci. **105** (2001), 2210–2219.
- [8] R. Kühnau, *The conformal module of quadrilaterals and rings*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), 99–129 (this Volume).
- [9] R. Kühnau, *Bibliography of Geometric Function Theorie*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), 809–828 (this Volume).
- [10] M.A. Lawrentjew und B.W. Schabat, *Methoden der komplexen Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1967) (transl. from the Russian).
- [11] O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, Berlin–Heidelberg–New York (1973).
- [12] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, Princeton (1951).
- [13] B. Riemann und H. Weber, *Die partiellen Differential-Gleichungen der mathematischen Physik* (nach Riemann’s Vorlesungen in vierter Auflage neu bearbeitet von Heinrich Weber), Bd. 1, Vieweg und Sohn, Braunschweig (1900).
- [14] R. Rothe, F. Ollendorf und K. Pohlhausen (Hrsg.), *Funktionentheorie und ihre Anwendung in der Technik*, Springer-Verlag, Berlin (1931); English transl.: Dover, New York (1961).
- [15] I.N. Sneddon, *Mixed boundary value problems in potential theory*, North-Holland, Amsterdam/Wiley, New York (1966).
- [16] G. Szegő, *On the capacity of a condenser*, Bull. Amer. Math. Soc. **51** (1945), 325–350.

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CHAPTER 14

Special Functions in Geometric Function Theory

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Introduction

In this chapter we shall review some of those aspects of the theory of special functions that are relevant for Geometric Function Theory, especially for conformal and quasiconformal maps. We restrict ourselves mainly to most important special functions, and briefly mention contexts where these functions occur. We do not attempt to give the results in their most general form, but rather our goal is to provide formulations which are easy to understand and adequate for many applications. As a rule, we have avoided giving proofs of results which are classical and available in the well-known monographs of the field – in such cases we have indicated references for the proof and possible generalizations or related results.

One of our favorite references of the field is [Ra] which covers all the aforementioned topics in a lucid and economical style. Another standard reference of the subject is [WW], a treatise that in its first edition was published at the end of 19th century. Some of the encyclopedic works of the field are [Bat1,Bat2,Bat3,AS]. The monumental three-volume series [Bat1,Bat2,Bat3] was, at its time of publication in the early 1950s, a state-of-the-art survey of the field and it is still the most comprehensive treatment of special functions. [AS] and [Bat1,Bat2,Bat3] are among the most frequently cited books on special functions. The reference [AS] is a most useful collection of formulas, graphs, and tabular data of numerical values of special functions. For a short excellent outline of the history of special functions see [C3, pp. 1–6].

The aforementioned standard references are rather old. Some welcome additions to the literature on special functions are the new books [AAR,MM,PSo,Tem,Wa].

The order of the sections in this chapter is as follows: We start with the Euler gamma function, then discuss hypergeometric functions and Legendre's complete elliptic integrals, and conclude with the Jacobian elliptic functions.

At the end of each section we include some remarks of general nature, some information about the numerical evaluation of special functions, some historical data, and a list of references to the literature. An extensive bibliography about the numerical computation of transcendental functions is given in [LO]. The reference [SO] provides a comprehensive display of the graphs of special functions. Some computer software for numerical evaluation of special functions with many graphs can be found in [Thom], where the organization of the contents follows [AS].

One of the reasons for the frequent occurrence of special functions is that solutions of extremal problems can often be expressed in terms of special functions. Another reason is that some important conformal mappings are given by special functions. For instance, the conformal mapping of an annulus onto the complement of two closed segments on the real axis and the conformal mapping of a square onto a rectangle are given by elliptic functions. The study of conformal invariants also often leads to special functions. Some examples of the use of special functions in Geometric Function Theory are also briefly mentioned.

This short review is far from complete. Some important topics not mentioned are orthogonal polynomials and their many applications to complex approximation and numerical conformal mapping. For these we refer to [Gai] and [Hen1–Hen3]. For those functions that we cover here, only the basic properties are discussed. The interested reader may find some further results in [AVV1] and [AVV2].

1. Gamma and beta functions

The gamma function first arose in connection with the interpolation problem for factorials. This problem of finding a function of a continuous variable x that equals $n!$ when $x = n \in \mathbb{N}$, was posed by Goldbach, Bernoulli and Stirling, and investigated by Euler in the 1720s. Its solution, the gamma function, is contained in Euler's letter of October 13, 1729, to Goldbach [AAR]. In the 19th century, the definition of the gamma function was extended to complex numbers. See [D] for an interesting account of the history of the gamma function. It is well known that this function is very important in mathematics, physics and engineering.

DEFINITION 1.1. The Euler *gamma function* is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0. \quad (1.1)$$

The closely related *beta function* $B(u, v)$ is defined as

$$\begin{aligned} B(u, v) &= \int_0^1 t^{u-1} (1-t)^{v-1} dt \\ &= 2 \int_0^{\pi/2} \sin^{2u-1} \varphi \cos^{2v-1} \varphi d\varphi, \quad \operatorname{Re} u > 0, \operatorname{Re} v > 0. \end{aligned} \quad (1.2)$$

The beta function can be expressed in terms of the gamma function as follows

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \quad (1.3)$$

We use Theorem 1.2 to define $\Gamma(z)$ for all $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.

1.1. Functional equalities

For $n \in \mathbb{N}$ and z such that the following expressions make sense:

$$\Gamma(z+n) = (z+n-1) \cdots (z+1)z\Gamma(z), \quad \Gamma(n+1) = n!, \quad (1.4)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (\text{Euler's reflection formula}), \quad (1.5)$$

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right), \quad (1.6)$$

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1) \cdots (z+n)}. \quad (1.7)$$

1.2. Euler–Mascheroni constant γ

This constant is defined by

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) = 0.5772156649 \dots, \quad (1.8)$$

or equivalently,

$$\gamma = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k} - \log \left(n + \frac{1}{2} \right) \right]. \quad (1.9)$$

The constant γ can also be expressed by

$$\begin{aligned} \gamma &= \lim_{n \rightarrow \infty} \left\{ \int_0^1 \left[1 - \left(1 - \frac{t}{n} \right)^n \right] \frac{dt}{t} - \int_1^n \left(1 - \frac{t}{n} \right)^n \frac{dt}{t} \right\} \\ &= \int_0^1 (1 - e^{-t} - e^{-1/t}) \frac{dt}{t} = \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} dt \\ &= \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}. \end{aligned} \quad (1.10)$$

THEOREM 1.2 (Weierstrass, Euler's infinite product [AAR]). *The function*

$$w(z) = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k} \right) e^{-z/k}$$

is analytic in the whole plane. Moreover, for $\operatorname{Re} z > 0$,

$$w(z) = \frac{1}{\Gamma(z)}.$$

For the graphs of $\Gamma(x)$ and $1/\Gamma(x)$, see Figure 1.

THEOREM 1.3 [AVV1, AAR, AnQ]. *The gamma function has the following properties:*

- (1) $\Gamma(x)$ is decreasing on $(0, x_0)$ and increasing on (x_0, ∞) , where $1 < x_0 < 2$.
- (2) $\Gamma(x)$ is log-convex on $(0, \infty)$.
- (3) $f(x) \equiv (1/x) \log \Gamma(x/2)$ is strictly increasing from $[2, \infty)$ onto $[0, \infty)$.
- (4) $\lim_{x \rightarrow \infty} \frac{1}{x \log x} \log \Gamma(1 + \frac{x}{2}) = \frac{1}{2}$.
- (5) $g(x) \equiv \frac{1}{x \log x} \log \Gamma(x+1)$ is strictly increasing and concave from $(1, \infty)$ onto $(1 - \gamma, 1)$. In particular, for $x \in (1, \infty)$,

$$x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}.$$

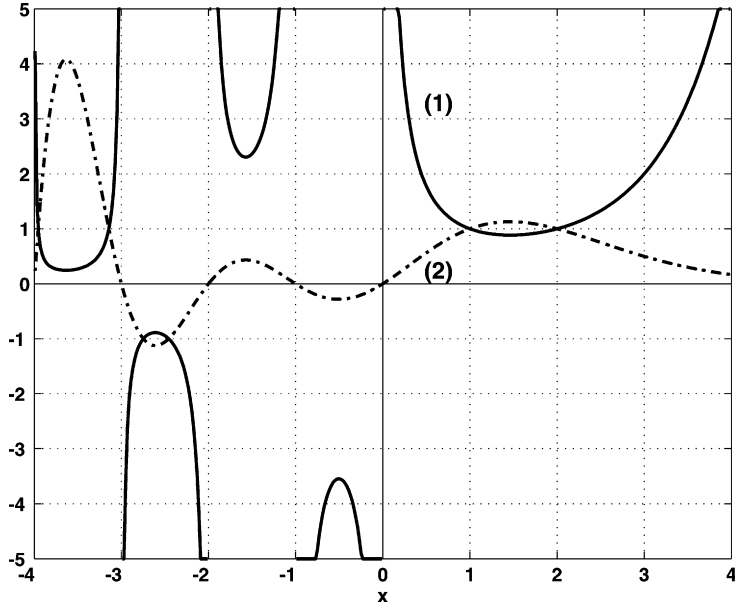


Fig. 1. Curve (1) $-\Gamma(x)$; curve (2) $-1/\Gamma(x)$.

1.3. The psi function

The logarithmic derivative $\Psi(z) = \Gamma'(z)/\Gamma(z)$ of $\Gamma(z)$, defined for $z \neq 0, -1, -2, \dots$, is called the *psi function*.

1.4. Functional equalities of $\Psi(z)$ [AS]

$$\Psi(z+1) = \Psi(z) + 1/z, \quad (1.11)$$

$$\Psi(1-z) = \Psi(z) + \pi \cot \pi z, \quad (1.12)$$

$$2\Psi(2z) = \Psi(z) + \Psi\left(z + \frac{1}{2}\right) + \log 4, \quad (1.13)$$

$$\begin{aligned} \Psi(z) &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt \quad (\operatorname{Re} z > 0) \\ &= \int_0^\infty \left[e^{-t} - \frac{1}{(1+t)^z} \right] \frac{dt}{t} \\ &= \int_0^\infty \frac{e^{-t} - e^{-zt}}{1-e^{-t}} dt - \gamma \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{1-t^{z-1}}{1-t} dt - \gamma \\
&= \log z - \frac{1}{2z} - 2 \int_0^\infty \frac{t dt}{(t^2+z^2)(e^{2\pi t}-1)} \quad \left(|\arg z| < \frac{\pi}{2} \right), \quad (1.14)
\end{aligned}$$

$$\Psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(z+k)}, \quad (1.15)$$

$$\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2} \quad (1.16)$$

for $z = 0, -1, -2, \dots$,

$$\Psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}} dt \quad (\operatorname{Re} z > 0). \quad (1.17)$$

1.5. Special values $\Gamma(z)$ and $\Psi(z)$

$$\begin{aligned}
(1) \quad & \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n)!}{4^n \cdot n!}, \\
(2) \quad & \Psi(1) = \Gamma'(1) = -\gamma, \quad \Psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}, \\
(3) \quad & \Psi\left(\frac{1}{2}\right) = -\gamma - \log 4, \quad \Psi\left(n + \frac{1}{2}\right) = -\gamma - \log 4 + 2 \sum_{k=1}^n \frac{1}{2k-1}, \\
(4) \quad & \Psi'(1) = \frac{\pi^2}{6}, \quad \Psi'\left(\frac{1}{2}\right) = \frac{1}{2}\pi^2, \\
(5) \quad & \Psi''(1) = -2\zeta(3), \quad \Psi''\left(\frac{1}{2}\right) = -14\zeta(3), \quad \Psi'''(1) = \frac{1}{15}\pi^4.
\end{aligned}$$

Here $\zeta(x)$ is the Riemann zeta function.

1.6. The Appell symbol

For $a \in \mathbb{C}$, $n \in \mathbb{N}$, the *Appell symbol* (a, n) stands for the shifted factorial

$$(a, n) = a(a+1) \cdots (a+n-1), \quad (a, 0) = 1 \text{ for } a \neq 0.$$

Clearly, $(a, n) = \Gamma(a+n)/\Gamma(a)$.

1.7. The Stirling and Wallis formulas

$$\lim_{x \rightarrow +\infty} x^{(1/2)-x} e^x \Gamma(x) = \sqrt{2\pi}, \quad x \in \mathbb{R}. \quad (\text{Stirling's formula})$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n!} \left(\frac{1}{2}, n \right) = \frac{1}{\sqrt{\pi}}, \quad \lim_{n \rightarrow \infty} \frac{(n!)^2}{(1/2, n)(3/2, n)} = \frac{\pi}{2}. \quad (\text{Wallis' formula})$$

1.8. Asymptotic formulas

$$\begin{aligned} \Gamma(z) &\sim e^{-z} z^{z-1/2} \sqrt{2\pi} \\ &\times \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \frac{571}{2488320z^4} + \cdots \right), \end{aligned} \quad (1.18)$$

$$\begin{aligned} \log \Gamma(z) &\sim \left(z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) \\ &+ \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \cdots, \end{aligned} \quad (1.19)$$

as $z \rightarrow \infty$ in $|\arg z| < \pi$.

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+1/2}, \quad (1.20)$$

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b}, \quad (1.21)$$

as $n \rightarrow \infty$, $n \in \mathbb{N}$. As $z \rightarrow \infty$ with $|\arg z| < \pi$,

$$\begin{aligned} \Psi(z) &\sim \log z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} \\ &= \log z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots, \end{aligned} \quad (1.22)$$

$$\Psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right], \quad (1.23)$$

where the B_{2m} are Bernoulli numbers.

REMARK 1.4. (1) The gamma and beta functions are convenient means for expressing certain important constants. For example, the n -dimensional volume Ω_n of the unit ball B^n and the $(n-1)$ -dimensional surface area ω_{n-1} of the unit sphere S^{n-1} in $\mathbb{R}^n$ can be written as

$$\Omega_n = \frac{2\pi}{n} \Omega_{n-2} = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \quad \omega_{n-1} = n\Omega_n = \frac{n\pi^{n/2}}{\Gamma(1+n/2)}. \quad (1.24)$$

(2) Hölder proved that, unlike many other special functions, the gamma function does not satisfy any differential equation with rational coefficients [C3, pp. 49–50], [Hö,Ru,Tot], [WW, p. 236].

(3) For further properties of the gamma function, see [Alz,Ar,AS,Bat1,Bat2,Bat3,C2, Hen1,Ra,SO,Tem,WW].

2. Hypergeometric functions

The hypergeometric function is one of the most important special functions, because of its many connections to other classes of special functions, and its numerous identities and expressions in terms of series and integrals.

DEFINITION 2.1. For parameters $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$, and $z \in \mathbb{C}$ with $|z| < 1$, the (Gaussian) hypergeometric function is defined by

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}. \quad (2.1)$$

The series on the right-hand side, called the hypergeometric series, is convergent for $|z| < 1$. Often the Pochhammer notation $(a)_n$ is used for (a, n) .

For nonnegative integers p and q , the generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^p (a_k, n)}{\prod_{k=1}^q (b_k, n)} \frac{z^n}{n!}, \quad (2.2)$$

provided that $(b_k, n) \neq 0$ for all k, n . The ${}_pF_q$ series is convergent for $|z| < 1$ if $p \leq q + 1$.

THEOREM 2.2 [Ch,Ra,WW]. If $c \neq 0, -1, -2, \dots$, then

- (1) Both the series in (2.1) and (2.2) converge absolutely and uniformly on each compact subinterval of $(-1, 1)$.
- (2) For $|z| < 1$, the function $F(a, b; c; z)$ depends analytically on a, b and c .

2.1. Integral representation [Ra]

For $\operatorname{Re} c > \operatorname{Re} b > 0$,

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt.$$

2.2. Elementary particular cases

Some of the simplest special cases of the hypergeometric functions are (see [AS,PBM]):

- (1) $(1-x)^{-a} = F(a, 1; 1; x)$.
- (2) $\log(1+x) = xF(1, 1; 2; -x)$.
- (3) $\arcsin x = xF(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2)$.
- (4) $\arctan x = xF(1, \frac{1}{2}; \frac{3}{2}; -x^2)$.
- (5) $\operatorname{arth} x = xF(\frac{1}{2}, 1; \frac{3}{2}; x^2)$.
- (6) $\log(x + \sqrt{1+x^2}) = x\sqrt{1+x^2}F(1, 1; \frac{3}{2}; -x^2)$.
- (7) $(1+x)^{-2a} + (1-x)^{-2a} = 2F(a, \frac{1}{2} + a; \frac{1}{2}; x^2)$.
- (8) $-2(x + \log(1-x)) = x^2F(1, 2; 3; x)$.
- (9) ${}_3F_2(1, 3, 3; 2, 2; x) = \frac{1}{4}(1-x)^{-3}(4-3x+x^2)$.
- (10) ${}_3F_2(1, 1, 1; \frac{3}{2}, 2; x) = \frac{1}{x} \arcsin^2 \sqrt{x}$.
- (11) $P_n(z) = F(-n, n+1; 1; \frac{1-z}{2})$, where $P_n(z)$ is the Legendre polynomial, $n = 0, 1, 2, \dots$
- (12) $T_n(x) \equiv \cos(n \arccos x) = F(n, -n; \frac{1}{2}; \frac{1-x}{2})$, $0 < x < 1$, $n \in \mathbb{N}$, where $T_n(x)$ is the Chebyshev polynomial.

2.3. Differentiation formula

For $n \in \mathbb{N}$, $|z| < 1$,

$$\frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a, n)(b, n)}{(c, n)} F(a+n, b+n; c+n; z).$$

2.4. Special values of the argument

There is a considerable literature about the evaluation of hypergeometric functions for certain values of the argument in terms of other special functions such as the gamma function. Here we only give three examples. Many special values of hypergeometric functions are given in [PBM,AS].

- (1) $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$
for $c \neq 0, -1, -2, \dots, \operatorname{Re}(c-a-b) > 0$.
- (2) $F\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}$
when the right-hand side makes sense.

$$(3) \quad F\left(a, 1-a; b; \frac{1}{2}\right) = \frac{2^{1-b} \sqrt{\pi} \Gamma(b)}{\Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{1+b-a}{2}\right)}$$

when the right-hand side makes sense.

2.5. Hypergeometric differential equation

Using Section 2.3, one may verify that $y = F(a, b; c; z)$ satisfies the hypergeometric differential equation

$$z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0. \quad (2.3)$$

The complete solution of this ordinary differential equation of the second degree is quite complicated, but under some special conditions on the coefficients the solutions are listed in [AS, pp. 562–564]. See also [Pr, Hen1, Hen2, Cara2]. A set of 24 solutions of (2.3) was first presented by Kummer. The study of (2.3) has led to very extensive theories; see, e.g., [Var].

2.6. Gauss' contiguous relations

The six functions $F(a \pm 1, b; c; z)$, $F(a, b \pm 1; c; z)$ and $F(a, b; c \pm 1; z)$ are called *contiguous* to $F(a, b; c; z)$. We use the notations

$$\begin{aligned} F &= F(a, b; c; z), \\ F(a+) &= F(a+1, b; c; z), \\ F(a-) &= F(a-1, b; c; z), \end{aligned} \quad (2.4)$$

together with similar notations $F(b+)$, $F(b-)$, $F(c+)$, $F(c-)$, for the other four of the six functions contiguous to F . Gauss proved that between F and each pair of its six contiguous functions there exists a linear relation with coefficients at most linear in z . There are 15 such relations. We now list the first five contiguous relations [Ra]:

$$(a-b)F = aF(a+) - bF(b+), \quad (2.5)$$

$$(a-c+1)F = aF(a+) - (c-1)F(c-), \quad (2.6)$$

$$[a + (b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+), \quad (2.7)$$

$$(1-z)F = F(a-) - c^{-1}(c-b)zF(c+), \quad (2.8)$$

$$(1-z)F = F(b-) - c^{-1}(c-a)zF(c+). \quad (2.9)$$

The remaining ten relations follow from these, and they are:

$$[2a - c + (b - a)z]F = a(1 - z)F(a+) - (c - a)F(a-), \quad (2.10)$$

$$(a + b - c)F = a(1 - z)F(a+) - (c - b)F(b-), \quad (2.11)$$

$$(c - a - b)F = (c - a)F(a-) - b(1 - z)F(b+), \quad (2.12)$$

$$(b - a)(1 - z)F = (c - a)F(a-) - (c - b)F(b-), \quad (2.13)$$

$$[1 - a + (c - b - 1)z]F = (c - a)F(a-) - (c - 1)(1 - z)F(c-), \quad (2.14)$$

$$[2b - c + (a - b)z]F = b(1 - z)F(b+) - (c - b)zF(b-), \quad (2.15)$$

$$[b + (a - c)z]F = b(1 - z)F(b+) - c^{-1}(c - a)(c - b)zF(c+), \quad (2.16)$$

$$(b - c + 1)F = bF(b+) - (c - 1)F(c-), \quad (2.17)$$

$$[1 - b + (c - a - 1)z]F = (c - b)F(b-) - (c - 1)(1 - z)F(c-), \quad (2.18)$$

$$\begin{aligned} [c - 1 + (a + b + 1 - 2c)z]F \\ = (c - 1)(1 - z)F(c-) - c^{-1}(c - a)(c - b)zF(c+). \end{aligned} \quad (2.19)$$

For instance, (2.10) follows from (2.7) and (2.8).

Using these contiguous relations, we now write the differentiation formula from Section 2.3 as in Theorem 2.3, which is useful in some studies. In particular, the differentiation formulas for the elliptic integrals in Section 3 follow from Theorem 2.3. A proof of Theorem 2.3 and its corollary is given in [AQVV].

THEOREM 2.3. *For $a, b, c > 0$, $r \in (0, 1)$, let $u = u(r) = F(a - 1, b; c; r)$, $v = v(r) = F(a, b; c; r)$, $u_1 = u(1 - r)$, $v_1 = v(1 - r)$. Then:*

$$(1) \quad r \frac{du}{dr} = (a - 1)(v - u),$$

$$(2) \quad r(1 - r) \frac{dv}{dr} = (c - a)u + (a - c + br)v,$$

$$\begin{aligned} (3) \quad r(1 - r) \frac{d}{dr}(uv_1 + u_1v - vv_1) \\ = (1 - a - b)[(1 - r)uv_1 - ru_1v - (1 - 2r)vv_1]. \end{aligned}$$

COROLLARY 2.4. *Under the conditions of Theorem 2.3,*

$$(1) \quad -r(1 - r)v^2 \frac{d}{dr} \left(\frac{v_1}{v} \right) = (c - a)(uv_1 + u_1v) + [b + 2(a - c)]vv_1$$

and

$$(2) \quad r \frac{d}{dr} [(1 - r)vv_1] = (c - a)[uv_1 - u_1v] + [(2b - 1)r - b]vv_1.$$

In particular, if $a \in (0, 1)$, $b = 1 - a < c$, then

$$(3) \quad uv_1 + u_1v - vv_1 = u(1) = \frac{\Gamma(c)^2}{\Gamma(c+a-1)\Gamma(c-a+1)}.$$

In addition, if $c = 1$, then:

$$(4) \quad uv_1 + u_1v - vv_1 = \frac{\sin(\pi a)}{\pi(1-a)}.$$

$$(5) \quad \frac{d}{dr} \left(\frac{v_1}{v} \right) = -\frac{\sin(\pi a)}{\pi r(1-r)v^2},$$

$$(6) \quad r \frac{d}{dr} ((1-r)vv_1) = \frac{\sin(\pi a)}{\pi} + v[(1-2a)rv_1 - 2(1-a)u_1].$$

REMARK 2.5. (1) The identity (3) in Corollary 2.4, which is the generalized Legendre relation, is a special case of Elliott's formula [El]

$$\begin{aligned} & F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu; 1 + \lambda + \mu; z\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - z\right) \\ & + F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu; 1 + \lambda + \mu; z\right) F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - z\right) \\ & - F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu; 1 + \lambda + \mu; z\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - z\right) \\ & = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \nu + \mu)}{\Gamma(\lambda + \mu + \nu + \frac{3}{2})}. \end{aligned} \quad (2.20)$$

This formula yields Corollary 2.4(3) if $\lambda = \nu = \frac{1}{2} - a$ and $\mu = c + a - \frac{3}{2}$.

(2) The formula (5) in Corollary 2.4 is a special case of the following formula of Ramanujan [Bern2, p. 87]: For $a, b, c, d > 0$ with $a + b + 1 = c + d$, the function $f(x) = mF(a, b; d; 1 - x)/F(a, b; c; x)$, where $m = \Gamma(a)\Gamma(b)/(\Gamma(c)\Gamma(d))$, has the derivative

$$f'(x) = -[x^c(1-x)^d F(a, b; c; x)^2]^{-1}. \quad (2.21)$$

(3) The contiguous relations given in Section 2.6 have generalizations also for ${}_pF_q$ (see [Hen1, Hen2]).

2.7. Transformation formulas

The hypergeometric function satisfies so-called linear and quadratic transformation formulas, which can be derived from the integral representations. The linear transformation

formulas connect the values of $F(a, b; c; z)$ to those of $F(a_1, b_1; c_1; w)$, where w is a linear fractional transformation of z , and a_1, b_1, c_1 are determined by a, b, c [AS, 15.3]. For a list of quadratic transformation formulas, we refer to [AS, 15.4]. Some third-order transformation formulas were proved by Goursat [Gou]. We give some examples for such linear and quadratic transformation formulas below.

If a, b and c are such that the functions involved are well defined, $y = \sqrt{1-z}$ and $|\arg(1-z)| < \pi$, then

$$\begin{aligned}
 (1) \quad & F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \\
 & = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \\
 & = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right), \\
 (2) \quad & F(a, b; 2b; z) = \left(\frac{1+y}{2}\right)^{-2a} F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; \left(\frac{1-y}{1+y}\right)^2\right), \\
 (3) \quad & F\left(a, a+\frac{1}{2}; c; z\right) = \left(\frac{1+y}{2}\right)^{-2a} F\left(2a, 2a-c+1; c; \frac{1-y}{1+y}\right), \\
 (4) \quad & F\left(a, b; a+b+\frac{1}{2}; z\right) = F\left(2a, 2b; a+b+\frac{1}{2}; \frac{1-y}{2}\right).
 \end{aligned}$$

2.8. Properties of coefficients of hypergeometric series

In the study of the hypergeometric functions, some properties of the coefficients of the hypergeometric series will be useful. An example of the coefficient properties is the following lemma (see [AVV1, pp. 19–20] for its proof).

LEMMA 2.6. (1) For $a, b > 0$, the sequence $f(n) \equiv (a, n)(b, n)/[(a+b, n)(n-1)!]$ is strictly increasing to the limit $1/B(a, b)$.

(2) For $a, b \in (0, 1)$, the sequence $g(n) \equiv f(n)(n+1)/n$ is decreasing to the limit $1/B(a, b)$. For $a, b \in (1, \infty)$, the sequence $g(n)$ is increasing to the limit $1/B(a, b)$. In particular, for $n \in \mathbb{N}$,

$$\frac{ab}{a+b} \leq f(n) < \frac{1}{B(a, b)} \quad \text{for all } a, b > 0, \quad (2.22)$$

$$\frac{1}{B(a, b)} < g(n) \leq \frac{2ab}{a+b} \quad \text{for all } a, b \in (0, 1), \quad (2.23)$$

$$\frac{2ab}{a+b} \leq g(n) < \frac{1}{B(a, b)} \quad \text{for all } a, b \in (1, \infty), \quad (2.24)$$

where the weak inequalities reduce to equalities if and only if $n = 1$.

2.9. Asymptotic and monotonicity properties of $F(a, b; c; z)$

The behavior of the hypergeometric function $F(a, b; c; z)$ at $z = 1$ depends on the expression $a + b - c$ as follows. If $a + b - c < 0$, then the function is bounded, by (1) in Section 2.4. If $a + b - c > 0$, then we see by (1) in Section 2.7 that $(1 - z)^{a+b-c} F(a, b; c; z)$ is bounded. The next two theorems deal with these cases. For Theorem 2.7, we refer to [WW, p. 299], [AAR, p. 63]. A proof of Theorem 2.8 is given in [AVV1, pp. 21–23].

THEOREM 2.7. *The function $F(a, b; c; x)$ has the following properties:*

- (1) *If a, b and c are neither 0 nor a negative integer and if $a + b > c$, then $F(a, b; c; x)$ is asymptotic to $D(1 - x)^{c-a-b}$ as $x \rightarrow 1$, where $D = \Gamma(c)\Gamma(a + b - c)/[\Gamma(a)\Gamma(b)]$.*
- (2) *If a, b and $a + b$ are neither 0 nor a negative integer, then $F(a, b; a + b; x)$ is asymptotic to $E \log(1 - x)$ as $x \rightarrow 1$, where $E = -1/B(a, b)$.*

THEOREM 2.8. *For $a, b > 0$, let $B = B(a, b)$ and let*

$$R = R(a, b) = -\Psi(a) - \Psi(b) - 2\gamma, \quad (2.25)$$

where γ is the Euler–Mascheroni constant. Then

- (1) *The function $f_1(x) \equiv (F(a, b; a + b; x) - 1)/\log(1/(1 - x))$ is strictly increasing from $(0, 1)$ onto $(ab/(a + b), 1/B)$.*
- (2) *The function $f_2(x) \equiv BF(a, b; a + b; x) + \log(1 - x)$ is strictly decreasing from $(0, 1)$ onto (R, B) .*
- (3) *The function $f_3(x) \equiv BF(a, b; a + b; x) + (1/x)\log(1 - x)$ is increasing from $(0, 1)$ onto $(B - 1, R)$ if $a, b \in (0, 1)$, and is decreasing from $(0, 1)$ onto $(R, B - 1)$ if $a, b \in (1, \infty)$.*
- (4) *The function $f_4(x) \equiv xF(a, b; a + b; x)/\log[1/(1 - x)]$ is decreasing from $(0, 1)$ onto $(1/B, 1)$ if $a, b \in (0, 1)$, and is increasing from $(0, 1)$ onto $(1, 1/B)$ if $a, b > 1$, and $f_4(x) = 1$ for all $x \in (0, 1)$ if $a = b = 1$.*

REMARK 2.9. (1) The asymptotic relation of Gauss (Theorem 2.7(2)) has been extended and generalized in various ways. Ramanujan established the relation [Bern2, pp. 33–34]

$$B(a, b)F(a, b; a + b; x) + \log(1 - x) = R(a, b) + O((1 - x)\log(1 - x)) \quad (2.26)$$

as $x \rightarrow 1$, which is a special case of formula (2.28).

(2) Theorem 2.8 is related to (2.26) and also to Theorem 2.7(2). For example, from Theorem 2.8(2) and (3), it follows that

$$R(a, b) < g(x) < R(a, b) + \frac{1 - x}{x} \log \frac{1}{1 - x} \quad (2.27)$$

for $a, b \in (0, 1)$ and $x \in (0, 1)$, where $g(x) = B(a, b)F(a, b; a + b; x) + \log(1 - x)$. This gives a more precise form of (2.26) when $a, b \in (0, 1)$.

2.10. More expansions

The following asymptotic expansions are useful for one to see the poles of $F(a, b; c; z)$ when $c = a + b \pm m$, $m \in \mathbb{N} \cup \{0\}$, and to find the limits of certain functions defined in terms of $F(a, b; c; z)$ or to study some other properties of $F(a, b; c; z)$: If $|1 - z| < 1$ and $|\arg(1 - z)| < \pi$, then [AS, 15.3.10–15.3.12]

$$F(a, b; a + b; z) = \frac{1}{B(a, b)} \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(n!)^2} C_n (1 - z)^n, \quad (2.28)$$

where $C_n = 2\Psi(n + 1) - \Psi(a + n) - \Psi(b + n) - \log(1 - z)$,

$$\begin{aligned} &F(a, b; a + b + m; z) \\ &= \frac{\Gamma(m)\Gamma(a + b + m)}{\Gamma(a + m)\Gamma(b + m)} \sum_{n=0}^{m-1} \frac{(a, n)(b, n)}{(1 - m, n)n!} (1 - z)^n \\ &\quad + \frac{\Gamma(a + b + m)}{\Gamma(a)\Gamma(b)} (z - 1)^m \sum_{n=0}^{\infty} \frac{(a + m, n)(b + m, n)}{n!(n + m)!} D_n (1 - z)^n, \end{aligned} \quad (2.29)$$

where $D_n = \Psi(n + 1) + \Psi(n + m + 1) - \Psi(a + m + n) - \Psi(b + m + n) - \log(1 - z)$, $m \in \mathbb{N}$,

$$\begin{aligned} &F(a, b; a + b - m; z) \\ &= \frac{\Gamma(m)\Gamma(a + b - m)}{\Gamma(a)\Gamma(b)} (1 - z)^{-m} \sum_{n=0}^{m-1} \frac{(a - m, n)(b - m, n)}{(1 - m, n)n!} (1 - z)^n \\ &\quad + \frac{(-1)^m \Gamma(a + b - m)}{\Gamma(a - m)\Gamma(b - m)} \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{n!(n + m)!} E_n (1 - z)^n, \end{aligned} \quad (2.30)$$

where $E_n = \Psi(n + 1) + \Psi(n + m + 1) - \Psi(a + n) - \Psi(b + n) - \log(1 - z)$, $m \in \mathbb{N}$.

2.11. Historical remarks

The term “hypergeometric series” was first used by J. Wallis in 1656 to refer to a generalization of the geometric series [Dut]. Many leading mathematicians of the 18th and 19th centuries, such as Euler, Gauss, Jacobi, Kummer, Fuchs, Riemann, Schwarz and Klein (cf. [K11, K12]) contributed to the study of hypergeometric series. In 1873 Schwarz [Sch] solved the problem of finding those values of the parameters a, b, c for which the hypergeometric function $F(a, b; c; x)$ reduces to an algebraic function of its fourth argument. This solution yields a connection between hypergeometric differential equation and group theory

that has found many applications. In the beginning of the 20th century Ramanujan proved numerous results for these functions, for instance, the identity

$${}_2F_1(a, b; c; y) {}_2F_1(a, b; d; y) = {}_4F_3\left(a, b, \frac{a+b}{2}, \frac{c+d}{2}; c, d, a+b; x\right),$$

where $d = a + b + 1 - c$ and $y = (1 - \sqrt{1-x})/2$ [Bern2, p. 59], [Ask]. A particular case of this identity is the Clausen formula

$${}_2F_1\left(a, b; a+b+\frac{1}{2}; x\right)^2 = {}_3F_2\left(2a, 2b, a+b; a+b+\frac{1}{2}, 2a+2b; x\right),$$

which played a crucial role in the solution of the Bieberbach conjecture by de Branges (see [Hen3]).

2.12. Classification of functions

Functions that occur frequently, and are therefore useful, are often called special functions. In many cases, special functions are solutions of ordinary or partial differential equations or of extremal problems. Traditionally, special functions are divided into two classes: (a) elementary functions and (b) higher transcendental functions.

Furthermore, those higher transcendental functions that are solutions of ordinary differential equations can be classified as algebraically transcendental or transcendently transcendental. See Rubel [Ru]. An alternative new classification of transcendental functions, with interesting consequences, is introduced in [WZ].

NOTE 2.10. For the numerical evaluation of the hypergeometric function, see the bibliography of [LO] and [Mo].

3. Complete elliptic integrals

Complete elliptic integrals arise in many fields of mathematics as well as in many physical problems.

3.1. Definition of complete elliptic integrals

For $0 < r < 1$, the functions

$$\mathcal{K} = \mathcal{K}(r) \equiv \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2 \sin^2 t}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2 t^2)}} \quad (3.1)$$

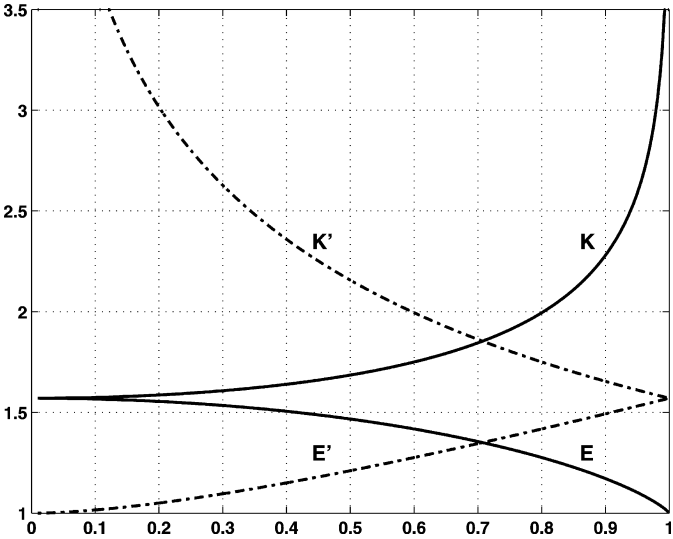


Fig. 2. Graphs of $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{E}$ and $\mathcal{E}'$.

and

$$\mathcal{E} = \mathcal{E}(r) \equiv \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t} \, dt = \int_0^1 \sqrt{\frac{1 - r^2 t^2}{1 - t^2}} \, dt \tag{3.2}$$

are known as *Legendre’s complete elliptic integrals of the first and second kind*, respectively. Let $\mathcal{K}(0) = \mathcal{E}(0) = \frac{\pi}{2}$, $\mathcal{K}(1-) = \infty$, and $\mathcal{E}(1) = 1$. The argument r is sometimes called *the modulus* of these integrals, and $r' = \sqrt{1 - r^2}$ its complement. We often denote (see Figure 2)

$$\mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \quad \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r').$$

3.2. Arc length of an ellipse

The perimeter $P(a, b)$ of the ellipse with semiaxes a and b , $a > b$, and eccentricity $e = (1/a)\sqrt{a^2 - b^2}$ is given by

$$P(a, b) = 2\pi a F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right) = 4a\mathcal{E}(e). \tag{3.3}$$

Many estimates have been obtained for $P(a, b)$. Some of them are given in [BPR].

3.3. Generalized elliptic integrals

For $r \in (0, 1)$, $a \in (0, 1)$ and $r' = \sqrt{1 - r^2}$, the generalized elliptic integrals are defined by

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1 - a; 1; r^2), \\ \mathcal{K}'_a = \mathcal{K}'_a(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \frac{\pi}{2}, \quad \mathcal{K}_a(1-) = \infty \end{cases} \quad (3.4)$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \frac{\pi}{2} F(a - 1, 1 - a; 1; r^2), \\ \mathcal{E}'_a = \mathcal{E}'_a(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \frac{\pi}{2}, \quad \mathcal{E}_a(1) = \frac{\sin(\pi a)}{2(1-a)}. \end{cases} \quad (3.5)$$

3.4. Identity and derivative formulas

It is clear that the complete integrals are special cases of $\mathcal{K}_a$ and $\mathcal{E}_a$ with $a = 1/2$. Therefore the formulas for $\mathcal{K}_a$ and $\mathcal{E}_a$ imply those for $\mathcal{K}$ and $\mathcal{E}$, and we below omit the latter. For instance, by Elliott's identity (2.20), the generalized elliptic integrals satisfy the remarkable identity

$$\mathcal{K}_a \mathcal{E}'_a + \mathcal{K}'_a \mathcal{E}_a - \mathcal{K}_a \mathcal{K}'_a = \frac{\pi \sin(\pi a)}{4(1-a)}. \quad (3.6)$$

Taking $a = 1/2$ in (3.6), we obtain Legendre's identity

$$\mathcal{K} \mathcal{E}' + \mathcal{K}' \mathcal{E} - \mathcal{K} \mathcal{K}' = \frac{\pi}{2}. \quad (3.7)$$

We now record the following derivative formulas, which follow from Theorem 2.3: For each $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$\frac{d\mathcal{K}_a}{dr} = \frac{2(1-a)}{rr'^2} (\mathcal{E}_a - r'^2 \mathcal{K}_a), \quad (3.8)$$

$$\frac{d\mathcal{E}_a}{dr} = \frac{2(a-1)}{r} (\mathcal{K}_a - \mathcal{E}_a), \quad (3.9)$$

$$\frac{d}{dr} (\mathcal{K}_a - \mathcal{E}_a) = \frac{2(1-a)r\mathcal{E}_a}{r'^2}, \quad (3.10)$$

$$\frac{d}{dr} (\mathcal{E}_a - r'^2 \mathcal{K}_a) = 2ar\mathcal{K}_a. \quad (3.11)$$

3.5. Particular values

By (3) in Section 2.4, we have

$$\mathcal{K}_a\left(\frac{1}{\sqrt{2}}\right) = \frac{c}{4\sqrt{\pi}} \sin(\pi a), \quad c = \Gamma\left(\frac{1-a}{2}\right) \Gamma\left(\frac{a}{2}\right), \quad (3.12)$$

while (3.12) and (3.6) yield

$$\mathcal{E}_a\left(\frac{1}{\sqrt{2}}\right) = \frac{4\pi^2 + (1-a)c^2 \sin(\pi a)}{8\sqrt{\pi}c(1-a)}. \quad (3.13)$$

Putting $a = 1/2$, we obtain

$$\begin{cases} \mathcal{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 = 1.85407467\dots, \\ \mathcal{E}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{8\sqrt{\pi}} [4\Gamma\left(\frac{3}{4}\right)^2 + \Gamma\left(\frac{1}{4}\right)^2] = 1.35064388\dots \end{cases} \quad (3.14)$$

3.6. Landen's identities

The functions $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the following identities due to Landen [BF, 163.01, 164.02]

$$\begin{cases} \mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), & \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1}{2}(1+r)\mathcal{K}'(r), \\ \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}-r^2\mathcal{K}}{1+r}, & \mathcal{E}\left(\frac{1-r}{1+r}\right) = \frac{\mathcal{E}'+r\mathcal{K}'}{1+r}. \end{cases} \quad (3.15)$$

The transformation $r \mapsto 2\sqrt{r}/(1+r)$ in the Landen identity is called *the ascending Landen transformation*, and its inverse $r \mapsto (1-r')/(1+r') = [r/(1+r')]^2$ is called *the descending Landen transformation*.

3.7. Elliptic integral algorithm

Numerical estimates for $\mathcal{K}(r)$ may be obtained very efficiently by the following recursive method. For $r \in (0, 1)$ let

$$\begin{cases} a_0 = 1, & b_0 = r', \\ a_{n+1} = \frac{a_n + b_n}{2}, & b_{n+1} = \sqrt{a_n b_n}. \end{cases} \quad (3.16)$$

Then the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ have the common limit $\pi/(2\mathcal{K}(r))$. The iteration is stopped when the desired accuracy is achieved. For more information about this algorithm and for a similar algorithm for $\mathcal{E}(r)$ see [AS, 17.6] and [AAR, p. 138]. We refer the reader to [Ki] for historical remarks concerning this algorithm.

3.8. The class Σ

In order to give an example of a concrete application of the elliptic integrals we consider the class of meromorphic functions in $|\zeta| > 1$ which are also univalent and have the expansion $F(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \dots$.

THEOREM 3.1 [Gol, p. 129]. *For a function $F \in \Sigma$ the following sharp inequality is valid for all $|\zeta| > 1$,*

$$\left| \frac{F(\zeta)}{\zeta} + 1 - \frac{1}{|\zeta|^2} - 2A \right| \leq 2 - 2A, \quad A = \mathcal{E}\left(\frac{1}{|\zeta|}\right) / \mathcal{K}\left(\frac{1}{|\zeta|}\right). \quad (3.17)$$

The inequality (3.17) determines the range of values of the quotient $F(\zeta)/\zeta$ for an arbitrary $|\zeta| > 1$.

3.9. Theta functions

Theta functions play an important role in the study of elliptic functions and elliptic integrals. They have been extensively studied since the beginning of the 19th century, and they often occur in mathematical physics and also in pure mathematics. In the literature there are several different notations used for them. For the sake of convenient reference, we give the definitions here [Law, pp. 5, 15]:

$$\begin{cases} \theta_1(z, q) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((2n+1)z), \\ \theta_2(z, q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos((2n+1)z), \\ \theta_3(z, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \\ \theta_4(z, q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz), \end{cases} \quad (3.18)$$

for all complex numbers z and q such that $|q| < 1$. Alternatively, these functions can be expressed as infinite products

$$\begin{cases} \theta_1(z, q) = 2q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos(2z) + q^{4n}), \\ \theta_2(z, q) = 2q^{1/4} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos(2z) + q^{4n}), \\ \theta_3(z, q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos(2z) + q^{4n-2}), \\ \theta_4(z, q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos(2z) + q^{4n-2}). \end{cases} \quad (3.19)$$

Jacobi and others found many formulas for theta functions; the classical results are summarized in [TM, Bat1, Bat2, Bat3, Law]. In order to demonstrate the connection of theta functions to elliptic integrals we give the following classical formula [AS, 16.38.5, p. 579], [BB, Theorem 2.1, p. 35], [Ra, pp. 325, 343]:

$$\sqrt{\frac{2\mathcal{K}(r)}{\pi}} = \theta_3(0, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2}, \quad (3.20)$$

where $q = \exp(-\pi \mathcal{K}(r')/\mathcal{K}(r))$ is the so-called *Jacobi's nome*.

3.10. Incomplete elliptic integrals

For $\varphi \geq 0$ and $r \in (0, 1)$, the incomplete elliptic integrals of the first and second kind are defined by

$$\mathcal{F}(\varphi, r) = \int_0^\varphi \frac{d\vartheta}{\sqrt{1-r^2 \sin^2 \vartheta}}, \quad \mathcal{E}(\varphi, r) = \int_0^\varphi \sqrt{1-r^2 \sin^2 \vartheta} d\vartheta,$$

respectively. Algorithms for the computation of these functions are given in [AS, 17.5]. When $r = 1$, these functions reduce to

$$\mathcal{F}(\varphi, 1) = \int_0^\varphi \frac{d\vartheta}{\cos \vartheta} = \log \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right) \quad \text{and} \quad \mathcal{E}(\varphi, 1) = \sin \varphi.$$

3.11. Historical remarks

The theory of elliptic integrals had its beginnings in 1718 with Fagnano's work on the computation of the arc length of a lemniscate [C2, p. 3], [Hou,PSo], and was developed by the 18th-century mathematicians Euler, Lagrange and Landen. In the 19th century, Gauss, Abel, Legendre and Jacobi made significant discoveries about elliptic integrals and their inverses, the elliptic functions. Numerous books discuss elliptic integrals and elliptic functions, for example, [AS,Bat1,Bat2,Bat3,BB,BF,Bo,Cay,Ch,C2,En,Gr,Ha,Hen2,Law,WW].

4. Quotients of elliptic integrals

A ring is a plane domain such that its complement consists of two components both containing at least two distinct points. It is well known that a ring may be mapped conformally onto the annulus $\{z \in \mathbb{C}: 1 < |z| < t\}$ for some $t > 1$; the modulus of the ring is defined to be $\log t$. The *Grötzsch ring* is the unit disk minus a radial slit along the real axis from the origin to a point $r \in (0, 1)$. The Grötzsch ring is one of the canonical ring domains that occurs frequently in the study of conformal invariants and quasiconformal maps. We next give a formula for its modulus. By the basic properties of $\mathcal{K}(r)$, the normalized quotient

$$\mu(r) = \frac{\pi \mathcal{K}'(r)}{2 \mathcal{K}(r)} \tag{4.1}$$

is a strictly decreasing homeomorphism of $(0, 1)$ onto $(0, \infty)$. This function represents the modulus of the Grötzsch ring and also has applications in number theory as we can see later. We always let $r' = \sqrt{1-r^2}$ for $r \in (0, 1)$. A graph of $\mu(r)$ appears in Figure 3.

4.1. Identities and derivative formulas

The function $\mu(r)$ satisfies the following identities

$$\mu(r)\mu(r') = \frac{\pi^2}{4}, \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right). \quad (4.2)$$

From the first and second identities or from the third identity, it follows that

$$\mu(r) = \frac{1}{2}\mu\left(\left(\frac{r}{1+r'}\right)^2\right) = \frac{1}{2}\mu\left(\frac{1-r'}{1+r'}\right). \quad (4.3)$$

Using the derivative formula for $\mathcal{K}(r)$ and Legendre's relation, we obtain

$$\frac{d\mu}{dr} = -\frac{\pi^2}{4rr^2\mathcal{K}(r)^2}, \quad \frac{d^2\mu}{dr^2} = \frac{\pi^2[2\mathcal{E}(r) - (1+r^2)\mathcal{K}(r)]}{4r^2r'^4\mathcal{K}(r)^3}. \quad (4.4)$$

Numerous properties have been obtained for $\mu(r)$. As examples, we gather some of them in the next theorem (see [AVV1] for the details). We denote arth by arth .

THEOREM 4.1. (1) *The function $\mu(r)$ is strictly decreasing from $(0, 1)$ onto $(0, \infty)$, has exactly one inflection point, and satisfies $\mu'(0+) = -\infty = \mu'(1-)$.*

(2) *The function $f(r) \equiv \mu(r) + \log r$ is strictly decreasing and concave from $(0, 1)$ onto $(0, \log 4)$ and satisfies $f'(0+) = 0$, $f'(1-) = -\infty$.*

(3) *The function $g(r) \equiv \mu(r) + \log(r/r')$ is strictly increasing and convex from $(0, 1)$ onto $(\log 4, \infty)$ and satisfies $g'(0+) = 0$, $g'(1-) = \infty$.*

(4) *The function $h(r) \equiv \mu(r) + \log[r/(1+r')]$ is strictly decreasing and concave from $(0, 1)$ onto $(0, \log 2)$.*

(5) *The function $F(r) \equiv \mu(r)/[\sqrt{r'}\log(4/r)]$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.*

(6) *The function $G(r) \equiv \mu(r)/\operatorname{arth} \sqrt[4]{r'}$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.*

(7) *The function $H(r) \equiv \mu(r)\operatorname{arth} \sqrt[4]{r}$ is strictly increasing from $(0, 1)$ onto $(0, \pi^2/4)$.*

(8) *The function $I(r) \equiv \mu(r) - \operatorname{arth} \sqrt{r'}$ is strictly decreasing and concave from $(0, 1)$ onto $(0, (1/2)\log 2)$.*

From Theorem 4.1 we obtain easily several inequalities for $\mu(r)$. For instance, part (4) gives, for $r \in (0, 1)$,

$$\log \frac{1+r'}{r} < \mu(r) < \log \frac{2(1+r')}{r} \quad (4.5)$$

while part (7) refines the lower bound in (4.5) to

$$\mu(r) > \operatorname{arth} \sqrt[4]{r'} > \log \frac{(1+\sqrt{r'})^2}{r} > \log \frac{1+3r'}{r}. \quad (4.6)$$

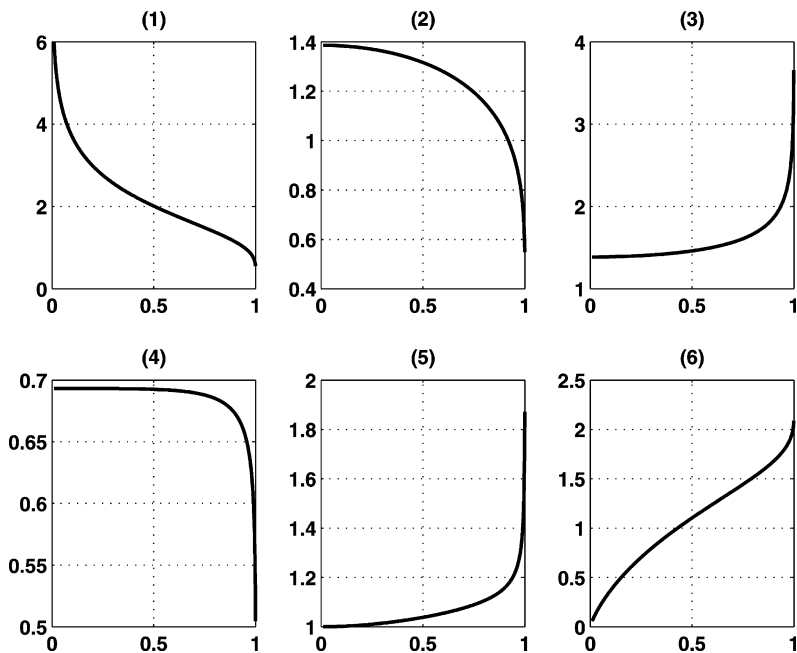


Fig. 3. (1) $\mu(r)$, (2) $\mu(r) + \log r$, (3) $\mu(r) + \log(r/r')$, (4) $\mu(r) + \log(r/(1+r'))$, (5) $\mu(r)/(\sqrt{r'} \log(4/r))$, (6) $\mu(r) \operatorname{arth} \sqrt{r}$.

From Theorem 4.1 we also see that $\mu(r)$ has a logarithmic singularity at $r = 0$, and hence it behaves “logarithmically” in the neighborhood of this singularity. The next theorem shows that $\mu(r)$ satisfies functional inequalities reminiscent of the basic property $\log(ab) = \log a + \log b$ of the logarithm. A proof of this theorem is given in [AVV1, pp. 83–84].

THEOREM 4.2. For $a, b \in (0, 1)$,

$$(1) \quad \mu(a) + \mu(b) \leq \mu\left(\frac{ab}{(1+a')(1+b')}\right) \leq 2\mu(\sqrt{ab}),$$

$$(2) \quad \mu\left(\frac{ab}{(1+c)^2}\right) \leq \mu(a) + \mu(b),$$

where $c = \min\{a', b'\}$. Equality holds in each case if and only if $a = b$.

The inequalities (4.5) and (4.6) are adequate for many approximation purposes. Note that arbitrarily good approximations to $\mu(r)$ can be obtained if we use the elliptic integral algorithm 3.7. We now indicate another approximation procedure. Fix $r \in (0, 1)$ and set

$r_0 = r$, $r_{k+1} = (r_k/(1+r'_k))^2$, $k = 0, 1, 2, \dots$. Then it follows from (4.5) and (4.3) that $\mu(r_{k+1}) = 2^{k+1}\mu(r)$ (cf. [AVV1, p. 81], [LV]) and

$$2^{-k-1} \log \frac{1}{r_{k+1}} < \mu(r) < 2^{-k-1} \log \frac{4}{r_{k+1}} \quad (4.7)$$

which implies

$$\mu(r) = \lim_{k \rightarrow \infty} 2^{-k-1} \log(4/r_{k+1}). \quad (4.8)$$

4.2. Expansions

In his foundational paper [Ja] on elliptic functions, Jacobi introduced the notation of an elliptic function and studied these functions using so-called theta series. A fundamental tool for his investigations was the use of infinite product expansions involving complete elliptic integrals. His work contains dozens of such expansions, two of which are given in the next theorem [Ja, pp. 148, 160]. Note that such expansions are, in general, valid for a restricted set of complex values of the argument, but we state these two expansions only for the case when the argument is real and between 0 and 1. We denote $\tanh$ by th .

THEOREM 4.3. *For $r \in (0, 1)$, $q = \exp(-2\mu(r))$, we have*

$$\begin{aligned} (1) \quad r &= 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^4, \\ (2) \quad r' &= \prod_{n=1}^{\infty} \left(\frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^4 = \prod_{n=1}^{\infty} \text{th}^4((2n-1)\mu(r)). \end{aligned}$$

COROLLARY 4.4. *For $r \in (0, 1)$, let $y = \mu(r)$. Then*

$$\begin{aligned} (1) \quad \mu^{-1}(y) &= 4e^{-y} \prod_{n=1}^{\infty} \left(\frac{1+e^{-4ny}}{1+e^{-(4n-2)y}} \right)^4, \\ (2) \quad \mu^{-1}(y)^2 &= 1 - \prod_{n=1}^{\infty} \text{th}^8((2n-1)y). \end{aligned}$$

In particular, for $y > 0$,

$$\sqrt{1 - \text{th}^8 y} < \mu^{-1}(y) < 4e^{-y} \left(\frac{1+e^{-4y}}{1+e^{-2y}} \right)^4. \quad (4.9)$$

Above we indicated two procedures for the numerical approximation of $\mu(r)$. For the approximation of $\mu^{-1}(y)$ we may use Corollary 4.4. A second method, presented in [AVV1, 5.32, p. 92] makes use of the Newton iteration and the derivative formula (4.4).

The next theorem gives a useful expansion for $\mu(r)$. For its proof, see, e.g., [QVu].

THEOREM 4.5. *For $r \in (0, 1)$, $n \in \mathbb{N}$, let $r_0 = r'$ and $r_n = 2\sqrt{r_{n-1}}/(1 + r_{n-1})$. Then*

$$\exp(\mu(r) + \log r) = \prod_{n=0}^{\infty} (1 + r_n)^{2^{-n}}. \quad (4.10)$$

4.3. Generalization of $\mu(r)$

Consider the decreasing homeomorphism $\mu_a : (0, 1) \rightarrow (0, \infty)$ defined by

$$\mu_a(r) \equiv \frac{\pi}{2 \sin(\pi a)} \frac{\mathcal{K}'_a(r)}{\mathcal{K}_a(r)} \quad (4.11)$$

for $a \in (0, 1/2]$. This function is a generalization of $\mu(r)$ since $\mu(r) = \mu_{1/2}(r)$, and occurs in the generalized modular equations, as shown later. Many properties of $\mu(r)$ have been extended to $\mu_a(r)$. We only record the following differentiation formula

$$\frac{d\mu_a(r)}{dr} = -\frac{\pi^2}{4rr'^2\mathcal{K}_a(r)^2}, \quad \frac{d\mu_a^{-1}(x)}{dx} = -\frac{4rr'^2\mathcal{K}_a(r)^2}{\pi^2}, \quad (4.12)$$

where $r = \mu_a^{-1}(x)$. For the details, see [AQVV].

4.4. The φ -distortion function

For $K > 0$ and $r \in [0, 1]$, let

$$\begin{cases} \varphi_K(r) = \mu^{-1}(\mu(r)/K) & \text{for } r \in (0, 1), \\ \varphi_K(0) = \varphi_K(1) - 1 = 0. \end{cases} \quad (4.13)$$

This function, called *the Hersch–Pfluger φ -distortion function*, plays an important role in quasiconformal theory, and also occurs in the theory of Ramanujan's modular equations. Its generalization

$$\varphi_K^a(r) \equiv \mu_a^{-1}(\mu_a(r)/K) \quad \text{for } a \in (0, 1/2] \quad (4.14)$$

occurs in the generalized modular equations. It follows from (4.12) that for $a \in (0, 1/2]$, $r \in (0, 1)$ and $K \in (0, \infty)$,

$$\frac{\partial s}{\partial r} = \frac{1}{K} \frac{ss'^2\mathcal{K}_a(s)^2}{rr'^2\mathcal{K}_a(r)^2}, \quad \frac{\partial s}{\partial K} = \frac{4}{(\pi K)^2} ss'^2\mathcal{K}_a(s)^2\mu_a(r), \quad (4.15)$$

where $s = \varphi_K^a(r)$. The partial derivatives of $\varphi_K(r)$ follow if we take $a = 1/2$ in (4.15).

4.5. Modular equations

For $p \in \mathbb{N}$, $r, s \in (0, 1)$, the equation

$$\mathcal{K}'(s)/\mathcal{K}(s) = p\mathcal{K}'(r)/\mathcal{K}(r) \quad (4.16)$$

is called *the (classical) modular equation of degree (or order) p* , while *the generalized modular equation with signature $1/a$ and degree (or order) p* is

$$\mathcal{K}'_a(s)/\mathcal{K}_a(s) = p\mathcal{K}'_a(r)/\mathcal{K}_a(r) \quad (4.17)$$

for $r, s \in (0, 1)$, $a \in (0, 1/2]$ and $p > 0$. Clearly, (4.16) and (4.17) can be rewritten as

$$\mu(s) = p\mu(r) \quad \text{and} \quad \mu_a(s) = p\mu_a(r), \quad (4.18)$$

respectively, so that their solutions are given by

$$s = \varphi_K(r) \quad \text{and} \quad s = \varphi_K^a(r), \quad K = 1/p, \quad (4.19)$$

respectively.

Modular equations were studied extensively by Ramanujan (see [BeBG]), who also gave numerous algebraic identities for the solution s for some rational values of a such as $1/6$, $1/4$ and $1/3$. As p increases, the solution becomes more involved, and there are no known explicit algebraic expressions for the solution of (4.18) for an arbitrary positive integer p . Many authors, however, have found algebraic identities satisfied by the solution when p is a small prime number (cf. [BB, pp. 103–109] and [BeBG, AQVV]).

4.6. Singular values

For each positive integer p , there exists a unique number $k_p \in (0, 1)$ such that

$$\mu(k_p) = \pi\sqrt{p}/2. \quad (4.20)$$

This number k_p is algebraic and is called *the p th singular value* of $\mathcal{K}(r)$, and can be rewritten as

$$k_p = \varphi_{1/\sqrt{p}}(1/\sqrt{2}). \quad (4.21)$$

Singular values have applications in number theory [SeC, BB], and many such values have been found explicitly. For $p = 1, \dots, 9$, the values of k_p are given in [BB, p. 139]. The values of $\mathcal{K}(k_p)$, $p = 1, \dots, 16$, appear in [BB, p. 298]; from these and (4.20) one can obtain $\mathcal{K}'(k_p)$, $p = 1, \dots, 16$.

4.7. Schwarz' lemma for quasiconformal maps

In the study of conformal invariants the special function $\mu(r)$ has several important applications. In particular, the study of quasiconformal maps often uses these functions, see [KK,LV]. The next result is a quasiconformal counterpart of the Schwarz lemma.

THEOREM 4.6 [LV, Section 10]. *Let $K \geq 1$ and let f be a K -quasiconformal mapping of the unit disk into itself with $f(0) = 0$. Then, for all $|z| < 1$,*

$$|f(z)| \leq \varphi_K(|z|). \quad (4.22)$$

From the basic properties of the function $\mu(r)$, (4.2) and (4.3), one may derive identities for $\varphi_K(r)$ as well. Following [AVV1, Theorem 10.5, p. 204] we list some of these.

For $K > 0$ and $r, s \in [0, 1]$,

$$\varphi_K(r)^2 + \varphi_{1/K}(s)^2 = 1 \quad \text{iff } r^2 + s^2 = 1, \quad (4.23)$$

$$\varphi_{1/K}(s) = \frac{1 - \varphi_K(r)}{1 + \varphi_K(r)} \quad \text{iff } s = \frac{1 - r}{1 + r}, \quad (4.24)$$

$$\varphi_K(s) = \frac{2\sqrt{\varphi_K(r)}}{1 + \varphi_K(r)} \quad \text{iff } s = \frac{2\sqrt{r}}{1 + r}, \quad (4.25)$$

$$\varphi_2(r) = \frac{2\sqrt{r}}{1 + r} \quad \text{and} \quad \varphi_{1/2}(r) = \left(\frac{r}{1 + r'} \right)^2 = \frac{1 - r'}{1 + r'}. \quad (4.26)$$

REMARK 4.7. (1) Many inequalities for $\varphi_K(r)$ and $\varphi_K^a(r)$ have been obtained. By [AVV1, Theorem 5.43, p. 93] we know for instance that $r^{1/K} \leq \varphi_K(r) \leq 4^{1-1/K} r^{1/K}$. The fact that $\varphi_{1/2}(r)$ and $\varphi_2(r)$ are the descending and ascending Landen transformations, respectively, suggests that there is a relation between $\varphi_K(r)$ and these transformations. One can also use this idea to derive bounds for $\varphi_K(r)$. For details, see [AVV1, AQVV].

(2) In the context of modular equations, the quotient $\mathcal{K}(r')/\mathcal{K}(r) = 2\mu(r)/\pi$ is a basic quantity in many papers of the 19th century.

The function $\varphi_K(r)$ also occurs in Schottky's theorem [AVV1, 10.16], [Hay, pp. 684–688].

5. Elliptic functions

In this section, we begin by defining the Jacobian elliptic functions, which are very useful in the study of conformal maps and invariants as well as in some other fields. A list of their basic properties follows. We also briefly mention the alternative definition of the Jacobian elliptic functions in terms of the theta functions. This connection is the source of numerous beautiful identities, for which the classical literature on elliptic functions, such as Jacobi's pioneering work [Ja], has become so famous.

5.1. Doubly-periodic functions

A nonconstant complex-valued function f of a complex variable z is said to be *periodic* if there is a nonzero complex number ω such that $f(z + \omega) = f(z)$ for all z , in which case ω is called a *period* of f . The sine, cosine, and tangent functions are familiar examples of periodic functions. A period ω is called *primitive* if $k\omega$ is not a period for $0 < k < 1$. If a function f has two periods ω_1, ω_2 whose ratio has nonzero imaginary part, then f is said to be *doubly-periodic*; and for each z , the parallelogram with vertices $z, z + \omega_1, z + \omega_2, z + \omega_1 + \omega_2$ is called a *period parallelogram* of f . If the interior of this period parallelogram contains no periods of f , then it is called a *fundamental period parallelogram* of f . In this case the complex plane is partitioned into a network of congruent period parallelograms determined by two systems of lines with directions ω_1, ω_2 .

5.2. Elliptic functions

A nonconstant doubly-periodic function which is meromorphic in the finite plane is called an *elliptic function*. Clearly such a function can have at most finitely many poles or zeros in any period parallelogram. Therefore, for contour integration, we may choose a period parallelogram P so that there are no poles or zeros on the boundary of P . Such a period parallelogram will be called a *cell*. If f is an elliptic function, we call its *order* the number of its poles in each cell, counted according to multiplicity. It is clear that a nonconstant elliptic function must have order at least one, since otherwise it would reduce to a constant by Liouville's theorem. In fact the order must be at least two.

THEOREM 5.1. *If f is an elliptic function, then the sum of the residues at its poles in any cell is zero. In particular, the order of each nonconstant elliptic function is at least two.*

An elliptic function of order two may have either a double pole or two simple poles in each cell. The Jacobian elliptic functions, which are defined below, are of order two with two simple poles in each cell. By Theorem 5.1, the residues at these poles must be negatives of each other. Throughout this section, we let $\mathcal{K} = \mathcal{K}(r)$, $\mathcal{K}' = \mathcal{K}'(r)$, and $r' = \sqrt{1 - r^2}$ for any $r \in [0, 1]$.

DEFINITION 5.2. Given $r \in (0, 1)$, the inverse Jacobian elliptic sine function sn^{-1} is first defined on $[-1, 1]$ as the incomplete elliptic integral of the first kind:

(1)

$$u = \operatorname{sn}^{-1} x = \operatorname{sn}^{-1}(x, r) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}} = \int_0^\vartheta \frac{dt}{\sqrt{1-r^2 \sin^2 t}},$$

$x = \sin \vartheta$, $\vartheta \in [-\pi/2, \pi/2]$, $x \in [-1, 1]$. Then sn^{-1} is a bijection from $[-1, 1]$ onto $[-\mathcal{K}, \mathcal{K}]$, so that the Jacobian elliptic sine function sn is defined on $[-\mathcal{K}, \mathcal{K}]$ as the inverse of sn^{-1} , that is,

$$(2) \operatorname{sn} u = \operatorname{sn}(u, r) = x = \sin \vartheta.$$

The functions $\operatorname{cn} u$, $\operatorname{dn} u$ and $\operatorname{tn} u$ are then defined on $[-\mathcal{K}, \mathcal{K}]$ by:

$$(3) \operatorname{cn} u = \operatorname{cn}(u, r) = \sqrt{1 - \operatorname{sn}^2 u},$$

$$(4) \operatorname{dn} u = \operatorname{dn}(u, r) = \sqrt{1 - r^2 \operatorname{sn}^2 u},$$

$$(5) \operatorname{tn} u = \operatorname{tn}(u, r) = \operatorname{sn} u / \operatorname{cn} u.$$

The graphs of sn , cn , dn are given in Figure 4.

5.3. Special values

Clearly,

$$\operatorname{sn} 0 = 0, \quad \operatorname{cn} 0 = 1, \quad \operatorname{dn} 0 = 1, \quad \operatorname{tn} 0 = 0, \quad (5.1)$$

$$\operatorname{sn} \mathcal{K} = 1, \quad \operatorname{cn} \mathcal{K} = 0, \quad \operatorname{dn} \mathcal{K} = r', \quad \operatorname{tn} \mathcal{K} = \infty, \quad (5.2)$$

$$\operatorname{sn}(u, 0) = \sin u, \quad \operatorname{cn}(u, 0) = \cos u, \quad \operatorname{dn}(u, 0) = 1, \quad \operatorname{tn}(u, 0) = \tan u, \quad (5.3)$$

$$\operatorname{sn}(u, 1) = \operatorname{th} u, \quad \operatorname{cn}(u, 1) = 1 / \operatorname{ch} u = \operatorname{dn}(u, 1), \quad \operatorname{tn}(u, 1) = \operatorname{sh} u, \quad (5.4)$$

where sh and ch stand for $\sinh$ and $\cosh$, respectively.

5.4. Squared relations

By Definition 5.2, the following identities hold on $[-\mathcal{K}, \mathcal{K}]$:

$$\begin{cases} \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, & \operatorname{dn}^2 u + r^2 \operatorname{sn}^2 u = 1, \\ r^2 \operatorname{cn}^2 u + r'^2 = \operatorname{dn}^2 u, & \operatorname{cn}^2 u + r'^2 \operatorname{sn}^2 u = \operatorname{dn}^2 u. \end{cases} \quad (5.5)$$

5.5. Derivatives

From Definition 5.2, it follows that

$$(1) \quad \frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \quad \frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u,$$

$$(2) \quad \frac{d}{du} \operatorname{dn} u = -r^2 \operatorname{sn} u \operatorname{cn} u, \quad \frac{d}{du} \operatorname{tn} u = \operatorname{dn} u / \operatorname{cn}^2 u.$$

REMARK 5.3. Glaisher [WW, p. 494] introduced the notation:

$$\operatorname{ns} u \equiv 1 / \operatorname{sn} u, \quad \operatorname{nc} u \equiv 1 / \operatorname{cn} u, \quad \operatorname{nd} u \equiv 1 / \operatorname{dn} u,$$

$$\operatorname{sc} u \equiv \operatorname{tn} u, \quad \operatorname{sd} u \equiv \operatorname{sn} u / \operatorname{dn} u, \quad \operatorname{cd} u \equiv \operatorname{cn} u / \operatorname{dn} u,$$

$$\operatorname{cs} u \equiv 1 / \operatorname{tn} u, \quad \operatorname{ds} u \equiv \operatorname{dn} u / \operatorname{sn} u, \quad \operatorname{dc} u \equiv \operatorname{dn} u / \operatorname{cn} u.$$

5.6. Addition formulas

For $u, v, u + v \in [-\mathcal{K}, \mathcal{K}]$,

$$\begin{aligned} (1) \quad \operatorname{sn}(u + v) &= \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - r^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ (2) \quad \operatorname{cn}(u + v) &= \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - r^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ (3) \quad \operatorname{dn}(u + v) &= \frac{\operatorname{dn} u \operatorname{dn} v - r^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - r^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \\ (4) \quad \operatorname{tn}(u + v) &= \frac{\operatorname{tn} u \operatorname{dn} v + \operatorname{tn} v \operatorname{dn} u}{1 - \operatorname{tn} u \operatorname{dn} v \operatorname{tn} v \operatorname{dn} u}. \end{aligned}$$

5.7. Double and half arguments

From Section 5.6, it follows that

$$\begin{aligned} (1) \quad \operatorname{sn} 2u &= \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - r^2 \operatorname{sn}^4 u}, \\ (2) \quad \operatorname{cn} 2u &= \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u}{1 - r^2 \operatorname{sn}^4 u}, \\ (3) \quad \operatorname{dn} 2u &= \frac{\operatorname{dn}^2 u - r^2 \operatorname{sn}^2 u \operatorname{cn}^2 u}{1 - r^2 \operatorname{sn}^4 u}, \\ (4) \quad \operatorname{tn} 2u &= \frac{2 \operatorname{tn} u \operatorname{dn} u}{1 - \operatorname{tn}^2 u \operatorname{dn}^2 u}. \end{aligned}$$

By (2), (3) and Section 5.4, we obtain:

$$\begin{aligned} (1) \quad \operatorname{sn}^2 \frac{u}{2} &= \frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}, \\ (2) \quad \operatorname{cn}^2 \frac{u}{2} &= \frac{\operatorname{dn} u + \operatorname{cn} u}{1 + \operatorname{dn} u}, \\ (3) \quad \operatorname{dn}^2 \frac{u}{2} &= \frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}, \\ (4) \quad \operatorname{tn}^2 \frac{u}{2} &= \frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}. \end{aligned}$$

In particular, we have

$$\operatorname{sn} \frac{\mathcal{K}}{2} = \frac{1}{\sqrt{1 + r'}}, \quad \operatorname{cn} \frac{\mathcal{K}}{2} = \sqrt{\frac{r'}{1 + r'}}, \quad \operatorname{dn} \frac{\mathcal{K}}{2} = \sqrt{r'}. \quad (5.6)$$

5.8. Jacobi's imaginary transformation

If we let $\sin t = i \tan \xi$ in Definition 5.2(1), then

$$\cos t = \sec \xi, \quad dt = i \sec \xi d\xi, \quad \sin \xi = -i \tan t, \quad \cos \xi = \sec t,$$

and

$$\frac{dt}{\sqrt{1-r^2 \sin^2 t}} = \frac{i d\xi}{\sqrt{1-r'^2 \sin^2 \xi}}.$$

Writing

$$\int_0^\vartheta \frac{dt}{\sqrt{1-r^2 \sin^2 t}} = i \int_0^\zeta \frac{d\xi}{\sqrt{1-r'^2 \sin^2 \xi}} = iu,$$

where $\sin \vartheta = i \tan \zeta$, we have $\operatorname{sn}(u, r) = \sin \zeta$ and $\operatorname{sn}(iu, r) = \sin \vartheta$. This motivates the following definition for imaginary argument: For $u \in [-\mathcal{K}, \mathcal{K}]$,

$$(1) \quad \operatorname{sn}(iu, r) = \frac{i \operatorname{sn}(u, r')}{\operatorname{cn}(u, r')},$$

$$(2) \quad \operatorname{cn}(iu, r) = \frac{1}{\operatorname{cn}(u, r')},$$

$$(3) \quad \operatorname{dn}(iu, r) = \frac{\operatorname{dn}(u, r')}{\operatorname{cn}(u, r')},$$

$$(4) \quad \operatorname{tn}(iu, r) = i \operatorname{sn}(u, r').$$

5.9. Complex arguments

By combining the addition formulas in Section 5.6 with the imaginary transformation in Section 5.8, we write

$$(1) \quad \operatorname{sn}(u + iv, r) = \frac{\operatorname{sn}(u, r) \operatorname{dn}(v, r') + i \operatorname{cn}(u, r) \operatorname{dn}(u, r) \operatorname{sn}(v, r') \operatorname{cn}(v, r')}{1 - \operatorname{sn}^2(v, r') \operatorname{dn}^2(u, r)},$$

$$(2) \quad \operatorname{cn}(u + iv, r) = \frac{\operatorname{cn}(u, r) \operatorname{cn}(v, r') - i \operatorname{sn}(u, r) \operatorname{dn}(u, r) \operatorname{sn}(v, r') \operatorname{dn}(v, r')}{1 - \operatorname{sn}^2(v, r') \operatorname{dn}^2(u, r)},$$

$$(3) \quad \operatorname{dn}(u + iv, r) = \frac{\operatorname{dn}(u, r) \operatorname{cn}(v, r') \operatorname{dn}(v, r') - ir^2 \operatorname{sn}(u, r) \operatorname{cn}(u, r) \operatorname{sn}(v, r')}{1 - \operatorname{sn}^2(v, r') \operatorname{dn}^2(u, r)}.$$

We remark that these formulas hold even if u and v are any complex numbers.

5.10. Moduli of the elliptic functions

By (1) in Section 5.9 and (5.4),

$$(1) \quad |\operatorname{sn}(u + iv, r)|^2 = \frac{1 - \operatorname{cn}^2(u, r) \operatorname{cn}^2(v, r')}{1 - \operatorname{dn}^2(u, r) \operatorname{sn}^2(v, r')},$$

$$(2) \quad |\operatorname{cn}(u + iv, r)|^2 = \frac{1 - \operatorname{sn}^2(u, r) \operatorname{dn}^2(v, r')}{1 - \operatorname{dn}^2(u, r) \operatorname{sn}^2(v, r')},$$

$$(3) \quad |\operatorname{dn}(u + iv, r)|^2 = \frac{\operatorname{dn}^2(v, r') - r^2 \operatorname{sn}^2(u, r)}{1 - \operatorname{dn}^2(u, r) \operatorname{sn}^2(v, r')}.$$

5.11. Periodicity properties

Using Sections 5.6 and 5.9, we can obtain the following conclusions: sn has periods $4\mathcal{K}$, $2i\mathcal{K}'$; cn has periods $4\mathcal{K}$, $2\mathcal{K} + 2i\mathcal{K}'$; and dn has periods $2\mathcal{K}$, $4i\mathcal{K}$.

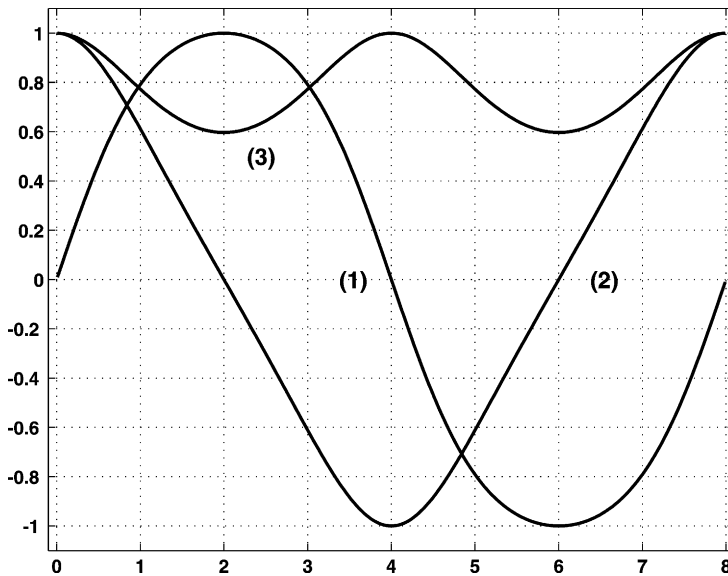


Fig. 4. Graphs of (1) $\operatorname{sn}(u, r)$, (2) $\operatorname{cn}(u, r)$ and (3) $\operatorname{dn}(u, r)$, $0 \leq u \leq 4\mathcal{K}(r)$, with $r = \mathcal{K}^{-1}(2) \approx 0.802406$. Notice that the graphs of $\operatorname{sn} u$ and $\operatorname{cn} u$ do not have the same shape, unlike those of their trigonometric analogs $\sin u$ and $\cos u$.

Table 1.
Table of periods, zeros

Function	Zeros	Poles (Residues)
sn	$0, 2\mathcal{K}$	$i\mathcal{K}' (1/r), \quad 2\mathcal{K} + i\mathcal{K}' (-1/r)$
cn	$\pm\mathcal{K}$	$i\mathcal{K}' (-i/r), \quad 2\mathcal{K} + i\mathcal{K}' (i/r)$
dn	$\mathcal{K} \pm i\mathcal{K}'$	$i\mathcal{K}' (-i), \quad -i\mathcal{K}' (i)$

5.12. Maclaurin’s series

By evaluating successive derivatives at $u = 0$, we obtain the following Maclaurin expansions of these elliptic functions:

(1)
$$\begin{aligned} \operatorname{sn} u &= u - (1 + r^2) \frac{u^3}{3!} + (1 + 14r^2 + r^4) \frac{u^5}{5!} \\ &\quad - \frac{(1 + 135r^2 + 135r^4 + r^6)u^7}{7!} + O(u^9), \end{aligned}$$

(2)
$$\operatorname{cn} u = 1 - \frac{u^2}{2!} + (1 + 4r^2) \frac{u^4}{4!} - (1 + 44r^2 + 16r^4) \frac{u^6}{6!} + O(u^8),$$

(3)
$$\operatorname{dn} u = 1 - r^2 \frac{u^2}{2!} + (4r^2 + r^4) \frac{u^4}{4!} - (16r^2 + 44r^4 + r^6) \frac{u^6}{6!} + O(u^8).$$

These expansions are valid for $|u| < \mathcal{K}'$.

5.13. Poles, residues and zeros

The poles, residues and zeros of sn, cn and dn in a fundamental period parallelogram are stated in Table 1. For the details, see [LeVa].

Each of these zeros and poles is simple.

5.14. Landen’s transformations

Substituting $t = (1 + r)x/(1 + rx^2)$ in Definition 5.2(1), and letting $r_1 = 2\sqrt{r}/(1 - r)$, we obtain

$$\int_0^t \frac{dt}{\sqrt{(1 - t^2)(1 - r_1^2 t^2)}} = (1 + r) \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - r^2 x^2)}},$$

so that

(1)
$$\operatorname{sn}\left((1 + r)u, \frac{2\sqrt{r}}{1 + r}\right) = \frac{(1 + r) \operatorname{sn}(u, r)}{1 + r \operatorname{sn}^2(u, r)}.$$

This, together with (3) and (4) in Definition 5.2, yields

$$(2) \quad \operatorname{cn}\left((1+r)u, \frac{2\sqrt{r}}{1+r}\right) = \frac{\operatorname{cn}(u, r) \operatorname{dn}(u, r)}{1+r \operatorname{sn}^2(u, r)},$$

$$(3) \quad \operatorname{dn}\left((1+r)u, \frac{2\sqrt{r}}{1+r}\right) = \frac{1-r \operatorname{sn}^2(u, r)}{1+r \operatorname{sn}^2(u, r)}.$$

Similarly the change of variable $t = (1+r')x\sqrt{1-x^2}/\sqrt{1-r^2x^2}$ gives:

$$(4) \quad \operatorname{sn}\left((1+r')u, \frac{1-r'}{1+r'}\right) = \frac{(1+r') \operatorname{sn}(u, r) \operatorname{cn}(u, r)}{\operatorname{dn}(u, r)},$$

$$(5) \quad \operatorname{cn}\left((1+r')u, \frac{1-r'}{1+r'}\right) = \frac{1-(1+r') \operatorname{sn}^2(u, r)}{\operatorname{dn}(u, r)},$$

$$(6) \quad \operatorname{dn}\left((1+r')u, \frac{1-r'}{1+r'}\right) = \frac{1-(1-r') \operatorname{sn}^2(u, r)}{\operatorname{dn}(u, r)}.$$

5.15. Theta function formulas for elliptic functions

The functions sn , cn , dn may be defined also as ratios of theta functions as follows:

$$\begin{cases} \operatorname{sn} u = \frac{\theta_3(0, q)\theta_1(z, q)}{\theta_2(0, q)\theta_4(z, q)}, \\ \operatorname{cn} u = \frac{\theta_4(0, q)\theta_2(z, q)}{\theta_2(0, q)\theta_4(z, q)}, \\ \operatorname{dn} u = \frac{\theta_4(0, q)\theta_3(z, q)}{\theta_3(0, q)\theta_4(z, q)}, \end{cases} \quad (5.7)$$

where $z = u/\theta_3(0, q)^2 \in \mathbb{C}$, $q = e^{-2\mu(r)}$, and $r \in (-1, 1)$ [Law, p. 24].

5.16. Occurrence in applications

Elliptic functions are used extensively in conformal mapping problems in physics and in Geometric Function Theory [Hen1, Ko, KS, Law, Neh, TD], and have applications to approximation theory [Ak].

5.17. Historical remarks

At the beginning of the 19th century, elliptic functions were discovered independently and almost simultaneously by Abel and Jacobi. The theta functions mentioned above in Section 5.15 and extensively studied by Jacobi were an essential tool in his work on elliptic functions. The diaries of Gauss indicate that he may have proved, in unpublished notes,

some results for the theory of elliptic functions before the works of Abel and Jacobi were published [Bü, pp. 87–90], [C1, pp. 497–498]. The development of elliptic functions is interwoven with the theory of elliptic integrals [MM]. Significant contributions to elliptic functions were also made by Klein and Weierstrass. A well-known treatise on these functions was written by Tannery and Molk [TM]. Valuable sources of historical data on elliptic functions are [Ch,Hou,K12].

5.18. Computer algorithms for elliptic functions

Computer programs for the computation of the theta functions are given in [Law] and [Thom]. With such programs, one can obtain numerical approximations for the Jacobian elliptic functions. See also [LO,Mo] and [ZJ].

References

- [AS] M. Abramowitz and I.A. Stegun (eds), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York (1965).
- [Ah1] L.V. Ahlfors, *Complex Analysis*, 3rd edn, McGraw-Hill, New York (1966).
- [Ah2] L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [Ak] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI (1990).
- [Alz] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.
- [ABRVV] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), 1713–1723.
- [AnQ] G.D. Anderson and S.-L. Qiu, *A monotonicity property of the gamma function*, Proc. Amer. Math. Soc. **125** (1997), 3355–3362.
- [AQVV] G.D. Anderson, S.-L. Qiu, M.K. Vamanamurthy and M. Vuorinen, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000), 1–37.
- [AVV1] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Mappings*, Wiley, New York (1997) (book with a software diskette).
- [AVV2] G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen, *Topics in special functions*, Papers on Analysis: A volume dedicated to Olli Martio on the occasion of his 60th birthday, J. Heinonen, T. Kilpeläinen and P. Koskela, eds, Report 83, Univ. Jyväskylä (2001), 5–26, ISBN 951-39-1120-9. Available at <http://www.math.jyu.fi/research/report83.html>.
- [AAR] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia Math. Appl., Vol. 71, Cambridge Univ. Press, Cambridge (1999).
- [Ar] E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, New York (1964).
- [Ask] R. Askey, *Variants of Clausen's formula for the square of a special ${}_2F_1$* , Number Theory and Related Topics, Tata Inst. Fund. Res., Bombay (1988), 1–12.
- [BPR] R.W. Barnard, K. Pearce and K.C. Richards, *A monotonicity property involving ${}_3F_2$ and comparisons of the classical approximations of elliptical arc length*, SIAM J. Math. Anal. **32** (2000), 403–419 (electronic).
- [Bern1] B.C. Berndt, *Ramanujan's Notebooks*, Vol. I, Springer-Verlag, Berlin (1985).
- [Bern2] B.C. Berndt, *Ramanujan's Notebooks*, Vol. II, Springer-Verlag, Berlin (1989).
- [Bern3] B.C. Berndt, *Ramanujan's Notebooks*, Vol. III, Springer-Verlag, Berlin (1991).
- [Bern4] B.C. Berndt, *Ramanujan's Notebooks*, Vol. IV, Springer-Verlag, Berlin (1993).
- [BeBG] B.C. Berndt, S. Bhargava and F.G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), 4163–4244.

- [BB] J.M. Borwein and P.B. Borwein, *Pi and the AGM*, Wiley, New York (1987).
- [Bot] U. Bottazzini, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*, Springer-Verlag, Berlin (1986).
- [Bo] F. Bowman, *Introduction to Elliptic Functions with Applications*, Dover, New York (1961).
- [Bü] W.K. Bühler, *Gauss, A Biographical Study*, Springer-Verlag, Berlin (1981).
- [BF] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd edn, Grundlehren Math. Wiss., Bd. 67, Springer-Verlag, Berlin (1971).
- [Cara1] C. Carathéodory, *Conformal Representation*, 2nd edn, Cambridge Tracts in Math., No. 28, Cambridge Univ. Press, Cambridge (1952).
- [Cara2] C. Carathéodory, *Theory of Functions of a Complex Variable*, Vols I and II, Chelsea, New York (1954).
- [C1] B.C. Carlson, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.
- [C2] B.C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York (1977).
- [C3] B.C. Carlson, *A table of elliptic integrals of the second kind*, Math. Comp. **49** (1987), 595–606, S13–S17.
- [Cay] A. Cayley, *An Elementary Treatise on Elliptic Functions*, Deighton, Bell and Sons, Cambridge (1876).
- [Ch] K. Chandrasekharan, *Elliptic Functions*, Grundlehren Math. Wiss., Bd. 281, Springer-Verlag, Berlin (1985).
- [D] P.J. Davis, *Leonard Euler's integral: A historical profile of the gamma function*, Amer. Math. Monthly **66** (1956), 849–869.
- [Dut] J. Dutka, *The early history of the hypergeometric function*, Arch. Hist. Exact Sci. **31** (1984–1985), 15–34.
- [El] E.B. Elliott, *A formula including Legendre's $EK' + KE' - KK' = \frac{1}{2}\pi$* , Messenger of Math. **33** (1904), 31–40.
- [En] A. Enneper, *Elliptische Functionen, Theorie und Geschichte*, 2nd edn, Louis Nebert, Halle an der Saale (1890).
- [Bat1] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi (eds), *Higher Transcendental Functions*, Vol. I, Bateman Manuscript Project, McGraw-Hill, New York (1953).
- [Bat2] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi (eds), *Higher Transcendental Functions*, Vol. II, Bateman Manuscript Project, McGraw-Hill, New York (1953).
- [Bat3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi (eds), *Higher Transcendental Functions*, Vol. III, Bateman Manuscript Project, McGraw-Hill, New York (1955).
- [Gai] D. Gaier, *Vorlesungen über Approximation im Komplexen*, Birkhäuser, Basel (1980).
- [Ga] C.F. Gauss, *Werke*, Hildesheim G. Olms, New York (1973).
- [Gau] W. Gautschi, *Computational methods in special functions – A survey*, Theory and Applications of Special Functions, R.A. Askey, ed., Academic Press, New York (1975), 1–98.
- [Gol] G.M. Golusin, *Geometrische Funktionentheorie*, Hochschulbücher für Mathematik, Bd. 31, VEB Deutscher Verlag Wiss., Berlin (1957) (in German).
- [Goo] A.W. Goodman, *Univalent Functions*, Vols I and II, Mariner, Tampa, FL (1983).
- [Gou] E. Goursat, *Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique*, Ann. Sci. École Norm. Sup. **10** (2) (1881), 3–142.
- [Gr] A.G. Greenhill, *The Applications of Elliptic Functions*, Dover, New York (1959).
- [Ha] H. Hancock, *Elliptic Integrals*, Wiley, New York (1917).
- [Hay] W.K. Hayman, *Subharmonic functions*, Vol. 2, London Math. Soc. Monogr., Vol. 20, Academic Press (Harcourt Brace Jovanovich, Publishers), London (1989).
- [Hen1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. I, Wiley, New York (1974).
- [Hen2] P. Henrici, *Applied and Computational Complex Analysis*, Vol. II, Wiley, New York (1977).
- [Hen3] P. Henrici, *Applied and Computational Complex Analysis*, Vol. III, Wiley, New York (1986).
- [Hö] O. Hölder, *Ueber die Eigenschaft der Gammafunction, keiner algebraischen Differentialgleichung zu genügen*, Math. Ann. **28** (1887), 1–13.
- [Hou] Ch. Houzel, *Elliptische Funktionen und Abelsche Integrale*, Geschichte Mathematik 1700–1900, J. Dieudonné, ed., VEB Deutscher Verlag Wiss., Berlin (1985), 422–539.

- [Ja] C.G.J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum* (1829), Jacobi's Gesammelte Werke, Vol. 1, Chelsea, New York (1969) (in Latin) (reprint of 1881–1891 edn).
- [J] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, corr. edn, *Ergeb. Math. Grenzgeb.*, Bd. 18, Springer-Verlag, Berlin–Heidelberg–New York (1965).
- [Ki] L.V. King, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge Univ. Press, Cambridge (1924).
- [K11] F. Klein, *Vorlesungen über die hypergeometrische Funktionen*, Teubner, Berlin (1933).
- [K12] F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Springer-Verlag, Berlin (1979).
- [Ko] H. Kober, *Dictionary of Conformal Representations*, 2nd edn, Dover, New York (1957).
- [KS] W. von Koppenfels and F. Stallmann, *Praxis der konformen Abbildung*, *Grundlehren Math. Wiss.*, Bd. 100, Springer-Verlag, Berlin (1959).
- [KK] S.L. Krushkal and R.Kühnau, *Quasiconformal Maps – New Methods and Applications*, Nauka, Novosibirsk (1984) (in Russian); Teubner, Leipzig (1983) (in German).
- [Küh] R. Kühnau, *Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag Wiss., Berlin (1974).
- [Kum] E.E. Kummer, *Über die hypergeometrische Reihe*, *J. Reine Angew. Math.* **15** (1836), 39–83, 127–172.
- [K1] G.V. Kuz'mina, *Moduli of Families of Curves and Quadratic Differentials*, *Trudy Mat. Inst. Steklov* **139** (1980); English transl.: *Proc. Steklov Inst. Math.* (1982), no. 1.
- [K2] G.V. Kuz'mina, *Methods of the geometric theory of functions. I, II, Erratum*, *Algebra i Analiz* **9** (1997), (3) 41–103, (5) 1–50; *Transl. St. Petersburg Math. J.* **9** (3) (1998), 455–507, *Algebra i Analiz* **9** (5) (1998), 889–930; *Transl. St. Petersburg Math. J.* **9** (5) (1998), 889–930, *Algebra i Analiz* **10** (3) (1998), 223; *Transl. St. Petersburg Math. J.* **10** (3) (1999), 577.
- [Law] D.F. Lawden, *Elliptic Functions and Applications*, *Appl. Math. Sci.*, Vol. 80, Springer-Verlag, New York (1989).
- [LeVa] W. Ledermann and S. Vajda, *Analysis, Handbook of Applicable Mathematics*, Vol. IV, Wiley, New York (1982).
- [LV] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edn, *Grundlehren Math. Wiss.*, Bd. 126, Springer-Verlag, New York (1973).
- [LO] D.W. Lozier and F.W.J. Olver, *Numerical evaluation of special functions*, *Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics*, W. Gautschi, ed., *Proc. Symp. Appl. Math.*, Vol. 48, Amer. Math. Soc., Providence, RI (1994), 79–125.
- [MM] H. McKean and V. Moll, *Elliptic Curves. Function Theory, Geometry, Arithmetic*, Cambridge Univ. Press, Cambridge (1997).
- [Mit] D.S. Mitrinović, *Analytic inequalities*, *Grundlehren Math. Wiss.*, Bd. 165, Springer-Verlag, Berlin (1970).
- [Mo] S.L. Moshier, *Methods and Programs for Mathematical Functions*, Horwood, Chichester (1989).
- [Neh] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York (1952).
- [PSo] V.V. Prasolov and Yu.P. Solov'yev, *Elliptic Functions and Elliptic Integrals*, *Transl. Math. Monographs*, Vol. 170, Amer. Math. Soc., Providence, RI (1997).
- [Pr] R.T. Prosser, *On the Kummer solutions of the hypergeometric equation*, *Amer. Math. Monthly* **101** (1994), 535–543.
- [PBM] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series, Vol. 3: More Special Functions*; *Transl. from the Russian by G.G. Gould, Gordon and Breach*, New York (1988); erratum: *Math. Comp.* **65** (1996), 1380–1384.
- [QVu] S.-L. Qiu and M. Vuorinen, *Infinite products and the normalized quotients of hypergeometric functions*, *SIAM J. Math. Anal.* **30** (5) (1999), 1057–1075.
- [Ra] E.D. Rainville, *Special Functions*, Macmillan, New York (1960).
- [Ram] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Springer-Verlag, New York (1988).
- [Ru] L.A. Rubel, *A survey of transcendently transcendental functions*, *Amer. Math. Monthly* **96** (1989), 777–788.
- [Sch] H.A. Schwarz, *Über diejenigen Fälle, in welchen die Gauss'sche hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt*, *J. Reine Angew. Math.* **75** (1873), 292–335.
- [SeC] A. Selberg and S. Chowla, *On Epstein's zeta function*, *J. Reine Angew. Math.* **227** (1967), 87–110.

- [SO] J. Spanier and K.B. Oldham, *An Atlas of Functions*, Hemisphere, New York (1987).
- [St] K.R. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA (1981).
- [TM] J. Tannery and J. Molk, *Functions Elliptiques*, Vols 1–4, Chelsea, New York (1972).
- [Tem] N. Temme, *Special functions, An Introduction to the Classical Functions of Mathematical Physics*, Wiley (1996).
- [Thom] W.J. Thompson, *Atlas for computing mathematical functions*, Wiley, Berlin (1997); Incl. 1 CD-ROM (with programs in C and Mathematica).
- [Ti] E.C. Titchmarsh, *The Theory of Functions*, 2nd edn, Oxford Univ. Press, London (1939).
- [Tot] V. Totik, *On Hölder's theorem that $\Gamma(x)$ does not satisfy algebraic differential equation*, Acta Sci. Math. (Szeged) **57** (1993), 495–496.
- [TD] L.N. Trefethen and T.A. Driscoll, *Schwarz–Christoffel mapping in the computer era*, Proc. Intern. Congress of Mathematicians, Vol. III, Berlin, 1998, Doc. Math., Extra Vol. III (1998), 533–542 (electronic).
- [Var] V.S. Varadarajan, *Linear meromorphic differential equations: A modern point of view*, Bull. Amer. Math. Soc. **33** (1996), 1–42.
- [Vu] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin (1988).
- [Wa] P. Walker, *Elliptic Functions, A Constructive Approach*, Wiley, New York (1996).
- [WW] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th edn, Cambridge Univ. Press, London (1958).
- [WZ] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities*, Invent. Math. **108** (1992), 575–633.
- [ZJ] S. Zhang and J. Jin, *Computation of Special Functions*, Wiley, New York (1996).

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CHAPTER 15

**Extremal Functions in Geometric Function Theory.
Higher Transcendental Functions. Inequalities**

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1. Introduction

If we have an extremal problem in a class of schlicht conformal mappings $w = w(z)$ of the unit disk $|z| < 1$ or of $|z| > 1$ (classes $\mathcal{S}$ and Σ), then of special interest are those cases in which the extremal mappings are slit mappings, where the slits are prescribed by a quadratic differential $Q(w)dw^2$ in form of

$$Q(w)dw^2 \geq 0. \quad (1.1)$$

Here $Q(w)$ is a rational function, where the poles and the order of the poles are in a simple way connected with the problem. This quadratic differential in these extremal problems of Grötzsch–Teichmüller type appear also with the general variational method of Schiffer.

Because the transformed quadratic differential $\tilde{Q}(z)dz^2$ in the z -plane satisfies at the unit circle the condition

$$\tilde{Q}(z)dz^2 \geq 0,$$

it can be constructed with a rational function $\tilde{Q}(z)$. So we arrive at an ordinary differential equation for the extremal functions $w(z)$ with the solution

$$\int^w \sqrt{Q(w)}dw = \int^z \sqrt{\tilde{Q}(z)}dz \quad (1.2)$$

(cf., e.g., [12]). This yields an implicit representation of $w(z)$. The appearance of accessory parameters is of course a great difficulty. And the possibility to get in the following a final result in form of a concrete (sharp) inequality for the corresponding functional depends essentially on the character of the integrals in (1.2). Of course the right-hand side integral in (1.2) is more complicated because the integrand has generally more singularities than the integrand on the left-hand side.

We have 3 possibilities for these integrals:

- (i) the integral represents an elementary function;
- (ii) the integral represents a “higher transcendental function”;
- (iii) for the integral does not exist a representation with these functions.

Fortunately we have for the most interesting extremal problems generally the case (i). But there are many cases in which higher transcendental functions occur.

An additional interesting thing is in many extremal problems the phenomenon that we obtain in the final (sharp) inequality different analytic expressions for different parameters. This is very surprising for the first moment. But the reason is simply the appearance of zeroes of the quadratic differential $Q(w)dw^2$ on the boundary slit as the image. Historically the first nice example was probably the famous Golusin “Drehungssatz” [3, p. 110], [5, Theorem 3.4].

If we replace the given domain $|z| < 1$ or $|z| > 1$ in the z -plane by an annulus, then the construction of $\tilde{Q}(z)$ in (1.2) generally needs elliptic functions, and the things become much more complicated.

There are many other situations in Geometric Function Theory in which special functions occur. In the case of classes of quasiconformal mappings the situation is a little bit another, although we have again a description of extremal mappings with quadratic differentials. Then a usual derivative of the mappings does not exist. But if we consider, e.g., the class $\mathcal{S}(Q)$ of those conformal mappings $w(z)$ of the class $\mathcal{S}$ which permit a continuous Q -quasiconformal extension for $|z| < 1$, then in the “conformal part” of the mappings also derivatives (and higher coefficients) are possible. For extremal functions in these mapping classes, which are also linked with quadratic differentials, we have again a general representation, but a much more complicated one than in (1.2); cf. [10, Part II, Chapter III]. Only in a few cases elementary functions are sufficient for the representation of the extremal functions.

In Section 2 we refer in examples about the appearance of higher transcendental functions in Geometric Function Theory, not only in connection with the above mentioned theory of extremal functions. A collection of such higher transcendental functions is listed in [30], with many references.

We mention also some problems of another type in which higher transcendental functions occur. The calculation of the conformal module of some special ring domains needs often elliptic integrals and elliptic functions [2]. And there are of course many applications of higher transcendental functions in connection with the classical Schwarz–Christoffel integral for the conformal mapping of polygonal domains or circular polygons (cf., e.g., [9,12,20]; of special interest is the doubly-connected case which usually needs elliptic functions), or in connection with the conformal mapping of triangulated Riemannian manifolds, defined as a set of triangles with a corresponding set of linear sewings (cf. [13], where, e.g., hypergeometric functions occur). We will leave this aside here.

Our collection in Section 2 is, of course, not exhaustive. Generally speaking, it is remarkable that elliptic integrals are the most frequently appearing higher transcendental functions in Geometric Function Theory. On the other side, it is striking that several higher transcendental functions (e.g., Bessel functions) practically did not appear until now.

2. Some examples of special functions in Geometric Function Theory

Elliptic integrals and elliptic functions. While the range of the values $f(z)$, for a fixed z , in the class $\mathcal{S}$ can be prescribed with elementary functions [3, Chapter IV, Section 1], this range of values in the class Σ needs the complete elliptic integrals $\mathcal{K}$, $\mathcal{E}$ of the first and of the second kind; cf. (3.1).

Further examples with $\mathcal{K}$, $\mathcal{E}$ and other elliptic integrals: [8, p. 424], [12, footnote 14], [21–23], [10, pp. 141–144], several papers of Kuz'mina (e.g., [26]) and of Solynin, e.g., [31,32]; in [24,25] in combination with the Jacobi functions sn , cn , etc.

Of a special significance in Geometric Function Theory is the so-called Grötzsch ring. This is the doubly connected domain between the unit circle and the segment $0, \dots, r$ ($0 < r < 1$). The corresponding conformal module is given by

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}. \quad (2.1)$$

And therefore this yields an extremely important application of elliptic integrals. Here we have an exhaustive collection of properties, inequalities, applications, etc. in the books [2,29] and in the article [30].

Closely related to the Grötzsch ring are, e.g., the Teichmüller ring and the Mori ring [2,29]. Here we mention also the application in the Teichmüller Verschiebungssatz [34] (there is a simple error in the calculation on p. 343, lower half: e.g., in the formula before the footnote the factor 2 should be deleted).

The *elliptic modular function*, *elliptic integrals of the first kind* and the *Weierstraß $\wp$ -function* occur already in [4] to prescribe the range of the values of the cross-ratio of the images of 4 fixed points in classes of conformal mappings; later many papers on special cases and limit cases without knowledge of [4]. In [11] the concrete formulas (additionally with hyperelliptic integrals) in the case of the mappings of the class Σ ; in [12, p. 107] for the mappings of an annulus. Also related is the analogous problem for quasiconformal mappings [33] (cf. also [1,15]). An application was given in [17]: Calculation of those Jordan curves through 4 given fixed points, for which the reflection coefficient has its smallest value.

Hypergeometric functions (and more general functions) appear in [14] in connection with extremal problems for functions of the class Σ with a quasiconformal extension. But here the dilatation bound is not a constant but depends on $|z|$ for $|z| < 1$.

Euler's Γ -function and the *Psi-function* $\psi = \Gamma'/\Gamma$ appear, e.g., in [10, p. 129], [27] and in several other papers of Kuz'mina and Solynin.

Orthogonal polynomials, *hypergeometric series*, *Euler's beta integral* and the *gamma function* appear also in the first proof of de Branges for the Bieberbach conjecture [6, p. 604].

Lobachevski's function, *Euler's dilogarithm* and *Clausen's integral* appear in [21].

Solutions of a *Riccati differential equation* occur in [19].

3. A curiosity: Deriving real inequalities from results in Geometric Function Theory

Here we will sketch how it is possible to obtain inequalities for higher transcendental functions from inequalities in Geometric Function Theory. The idea is quite simple. If we have an inequality for a functional in a class of (conformal or quasiconformal) mappings, then we obtain an inequality for the involved, e.g., special higher transcendental functions, if we insert a special, not too complicated mapping. We can add an inequality in the other direction if we have an arbitrary nonsharp estimate of our functional.

We will illustrate this with an example.

In [3, Chapter IV, Section 3] the following sharp inequality for the mappings $F(\zeta)$ of the class Σ was given:

$$\left| \frac{F(\zeta)}{\zeta} + 1 - \frac{1}{|\zeta|^2} - 2 \frac{\mathcal{E}(1/|\zeta|)}{\mathcal{K}(1/|\zeta|)} \right| \leq 2 - 2 \frac{\mathcal{E}(1/|\zeta|)}{\mathcal{K}(1/|\zeta|)}. \quad (3.1)$$

Here $\mathcal{K}$, $\mathcal{E}$ denote the complete elliptic integrals of the first kind and of the second kind. (For the history of (3.1), which goes back to Grötzsch, cf. [16]; other proofs of (3.1)

in [7, p. 97], [28, p. 251].) The inequality (3.1) shows that the possible values $F(\zeta)$ for fixed $\zeta = \zeta^*$ fill a closed disk. All boundary points are attained for exactly one mapping. The extremal mapping $F(\zeta)$, which corresponds to the point of this disk with the smallest real value, transforms $|\zeta| = 1$ onto a parabolic slit symmetric to the real axis and with the focal point $F(\zeta^*)$. By using the mapping

$$F(\zeta) = \zeta - \frac{1}{\zeta}, \quad (3.2)$$

as a (very rough) approximation of this slit mapping, we obtain for $\zeta = \zeta^* = \frac{1}{k} > 1$ from (3.1) (used only with the real part) the rough inequality

$$\frac{\mathcal{E}(k)}{\mathcal{K}(k)} \leq 1 - \frac{k^2}{2}, \quad 0 < k < 1. \quad (3.3)$$

We can expect that we will get a much more better estimate if we use a “better” mapping $F(\zeta)$, that means a slit mapping, for which the slit is a better approximation of the parabolic slit. For this reason we use now a circular slit mapping:

$$F(\zeta) = \zeta \frac{\zeta - \zeta_0}{\zeta - \frac{1}{\zeta_0}} + \zeta_0 - \frac{1}{\zeta_0} \in \Sigma \quad (3.4)$$

with a “suitable” $\zeta_0 > 1$. For $\zeta^* = \frac{1+\zeta_0}{2}$, that means by the choice $\zeta_0 = 2\zeta^* - 1$, we get a circular slit with the same curvature as our parabolic slit at the vertex, in the limit case $\zeta^* \rightarrow \infty$. This leaves us, by using of this “better” mapping (3.4), for $\zeta = \zeta^* = \frac{1}{k}$ after (3.1) with

$$\frac{F(\zeta^*)}{\zeta^*} + 3 - \frac{1}{\zeta^{*2}} = \frac{16\zeta^{*4} - 12\zeta^{*2} + 1}{(4\zeta^{*2} - 1)\zeta^{*2}} \geq 4 \frac{\mathcal{E}(k)}{\mathcal{K}(k)},$$

therefore

$$\frac{\mathcal{E}(k)}{\mathcal{K}(k)} \leq \frac{1}{4} \frac{16 - 12k^2 + k^4}{4 - k^2} \quad \text{for } 0 < k < 1. \quad (3.5)$$

Here the difference between both sides, e.g., for $k = 0.5$ is smaller than $4 \cdot 10^{-4}$, for $k = 0.1$ smaller than $2 \cdot 10^{-8}$.

In a retrospect it is not immediately to answer the question: What did we use altogether in proving (3.5)?

To obtain an estimate in the other direction we use the simple inequality

$$|F(\zeta) - \zeta| \leq \frac{3}{|\zeta|} \quad \text{for } F(\zeta) \in \Sigma. \quad (3.6)$$

This follows with the maximum principle by using the function $\zeta[F(\zeta) - \zeta]$ (analytically also at $\zeta = \infty$) and $|F(\zeta)| \leq 2$ for $|\zeta| = 1$.

If we use in (3.6) the above-mentioned parabolic slit mapping $F(\zeta)$ with

$$F(\zeta^*) - \zeta^* = -4\zeta^* + \frac{1}{\zeta^*} + 4\zeta^* \frac{\mathcal{E}(1/\zeta^*)}{\mathcal{K}(1/\zeta^*)}, \quad \zeta^* = \frac{1}{k},$$

we obtain the desired rough estimate in the other direction:

$$\frac{\mathcal{E}(k)}{\mathcal{K}(k)} \geq 1 - k^2 \quad \text{for } 0 < k < 1. \quad (3.7)$$

We add the *general remark*, that it is possible to prove inequalities for expressions in higher transcendental functions (e.g., also (3.5)), to get simpler expressions in elementary functions, with a simple algorithm [18] by using a computer. (This yields a strong mathematical proof, not only for a finite number of parameters.) This sometimes can be useful if the sharp estimate for a functional in a class of mappings is too complicated.

Further we add as a curiosity the remark that it is possible to use inequalities in Geometric Function Theory, in which higher transcendental functions are involved, also in the other direction, namely for a variational characterization of these functions. We explain this again with (3.1). This inequality can be read also as follows: The values of the function $4\mathcal{E}(k)/\mathcal{K}(k)$ can be characterized for all fixed k ($0 < k < 1$) as the infimum of all values

$$3 - k^2 + k \Re F(1/k) \quad (3.8)$$

in the class of all mappings $F \in \Sigma$.

Finally we add the remark that often distinct expressions in higher transcendental functions appear if we solve an extremal problem in Geometric Function Theory with different methods. This means that we have proved an identity in higher transcendental functions, if both inequalities are sharp; cf. as an example (3.1) with Corollary 6.11 in [7].

References

- [1] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, NJ (1966).
- [2] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Mappings*, Wiley, New York (1997).
- [3] G.M. Golusin, *Geometrische Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1957); 2nd edn of the Russian original: Nauka, Moskva (1966); English transl.: Amer. Math. Soc., Providence, RI (1969).
- [4] H. Grötzsch, *Die Werte des Doppelverhältnisses bei schlichter konformer Abbildung*, Sitz. Preuß. Akad. Wiss. Phys.-Math. Kl. (1933), 501–515.
- [5] W.K. Hayman, *Univalent and multivalent functions*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier, Amsterdam (2002), 1–36.
- [6] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 3, Wiley, New York (1986).
- [7] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1958).
- [8] J.A. Jenkins, *The method of the extremal metric*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, Elsevier, Amsterdam (2002), 393–456.

- [9] W. von Koppenfels and F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1959).
- [10] S.L. Krushkal and R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner, Leipzig (1983); in Russian: Nauka Sibirsk. Otd., Novosibirsk (1984).
- [11] R. Kühnau, *Berechnung einer Extremalfunktion der konformen Abbildung*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **9** (1960), 285–287.
- [12] R. Kühnau, *Über die analytische Darstellung von Abbildungsfunktionen, insbesondere von Extremalfunktionen der Theorie der konformen Abbildung*, J. Reine Angew. Math. **228** (1967), 93–132.
- [13] R. Kühnau, *Triangulierte riemannsche Mannigfaltigkeiten mit ganz-linearen Bezugssubstitutionen und quasikonforme Abbildungen mit stückweise konstanter komplexer Dilation*, Math. Nachr. **46** (1970), 243–261.
- [14] R. Kühnau, *Extremalprobleme bei quasikonformen Abbildungen mit kreisringweise konstanter Dilatationsbeschränkung*, Math. Nachr. **66** (1975), 269–282.
- [15] R. Kühnau, *Über die Werte des Doppelverhältnisses bei quasikonformer Abbildung*, Math. Nachr. **95** (1980), 237–251.
- [16] R. Kühnau, *Zur ebenen Potentialströmung um einen porösen Kreiszylinder*, Z. Angew. Math. Phys. **40** (1989), 395–409.
- [17] R. Kühnau, *Möglichst konforme Jordankurven durch vier Punkte*, Rev. Roumaine Math. Pures Appl. **36** (1991), 383–393.
- [18] R. Kühnau, *Eine Methode, die Positivität einer Funktion zu prüfen*, Z. Angew. Math. Mech. **74** (1994), 140–142.
- [19] R. Kühnau, *Ersetzungssätze bei quasikonformen Abbildungen*, Ann. Univ. Mariae Curie-Skłodowska Lublin Sect. A **52** (1998), 65–72.
- [20] R. Kühnau, *Bibliography of Geometric Function Theory (GFT)*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), this Volume.
- [21] R. Kühnau and E. Hoy, *Abschätzung des Wertebereichs einiger Funktionale bei quasikonformen Abbildungen*, Bull. Soc. Sci. Lett. Łódź **29** (1979), 4, 1–9.
- [22] R. Kühnau and E. Hoy, *Bemerkungen über quasikonform fortsetzbare schlichte konforme Abbildungen*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **31** (1982), 129–133.
- [23] R. Kühnau and W. Niske, *Abschätzung des dritten Koeffizienten bei den quasikonform fortsetzbaren schlichten Funktionen der Klasse S* , Math. Nachr. **78** (1977), 185–192.
- [24] R. Kühnau and B. Thüring, *Berechnung einer quasikonformen Extremalfunktion*, Math. Nachr. **79** (1977), 99–113.
- [25] G.V. Kuz'mina, *Moduli of Families of Curves and Quadratic Differentials*, Proc. Steklov Inst. Math. **139** (1982) (in Russian); English transl.: Amer. Math. Soc., Providence, RI (1980).
- [26] G.V. Kuz'mina, *The module problem for families of classes of curves in a circular ring*, Zap. Nauchn. Sem. Leningr. Otdel. Mat. Inst. Steklov **144** (1985), 115–127 (in Russian).
- [27] G.V. Kuz'mina, *Methods of Geometric Function Theory, II*, St. Petersburg Math. J. **9** (1998), 889–930.
- [28] N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Nauka, Moskva (1972) (in Russian).
- [29] O. Lehto and K.I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edn, Springer-Verlag, Berlin–Heidelberg–New York (1973).
- [30] S.-L. Qiu and M. Vuorinen, *Special functions in Geometric Function Theory*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), 621–659 (this Volume).
- [31] A.Yu. Solynin, *Solution of the Pólya–Szegő isoperimetric problem*, Zap. Nauchn. Sem. LOMI **168** (1988), 140–153; J. Soviet. Math. **53** (1991), 311–320.
- [32] A.Yu. Solynin, *Modules and extremal metric problems*, St. Petersburg Math. J. **11** (1) (2000), 1–65.
- [33] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuß. Akad. Wiss. Math.-Nat. Kl. **22** (1939), 1–179.
- [34] O. Teichmüller, *Ein Verschiebungssatz der quasikonformen Abbildung*, Deutsche Math. **7** (1944), 336–343.

CHAPTER 16

Eigenvalue Problems and Conformal Mapping

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1. Introduction

Conformal maps are a powerful tool for solving eigenvalue problems in the plane. Sometimes one is interested in isoperimetric inequalities, sometimes one looks for approximate methods in order to obtain upper and lower bounds. One of the basic ideas in both problems is to transform conformally the problem to a simpler geometric configuration. If we do so we have to replace the independent variable in a given function by a conformal mapping. This method is called conformal transplantation. The eigenvalue problems which we consider are characterized by a Rayleigh quotient, which contains the Dirichlet integral. The Rayleigh principle or the Poincaré principle provide variational characterizations of the eigenvalues [12]. The invariance of the Dirichlet integral under conformal maps implies that the method of conformal transplantation is often useful for getting upper and lower isoperimetric bounds and for approximation methods in eigenvalue problems.

This survey deals with isoperimetric inequalities and related questions for some classical eigenvalue problems. Of course, it is not possible to comprehensively treat this field in here. It is at best possible to point out only a few results and to restrict ourselves to some of the eigenvalue problems: the membrane problems and the Stekloff problems. According to the aim of this Handbook we focus our attention to conformal mapping techniques. For a more complete consideration of the topic let us mention the book by Pólya and Szegő [84], that by Bandle [12] and the survey papers by Payne, Osserman and Schnitzer [73,74,76,88]. The comments on Pólya's work given by Hersch [51] are also essential for the topics mentioned here.

2. Isoperimetric inequalities

2.1. Membrane problems

Let $D \subset \mathbb{R}^2$ be a bounded domain. We consider the eigenvalue problem of the fixed membrane [12]

$$\begin{aligned} \Delta u + \lambda u &= 0 & \text{in } D, \\ u &= 0 & \text{on } \partial D \end{aligned} \tag{1}$$

and the free membrane eigenvalue problem [12]

$$\begin{aligned} \Delta u + \mu u &= 0 & \text{in } D, \\ \frac{\partial u}{\partial n} &= 0 & \text{on } \partial D, \end{aligned} \tag{2}$$

where n stands for the normal to ∂D , λ and μ for the eigenvalue parameters.

It is well known that there exists an infinity of eigenvalues with finite multiplicity

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \tag{3}$$

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \dots. \tag{4}$$

The eigenvalues are the stationary values of the Rayleigh quotient

$$R[v] = \frac{\iint_D |\nabla v|^2 dx dy}{\iint_D v^2 dx dy}, \quad (5)$$

where the infimum is taken in an appropriate Sobolev space.

The basic isoperimetric result on vibrating membranes with fixed boundary is that of Rayleigh, Faber and Krahn [12]:

THEOREM 1. *Among all membranes of a given area, the circle has the lowest first eigenvalue.*

Proofs were given independently by Faber and Krahn.

The minimum of the second eigenvalue of all membranes of a given area is realized by two disks having the same area, cf. Krahn [56, p. 43]. Later this result was also obtained by Szegő [81, p. 336]. For this reason it is a natural question to ask for domains which minimize the second eigenvalue of all *convex* domains of the same area. It was conjectured by Troesch [96] that the convex hull of two identical tangent disks is a minimizer. This is not the case. Recently Henrot and Oudet [44] refuted this conjecture. The problem is still open but a lot of partial results are given there.

The eigenvalue problem for the Beltrami operator on a surface is equivalent to the eigenvalue problem for an inhomogeneous membrane. Using a symmetrization procedure for inhomogeneous membranes whose mass density distribution p is such that $\log p$ is subharmonic, Nehari [71] proved

$$\lambda_1 \geq \lambda_0, \quad (6)$$

where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass. It was also shown that this is not true if the restriction to p is replaced by the weaker condition that p is subharmonic. For a superharmonic density distribution an opposite inequality is also proven. This result is equivalent to a result obtained by Peetre [79] for domains on a simply-connected surface with the Gaussian curvature $K \leq 0$. Bandle [8, 11] generalized the results of Nehari and Peetre using a general inequality of Alexandrow.

A generalization of Theorem 1 and of a result by Szegő [94] on the inhomogeneously fixed membranes is given by Schwarz [88] Using symmetrization techniques also proves an opposite result.

For more generalizations of the Faber–Krahn result see [3, 10, 76].

Proving a conjecture of Hersch [45], a new characterization of λ_1 via extremal length ideas was given by Bossel [17].

THEOREM 2. *Let the domain D be simply connected and let G be a domain with $G \subset \bar{D}$. Then it holds*

$$\lambda_1 = \max_p \inf_{G \subset D} \frac{\int_{\partial G} p ds - \iint_G p^2 dA}{A_G}, \quad (7)$$

where p is a nonnegative real-valued function in G , and A_G is the area of G .

A consequence of Theorem 2 is a new proof of Theorem 1. This proof uses neither Rayleigh's principle nor Schwarz's symmetrization.

A counterpart of the Faber–Krahn inequality was derived using conformal mappings by Pólya and Szegő [84, p. 97].

THEOREM 3. *For all simply-connected domains D in the plane holds*

$$\lambda_1 \dot{r}^2 \leq j^2, \quad (8)$$

where j denotes the first positive root of the Bessel function J_0 , and $\dot{r}$ means the maximum value for the inner conformal radius of D . Equality holds if and only if D is a disk.

To prove inequality (8) requires one, to choose a suitable point such that the inner conformal radius attains its maximum. It is not easy to determine the points where the inner conformal radius is maximized. This problem has been discussed by Haegi [41] and Kühnau [60]. If D is convex and has a center of symmetry the maximum is attained in the center [41,60]. Pólya and Szegő [84, p. 16] noticed that $\sqrt{\lambda_1} \dot{r}$ has a positive lower bound for convex domains and raised the conjecture that this lower bound is 2, which is attained for a strip, but they asked for a lower bound for unrestricted plane domains. For convex domains Hersch proved $\lambda_1 \geq (\pi/2)^2 d^{-2}$, where d is the radius of the largest disk contained in D [45], [12, p. 158], with equality if and only if D is an infinite strip. For simply-connected domains in the plane Makai¹ [69] proved $\lambda_1 \geq (1/4)d^{-2}$ (for more details see [15]). Later it was proven by Hayman [42] not knowing Makai's result $\lambda_1 > (1/900)d^{-2}$, where d is the radius of the largest disk contained in D . The higher-dimensional case has also been discussed. An improvement of Hayman's result in the constant and an extension to multiply-connected domains and to domains on surfaces had been provided by Osserman [72]. Again, not knowing Makai's result, he obtained $\lambda_1 \geq (1/4)d^{-2}$ (see also [51, pp. 527–528] and [86, p. 194]). Bañuelos and Carroll [13] have considered this problem together with two other problems, one concerning the expected lifetime of the Brownian motion and the other one concerning the density of the hyperbolic metric or the problem of the schlicht Bloch–Landau constant. Using the best known lower bound for the schlicht Bloch–Landau constant it follows $\lambda_1 \geq 0.6197d^{-2}$ [13].

Encouraged by a conjecture of Kornhauser and Stakgold, Szegő [95] proved that of all simply-connected domains of a given area, the circle yields the maximum values of μ_2 . The proof used conformal mappings and a fixed point argument. G. Szegő and Weinberger [97, p. 634] noticed that the same proof causes the disk to minimize

$$\frac{1}{\mu_2} + \frac{1}{\mu_3} \quad (9)$$

¹The author wishes to thank W.K. Hayman and T. Carroll for drawing his attention to these facts.

among all simply connected domains of a given area. The proof used the invariance of the Dirichlet integral under conformal mapping and the Poincaré principle [12, pp. 97–100]. Bandle [12] extended this to inhomogeneous free membranes using a variational principle for the reciprocals and a special conformal mapping, which fulfilled the orthogonality condition. The idea to choose a conformal mapping, such that the orthogonality condition arising from the variational characterization of the eigenvalues is met by some auxiliary functions, goes back to Szegő [95, p. 351] (see also Theorem 10). This idea and the well-known max–min principle have been used by Nadirashvili [70] in order to sharpen the above inequality. Let $\alpha(E) = \inf_{x \in \mathbb{R}^2} \text{meas}(E \setminus E_x^*)$, where E_x^* is the disk equimeasurable with E and centered in x . Then holds [70]

$$\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} \geq \frac{1}{\mu_2(\Omega^*)} + \frac{1}{\mu_3(\Omega^*)} + C\alpha(\Omega)^2 \quad (10)$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ with $\text{meas}(\Omega) = \pi$, where C is a positive constant independently of Ω and Ω^* is the unit disk.

Using the conformal transplantation and a result of Carleman, Gasser and Hersch [38] proved an isoperimetric result for doubly-connected membranes of a given modulus and of a given area of the hole.

THEOREM 4. *Among all doubly-connected membranes of a given modulus and of a given area of the hole, the annulus has the greatest first eigenvalue of the fixed membrane problem (1) and of the membrane problem with mixed boundary conditions.*

A generalization is given by Bandle [8, p. 209], a similar result for the Fredholm eigenvalues was obtained by Hoy [54].

Of course, the products $\lambda_n \dot{r}^2$ and $\mu_n \dot{r}^2$ depend only on the shape of the domain D , not on its size. For higher values of n the products $\lambda_n \dot{r}^2$ and $\mu_n \dot{r}^2$ do not attain the maximum for the disk, Pólya [81, p. 332], and so do λ_n and μ_n for domains with the same $\dot{r}$. But for domains with a sufficiently high symmetry the circle yields the maximum of $\lambda_n \dot{r}^2$ and $\mu_n \dot{r}^2$, more precisely [81].

THEOREM 5. *If the domain D is convex and possesses a symmetry of the order not lower than $2n + 1$, then*

$$\lambda_1 \dot{r}^2, \lambda_2 \dot{r}^2, \dots, \lambda_{l(n)} \dot{r}^2, \quad \mu_1 \dot{r}^2, \mu_2 \dot{r}^2, \dots, \mu_{m(n)} \dot{r}^2, \quad (11)$$

where $l(n)$ and $m(n)$ are strictly monotonously increasing functions of n , will be inferior to the corresponding quantity belonging to a circle.

For more references, see Hersch's comment on Pólya's paper [51, p. 519].

On the other hand, from Weyl's well-known asymptotic result [101]

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \lim_{n \rightarrow \infty} \frac{\mu_n}{n} = \frac{4\pi}{A}, \quad (12)$$

where $A = |D|$ is the area of D , it follows that for sufficiently large n it holds $\lambda_n(D) < \lambda_n(D_1)$ and $\mu_n(D) < \mu_n(D_1)$ if D is the conformal image of the unit disk D_1 under a normalized conformal map such that $\dot{r}$ is the same because then $|D| > |D_1|$.

Pólya and Schiffer derived an interesting result. They use conformal transplantation and convexity arguments in order to prove [83, pp. 304–307].

THEOREM 6. *For any n ,*

$$\frac{1}{\dot{r}^2} \sum_{i=1}^n \frac{1}{\lambda_i} \geq \sum_{i=1}^n \frac{1}{\tilde{\lambda}_i}, \quad (13)$$

where $\tilde{\lambda}_i$ denotes the eigenvalues of the unit disk. Equality holds if and only if D is the unit disk.

Hersch [47] proved the Pólya–Schiffer inequality using only conformal mapping techniques and the Poincaré principle.

Bandle [9,12] generalized (13) to the problem of an inhomogeneous membrane where instead of the Laplace operator the Beltrami operator is used, implying also a sharper form for a class of elastically supported membranes. The proofs are based on Hersch's method and a general isoperimetric inequality of Alexandrow.

The Pólya–Schiffer inequality has been extended by Laugesen and Morpurgo [67]. Instead of (1) Laugesen and Morpurgo consider the problem

$$\begin{aligned} \frac{1}{w} \Delta u + \lambda u &= 0 & \text{in } D, \\ u &= 0 & \text{on } D, \end{aligned} \quad (14)$$

where w is a positive function defined on D , and there is a conformal map $f(z)$ from D onto a domain D' and a positive function $h, h \in \mathbb{C}^2$, with h and $1/h$ bounded and $w = (h \circ f)|f'|^2$. Such functions w are called admissible. The eigenvalues of this problem are denoted by $\lambda_n(w)$.

THEOREM 7. *Let D be a disk or an annulus. Then for any admissible function w and a radial function v with $\int_0^{2\pi} w(re^{i\varphi}) d\varphi \geq 2\pi v(r)$ for all r it holds*

$$\sum_{j=1}^n \frac{1}{\lambda_j(w)} \geq \sum_{j=1}^n \frac{1}{\lambda_j(v)}, \quad (15)$$

with strict inequality unless $\int_0^{2\pi} w(re^{i\varphi}) d\varphi = 2\pi v(r)$ for all r .

The proof closely follows Hersch [47].

Using a general majorization result of Hardy, Littlewood and Pólya [67] the following is derived.

THEOREM 8. Let $\Phi(a)$ be convex and increasing for $a \geq 0$. Then

$$\sum_{j=1}^n \Phi\left(\frac{1}{\lambda_j(\Omega)}\right) \geq \sum_{j=1}^n \Phi\left(\frac{1}{\lambda_j(D)}\right), \quad (16)$$

where n is either a positive integer or $+\infty$, and Ω is the conformal image of the unit disk D_1 under the conformal mapping f with $|f'(0)| = 1$, with strict inequality unless Ω equals the unit disk.

In [65], this is extended to doubly-connected domains to the eigenvalue problem under Dirichlet boundary conditions on the outer boundary curve and under Neumann conditions on the inner boundary curve. In [66] it is extended to inhomogeneous membranes under Dirichlet boundary conditions in a ball in $\mathbb{R}^n$.

Using a well-known property of subharmonic functions [43, Theorem 2.12] Laugesen and Morpurgo [67] proved that the functional on the left-hand side of the equation in Theorem 8 is a continuous, strictly increasing function of ξ and strictly convex as a function of $\log \xi$ with $\Omega = \Omega(\xi) = f(\xi D_1)/\xi$, $0 < \xi \leq 1$. Such kind of monotony results as for the Fredholm eigenvalues was derived by Schiffer [87, pp. 1209–1210].

In [67] isoperimetric results are reported similar to those in Theorem 8 for regions on cylinders, cones and for simply- and doubly-connected surfaces with a curvature bounded above.

Luttinger [68] showed a complementary upper bound: $\sum_{j=1}^{\infty} \lambda_j(\Omega)^{-s} \leq \sum_{j=1}^{\infty} \lambda_j(\Omega^*)^{-s}$, where Ω^* is the disk of the same area as Ω .

As to the free membrane problem (2), the situation seems to be more difficult [67, p. 91]. Another approach based on conformal transplantation and an integral equation for problem (2) is given by Dittmar [25]. For the sum of all squares of reciprocals a formula is derived with the following isoperimetric result:

THEOREM 9. For all simply connected plane domains D with $i = 1$ the disk yields the minimum of

$$A^2 \sum_{j=2}^{\infty} \frac{1}{\mu_j^2}, \quad (17)$$

where A denotes the area of D .

The minimal value in Theorem 9 is $\frac{5}{48}\pi^2 - \frac{155}{192}\pi^2$ [25], see also [27].

A still open conjecture of Pólya is $\mu_n < 4\pi n A^{-1} < \lambda_n$ [80, p. 52]. It was proven by Kröger [57] that $\mu_n < 8\pi n A^{-1}$. Upper bounds for the free membrane eigenvalues for convex domains in terms of the diameter are given by Kröger [58]. A partial proof of Pólya's conjecture is given by himself [82] and by Kellner [55]. For more details see Hersch's comments on Pólya's paper [51, p. 525]. The isoperimetric result above does not follow from Kröger's result. Using Kröger's result and the minimal value given above proves that a minimizer of $A^2 \sum \lambda_j^{-2}$ must have an area less than $2.163 \dots \pi$.

Between the fixed and the free membrane eigenvalues a number of inequalities occurred. Friedlander [37] obtained

$$\mu_{k+1} \leq \lambda_k \quad (18)$$

for any domain, and Horgan and Payne [53] proved

$$\mu_{2k} \geq \lambda_k C_D^{-1}, \quad (19)$$

where C_D is a constant depending only on domain D . Lower bounds for free membrane eigenvalues are also given by Kuttler and Sigillito [62,63].

2.2. Stekloff problem

We restrict ourselves to the following classical eigenvalue problem

$$\begin{aligned} \Delta u &= 0 && \text{in } D, \\ \frac{\partial u}{\partial n} &= \nu p(s)u && \text{on } \partial D, \end{aligned} \quad (20)$$

where $p(s)$ is a given nonnegative function and domain D may also be unbounded with a bounded boundary.

The problem of $p \equiv 1$ for some part of the boundary and of $p \equiv 0$ for the other part is called the sloshing problem. The sloshing problem is a classical problem in fluid mechanics. Fox and Kuttler [36] gave a summary of the results especially by using of conformal mapping techniques for numerical calculations based on the powerful method of intermediate problems. Problem (20) is called the Stekloff problem [12,93]. An infinity of eigenvalues also exists with finite multiplicity

$$0 = \nu_1 < \nu_2 \leq \nu_3 \leq \dots \quad (21)$$

Using the conformal transplantation and a fixed point argument Weinstock [100] proved the following isoperimetric inequality, based on Szegő's idea:

THEOREM 10. *For any nonnegative, piecewise-continuous function $p(s)$ and all simply connected domains D holds*

$$\nu_2 \leq \frac{2\pi}{M}, \quad (22)$$

where $M = \int_{\partial D} p(s) ds$. Equality occurs for the disk with constant p .

Hersch and Payne [49] noticed that Weinstock's proof contains the sharper inequality

$$\left(\frac{1}{\nu_2} + \frac{1}{\nu_3} \right) \frac{1}{M} \geq \pi^{-1}, \quad (23)$$

and Hersch, Payne and Schiffer [50] stated

$$v_2 v_3 \leq \frac{4\pi^2}{M^2}. \quad (24)$$

For an n -fold connected domain $D \ni \infty$ with $\partial D = \bigcup_{i=1}^n C_i$, Hersch, Payne and Schiffer proved

$$v_2 v_3 \leq \frac{4\pi^2}{\max M_i}, \quad (25)$$

where $M_i = \int_{C_i} p(s) ds$. An isoperimetric inequality for the product $v_2 v_3$ with conformal quantities in the multiply-connected domain $D \ni \infty$ is given by Dittmar [20, eq. (15)].

$$v_2 v_3 \leq \frac{\int_{\partial D} p^{-1} \dot{x}^2 ds \int_{\partial D} p^{-1} \dot{y}^2 ds - (\int_{\partial D} p^{-1} \dot{x} \dot{y} ds)}{A^2 + 4\pi r(\pi r - A - \pi |m|^2 r^{-1})}, \quad (26)$$

where $\dot{x} = \frac{dx}{ds}$, and s denotes the length of the boundary curve in the $z = (x + iy)$ -plane, A is the area of the holes, and m and r , respectively, are the center and the radius of the disk, which is given by all coefficients a_1 for univalent conformal maps $f(z) = z + a_1 z^{-1} + \dots$ of domain D [40]. Equality occurs if D is simply-connected and bounded by the unit circle. It is well known that $\pi r - A - \pi |m|^2 r^{-1} \geq 0$ [34,59].

For simply-connected domains, which are symmetric with respect to a rotation, Bandle [6,12] derived sharp upper bounds for higher eigenvalues using a concept of Pólya [81].

An asymptotic result was proven by Shamma [92]

$$v_{2n} = \frac{2n\pi}{M} + \varepsilon_n, \quad v_{2n+1} = \frac{2n\pi}{M} + \varepsilon'_n, \quad (27)$$

where ε_n and ε'_n approach 0 with $n \rightarrow \infty$.

Transplanting the eigenvalue problem (20) conformally in the unit disk and using the Neumann function of the unit disk and the well-known fact that the Stekloff eigenfunctions are eigenfunctions of the Neumann function of the boundary, Dittmar [24,26] obtained encouraged by Bergman and Schiffer [16, p. 395], a formula for the sum of all squares of reciprocals.

THEOREM 11. *If f is the univalent conformal mapping of the interior of the unit disk onto the domain D with $f(0) = f'(0) - 1 = 0$, then for the eigenvalues of problem (20) with $p \equiv 1$ we get*

$$\sum_{i=2}^{\infty} \frac{1}{v_i^2} = \int_0^{2\pi} \int_0^{2\pi} N_D^2(f(\xi), f(\eta)) |f'(\xi)| |f'(\eta)| ds_{\xi} ds_{\eta}, \quad (28)$$

where N_D is Neumann's function of the domain D .

One result of Theorem 11 is that the circle minimizes the sum of all reciprocals. The idea to derive isoperimetric inequalities for sums of reciprocals of eigenvalues using the theory of integral equation was realized by Schwarz [90] for ordinary differential equations.

Edward [31] obtained $\sum_{j=1}^{\infty} (v_j^2 - (v'_j)^2) \geq 0$, where v'_j denote the eigenvalues of the unit disk with equality only for the unit disk. The proof is based on properties of the zeta function $\zeta(z) = \sum_{j=1}^{\infty} v_j^{-z}$, which is defined in such a way that the numbers $j = 1, 2, \dots$ in the well-known Riemann zeta function $\zeta_R(z) = \sum_{j=1}^{\infty} j^{-z}$ are replaced by the eigenvalues v_j .

We look for lower bounds. It is easy to see that there are no lower ones for the eigenvalues of problem (20) with constant mass if the boundary curve is not smooth enough [20, p. 157]. For analytically-bounded convex domains Payne [75] proved for $p \equiv 1$

$$v_2 \geq \min k(s), \quad (29)$$

where $k(s)$ denotes the curvature of the boundary curve. Equality holds only for the circle. Bounds for higher eigenvalues for convex domains are given by Dittmar [22] proving also a monotony result.

It is easy to obtain roughly lower bounds using the following *monotonicity principle*, which arises from the Poincaré principle [12] (this principle is also applicable to the membrane problem).

If $0 \leq p_1(s) \leq p_2(s)$ in the same domain D , then $v_j(p_1) \geq v_j(p_2)$.

For the unit disk and $p \equiv 1$, the eigenvalues are given by $0, 1, 2, 3, \dots$ [12, p. 96]. The eigenvalue problem (20) with $p \equiv 1$ in a given domain D is equivalent to problem (20) in the unit disk with $p = |f'(z)|$, where $f(z)$ denotes a conformal mapping of the unit disk onto the given domain D . Using the monotonicity principle, Dittmar [20] obtained lower bounds for the eigenvalues in D using upper bounds for $|f'(z)|$ on the unit circle. A simple consequence is that we have to distinguish between the eigenvalues of the exterior and interior domains (a consequence of the difference between the conformal mapping of the exterior and interior domains) [20, p. 160, Example 4]. We denote the eigenvalues of the exterior domain by $v_{2e}, v_{3e}, \dots$ and those of the interior domain by $v_{2i}, v_{3i}, \dots$. In contrast to (29), which holds for the eigenvalues of the interior domain, for exterior domains with a convex complement and the outer conformal radius 1 it follows [20]

$$v_{2e} > 1/2. \quad (30)$$

Isoperimetric inequalities between v_{2e} and v_{2i} are given in [20].

Let Λ_C be the Fredholm eigenvalue of a closed Jordan curve C then the following isoperimetric inequality holds [20]

$$\frac{\Lambda_C - 1}{\Lambda_C + 1} \leq \frac{v_{2e}}{v_{2i}} \leq \frac{\Lambda_C + 1}{\Lambda_C - 1}. \quad (31)$$

Equality holds for a disk. Kuttler and Sigillito [62] obtained lower bounds for λ_2 using Hersch's method of one-dimensional auxiliary problems and for all eigenvalues of a convex interior domain which is starlike [63].

Bramble and Payne [18] derived bounds for the inner Neumann problem using bounds for the Stekloff eigenvalues. Kuttler and Sigillito [63] yielded inequalities between membrane and Stekloff eigenvalues. The annulus case was considered by Dittmar [23]. The eigenvalues are known explicitly [23] for $p \equiv 1$. In a ring domain there is an eigenfunction with a closed nodal line. There is also an inverse result proven: For $p \equiv 1$, among all ring domains only the annulus has only one eigenfunction with a multivalent conjugate harmonic [23].

Based on the papers of Serrin [91] and Weinberger [98], results for the overdetermined Stekloff eigenvalue problem have been obtained by Payne and Philippin [77] and Allessandrini and Magnanini [1,2].

2.3. Mixed Stekloff eigenvalues

The following eigenvalue problem, which is closely related to the preceding one, has been considered by several authors [24,28,30,49]:

$$\begin{aligned} \Delta u &= 0 && \text{in } D, \\ \frac{\partial u}{\partial n} &= \sigma p(s)u && \text{on } C_1, \\ u &= 0 && \text{on } C_2, \end{aligned} \tag{32}$$

where $p(s)$ is a given nonnegative function of C_1 and $\partial D = C = C_1 \cup C_2$. There is an infinity of eigenvalues with finite multiplicity

$$0 < \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \dots \tag{33}$$

If D is a ring domain and C_2 is one of the components of the boundary curves, Hersch and Payne [49, p. 808] found that the conformal modulus $\text{mod}(D)$ of D can be characterized by

$$\frac{1}{\text{mod}(D)} = \max_{\text{choice of } p} \sigma_1 M, \tag{34}$$

where $M = \int_{C_1} p(s) ds$. The modulus is usually characterized by a Dirichlet problem or via extremal length methods (for more details, see [61]). An analogous eigenvalue problem for a quadrilateral and a characterization of the modulus of this quadrilateral are also given. Using the well-known property that reciprocal quadrilaterals have reciprocal moduli, both side estimations for the modulus are given by the eigenvalues for the corresponding eigenvalue problems.

A consequence of (34) is

$$\sigma_1 \leq \frac{1}{M \text{mod}(D)}. \tag{35}$$

Hersch and Payne derived lower bounds for the first eigenvalue applying the method of one-dimensional auxiliary problems [49].

It is easy to see that σ_1 tends to 0 for special densities with constant mass if the modulus is fixed [30]. Using circular symmetrization, Dittmar and Solynin [30] have derived lower bounds. We mention here only the following result for the Grötzsch ring domain $R_D(x_0)$. This is the doubly-connected domain, which we obtain from the unit disk if we omit a straight slit from 0 to x_0 , $0 < x_0 < 1$ and which is conformally equivalent to D .

THEOREM 12. *Let D be a ring domain, $0 \notin D$, with boundary components C_1, C_2 , where C_1 is the unit circle and C_2 is a subset of the unit disk. If $p \equiv 1$, it follows*

$$\sigma_1(D) > \sigma_1(R_D(x_0)), \quad (36)$$

where $R_D(x_0)$ is the Grötzsch ring domain and D is not the Grötzsch ring or a rotation of this.

For the Grötzsch ring $R_D(x_0)$ it follows $\sigma_{2n} = n$, $n = 1, 2, \dots$, and $n - 1 < \sigma_{2n-1} < n$ [30]. Other lower bounds are also given.

Using conformal mappings, transplantation techniques and the reflection principle Dittmar and Kühnau [28] present a method for constructing an analytic function the real part of which is the harmonic eigenfunction for the mixed problem (32). In the Grötzsch ring case and a simply-connected limiting case this has been outlined [28, Satz 5], with given eigenfunctions. Here we mention the simply-connected limiting case if the slit on the real axis touches the unit circle. Let the unit circle in problem (32) be C_1 and let the slit on the real axis between x_0 and 1, $0 < x_0 < 1$, C_2 . For the analytic function $f(z)$, the real part of which is the eigenfunction with the eigenvalue σ , the following holds [28]

$$f(z) = z^\sigma \left\{ \int_{x_0}^z \frac{A}{2} \left(\sqrt{\frac{x_0 - z}{1 - x_0 z}} - \sqrt{\frac{1 - x_0 z}{x_0 - z}} \right) z^{-\sigma-1} dz + \text{const} \right\} + i \frac{B}{\sigma}, \quad (37)$$

where A and B are real constants, resulting in an equation for the eigenvalues, which is used for numerical calculations.

According to Theorem 11 for the mixed problem (32) the following result has been obtained [24].

THEOREM 13. *If f is the univalent conformal mapping of the annulus A , bounded by two concentric circles with the center in the origin and of radii 1 and ρ , onto domain D bounded by $C = C_1 \cup C_2$, such that the circle of radius ρ corresponds to curve C_1 . Then, for the eigenvalues of problem (32) in D with $p \equiv 1$, it holds*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} &= \int_{|z|=\rho} \int_{|z|=\rho} R^2(z, \zeta) |f'(z)| |f'(\zeta)| ds_z ds_\zeta \\ &= \frac{B_0 L^2}{4} + \frac{\pi}{2} \sum_{n=1}^{\infty} B_n (\alpha_n^2 + \beta_n^2), \end{aligned} \quad (38)$$

where $R(z, \zeta)$ is the Robin function of the annulus A

$$|f'(z)|_{|z|=\rho} \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\varphi + \beta_n \sin n\varphi, \quad (39)$$

L is the length of C_1 and $B_k, k = 0, 1, \dots$, are given numbers.

Here also the annulus minimizes the sum of all reciprocals for this mixed problem under consideration.

2.4. Trilaterals, quadrilaterals and their eigenvalues

A trilateral is a simply-connected Jordan domain $D = D(A, B, C)$ with three distinguished points A, B and C on its boundary T . All trilaterals are conformally equivalent. Hersch introduced the following eigenvalue problems [12,46]:

$$\begin{aligned} \Delta u + \varepsilon p u &= 0 && \text{in } D(A, B, C), \\ u &= 0 && \text{on } C_0, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } T \setminus C_0, \end{aligned} \quad (40)$$

with p being a given positive function of D and $T = \partial D(A, B, C)$. The first eigenvalues are denoted by ε_{AB} and ε_{CA} , if C_0 coincides with the boundary curve through the points A, B and C, A , respectively.

Hersch [46] derived

$$\frac{1}{\varepsilon_{AB}} + \frac{1}{\varepsilon_{BC}} + \frac{1}{\varepsilon_{CD}} \geq \frac{3M}{\pi}, \quad (41)$$

where $M = \int_D p \, dx \, dy$, and equality holds for the triangle $x_1 > 0, x_2 > 0, x_3 > 0$ on the sphere $x_1^2 + x_2^2 + x_3^2 = R^2$ with $p \equiv 1$.

Let now $D = D(A, B, C, D)$ be a quadrilateral with four distinguished points A, B, C, D on the boundary Q (for the definition and properties see [61]). For quadrilaterals, similar eigenvalue problems had been considered by Hersch [46]. For the rectangles with constant mass density the eigenfunctions are known. Via conformal transplantation Hersch [46] obtained the following isoperimetric inequality

$$\frac{1}{\varepsilon_{AB}} + \frac{1}{\varepsilon_{CD}} \geq \frac{8\mu M}{\pi^2}, \quad (42)$$

where μ denotes the conformal modulus of D and $M = \int_D p \, dx \, dy$. Equality holds for the rectangles with constant mass density p . Other isoperimetric results especially those like the Pólya–Schiffer inequality Theorem 6 are given in [46]. Some of these results are

extended by Hersch and Sigrist [52] to an elastically supported vibrating membrane. Upper bounds for all eigenvalues of such kind are given by Bandle [7]. There are also indicated extensions to more general eigenvalue problems.

2.5. Numerical calculation

Since the eigenvalues are not explicitly known in most cases, several methods for their approximation have been developed. The oldest method to obtain numerical upper bounds is the variational method originally known as Rayleigh's principle, which was further developed to the Rayleigh–Ritz method. Both methods provide upper bounds. A far more difficult problem is to find lower bounds. For this purpose the method of intermediate problems had been developed. There are many self-contained expositions of numerical methods for eigenvalue problems: the books of Gould [39], Fichera [35] and Weinstein and Stenger [99], and the survey by Babuška and Osborn [5]. The method of intermediate problems was studied in detail by Fox and Kuttler [36] especially for the sloshing frequencies. The basic idea is also to transform the eigenvalue problem by conformal maps into weighted problems on standard domains, which also imply a solution via intermediate problems. All eigenvalue problems mentioned above could be solved numerically in this way. Another straightforward approach for the Stekloff problem had been given by Dittmar [21] and Dittmar and Schenk [29]. Ennenbach [32,33] used a special inclusion theorem for computing Stekloff eigenvalues.

References

- [1] G. Alessandrini and R. Magnanini, *Symmetry and non-symmetry for the overdetermined Stekloff eigenvalue problem*, Z. Angew. Math. Phys. **45** (1994), 44–52.
- [2] G. Alessandrini and R. Magnanini, *Elliptic equations in divergence form, geometric critical points of solutions, and Stekloff eigenfunctions*, SIAM J. Math. Anal. **25** (1994), 1259–1268.
- [3] M.S. Ashbaugh and R.D. Benguria, *Universal bounds for the low eigenvalues of Neumann Laplacians in N dimensions*, SIAM J. Math. Anal. **24** (1993), 557–570.
- [4] M.S. Ashbaugh and R.D. Benguria, *Isoperimetric inequalities for eigenvalue*, Partial Differential Equations of Elliptic Type, A. Alvino et al., eds, Proc. Conf. October 12–16, 1992, Cortona, Italy, Symp. Math., Vol. 35, Cambridge Univ. Press (1994), 1–36.
- [5] I. Babuška and J. Osborn, *Eigenvalue Problems*, Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions, eds, Elsevier, North-Holland (1991), 643–787.
- [6] C. Bandle, *Über das Stekloffsche Eigenwertproblem; Isoperimetrische Ungleichungen für symmetrische Gebiete*, Z. Angew. Math. Phys. **19** (1968), 627–637.
- [7] C. Bandle, *Obere Schranken für die Eigenwerte einer stückweise freien Membrane auf einem "Vierseit"*, Z. Angew. Math. Phys. **21** (1970), 1072–1077.
- [8] C. Bandle, *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie*, Comment. Math. Helv. **46** (1971), 182–213.
- [9] C. Bandle, *Extensions of an inequality by Pólya and Schiffer for vibrating membranes*, Pacific J. Math. **42** (1972), 543–555.
- [10] C. Bandle, *Isoperimetric inequality for some eigenvalues of an inhomogeneous, free membrane*, SIAM J. Appl. Math. **22** (1972), 142–147.
- [11] C. Bandle, *A geometrical isoperimetric inequality and applications to problems of mathematical physics*, Comment. Math. Helv. **49** (1974), 496–511.

- [12] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, London (1980).
- [13] R. Bañuelos and T. Carroll, *Brownian motion and the fundamental frequency of a drum*, Duke Math. J. **75** (1994), 575–602.
- [14] R. Bañuelos and T. Carroll, *An improvement of the Osserman constant for the bass note of a drum*, Proc. Sympos. Pure Math. **57** (1995), 3–10.
- [15] R. Bañuelos and T. Carroll, *Addendum to “Brownian motion and the fundamental frequency of a drum”*, Duke Math. J. **82** (1996), 227.
- [16] S. Bergman and M. Schiffer, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, New York (1953).
- [17] M.-H. Bossel, *Longueurs extrémales et fonctionnelles de domaine*, Complex Var. **6** (1986), 203–234.
- [18] J.H. Bramble and L.E. Payne, *Bounds in the Neumann problem for second order uniformly elliptic operators*, Pacific J. Math. **12** (1962), 823–833.
- [19] F. Brock, *An isoperimetric inequality for eigenvalues of the Stekloff problem*, Z. Angew. Math. Mech. **81** (2001), 69–71.
- [20] B. Dittmar, *Stekloffsche Eigenwerte und konforme Abbildungen*, Z. Anal. Anwendungen **7** (1988), 149–163.
- [21] B. Dittmar, *Zur Berechnung Stekloffscher Eigenwerte*, Z. Angew. Math. Phys. **39** (1988), 351–366.
- [22] B. Dittmar, *Monotonie bei Stekloffschen Eigenwerten*, Z. Angew. Math. Phys. **40** (1989), 603–607.
- [23] B. Dittmar, *Zu einem Stekloffschen Eigenwertproblem in Ringgebieten*, Mitt. Math. Sem. Giessen **228** (1996), 1–7.
- [24] B. Dittmar, *Isoperimetric inequalities for the sums of reciprocal eigenvalues*, Progress in Partial Differential Equations, Pont-à-Mousson 1997, Vol. 1, H. Amann, C. Bandle, M. Chipot, F. Conrad and I. Shafrir, eds, Pitman Res. Notes in Math. Ser., Vol. 383, Addison-Wesley–Longman, Harlow, Essex (1998), 78–87.
- [25] B. Dittmar, *Sums of reciprocal eigenvalues of the Laplacian*, Math. Nachr. **237** (2002), 45–61.
- [26] B. Dittmar, *Sums of reciprocal Stekloff eigenvalues*, Math. Nachr. **268** (2004), 44–49.
- [27] B. Dittmar, *J. Anal.* **95** (2005), to appear.
- [28] B. Dittmar and R. Kühnau, *Zur Konstruktion der Eigenfunktionen Stekloffscher Eigenwertaufgaben*, Z. Angew. Math. Phys. **51** (2000), 806–819.
- [29] B. Dittmar and B. Schenk, *Zur Berechnung Stekloffscher Eigenwerte-numerische Beispiele*, Z. Angew. Math. Phys. **42** (1991), 497–505.
- [30] B. Dittmar and A.Yu. Solynin, *Mixed Stekloff eigenvalue problem and new extremal properties of the Grötzsch ring*, Zap. Nauchn. Sem. St. Petersburg. Otdel. Mat. Inst. Steklov (POMI) **270** (2000) (in Russian), Issled. po Linein. Oper. i Teor. Funkts. **28**, 51–79, 365; transl.: J. Math. Sci. **115** (2) (2003), 2119–2134.
- [31] J. Edward, *An inequality for Steklov eigenvalues for planar domains*, Z. Angew. Math. Phys. **45** (1994), 493–496.
- [32] R. Ennenbach, *The inclusion of Stekloff eigenvalues by boundary data approximation*, Z. Angew. Math. Phys. **45** (1994), 399–413.
- [33] R. Ennenbach, *The inclusion of Stekloff eigenvalues for nonsmooth bounded domains*, Z. Angew. Math. Phys. **45** (1994), 827–831.
- [34] J. Epstein, *Some inequalities relating to conformal mapping upon canonical slit-domains*, Bull. Amer. Math. Soc. **53** (1947), 813–819.
- [35] G. Fichera, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Math., Vol. 8, Springer-Verlag, New York (1965).
- [36] D.W. Fox and J.R. Kuttler, *Sloshing frequencies*, Z. Angew. Math. Phys. **34** (1983), 668–696.
- [37] L. Friedlander, *Some inequalities between Dirichlet and Neumann eigenvalues*, Arch. Ration. Mech. Anal. **116** (1991), 153–160.
- [38] T. Gasser and J. Hersch, *Über Eigenfrequenzen einer mehrfach zusammenhängenden Membran: Erweiterung von isoperimetrischen Sätzen von Pólya und Szegő*, Z. Angew. Math. Phys. **19** (1968), 672–675.
- [39] S.H. Gould, *Variational Methods for Eigenvalue Problems: An Introduction to the Weinstein Method of Intermediate Problems*, 2nd edn, Univ. Toronto Press (1966).
- [40] H. Grötzsch, *Über das Parallelschlitztheorem der konformen Abbildung schlichter Bereiche*, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig **84** (1932), 15–36.

- [41] H.R. Haegi, *Extremalprobleme und Ungleichungen konformer Gebietsgrößen*, Compos. Math. **8** (1950), 81–111.
- [42] W.K. Hayman, *Some bounds for principal frequency*, Appl. Anal. **7** (1978), 247–254.
- [43] W.K. Hayman and P.B. Kennedy, *Subharmonic Functions*, Vol. 1, Academic Press, London (1976).
- [44] A. Henriot and E. Oudet, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Arch. Ration. Mech. Anal. **169** (2003), 73–87.
- [45] J. Hersch, *Sur le fréquence fondamentale d'un membrane vibrante. Evaluation par défaut et principe de maximum*, Z. Angew. Math. Mech. **11** (1960), 387–413.
- [46] J. Hersch, *Une inégalité isopérimétrique pour les membranes vibrantes sur un 'trilatère'*, C. R. Acad. Sci. Paris **261** (1965), 2443–2445.
- [47] J. Hersch, *On symmetric membrans and conformal radius: Some complements to Pólya's and Szegő's inequalities*, Arch. Ration. Mech. Anal. **20** (1965), 378–395.
- [48] J. Hersch, *Anwendungen der konformen Abbildung auf isoperimetrische Sätze für Eigenwerte*, Festband zum 70. Geburtstag von Rolf Nevanlinna (Herausgeber: H.P. Künzi und A. Pfluger), Springer-Verlag, Berlin–Heidelberg–New York (1966), 25–34.
- [49] J. Hersch and L.E. Payne, *Extremal principles and isoperimetric inequalities for some mixed problems of Stekloff's type*, Z. Angew. Math. Phys. **19** (1968), 802–819.
- [50] J. Hersch, L.E. Payne and M.M. Schiffer, *Some inequalities for Stekloff eigenvalues*, Arch. Ration. Mech. Anal. **57** (1974), 99–114.
- [51] J. Hersch and G.-C. Rota (eds), *George Pólya, Collected Papers, Vol. III: Analysis*, MIT Press, Cambridge, MA (1984).
- [52] J. Hersch and K. Sigrist, *Isoperimetrische Ungleichungen für elastisch gebundene Membranen auf Dreiseiten und auf Vierseiten*, Z. Angew. Math. Phys. **21** (1970), 836–841.
- [53] C.O. Horgan and L.E. Payne, *Lower bounds for free membrane and related eigenvalues*, Rend. Mat. **10** (1990), 457–491.
- [54] E. Hoy, *Area theorems and Fredholm eigenvalues*, Ann. Acad. Sci. Fenn. Ser. A I Math. **14** (1989), 137–148.
- [55] R. Kellner, *On a theorem of Pólya*, Amer. Math. Monthly **73** (1966), 856–858.
- [56] E. Krahn, *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Dorpat. **A9** (1926), 1–44.
- [57] P. Kröger, *Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space*, J. Funct. Anal. **106** (1992), 353–357.
- [58] P. Kröger, *On upper bounds for high order Neumann eigenvalues of convex domains in Euclidean space*, Proc. Amer. Math. Soc. **127** (6) (1999), 1665–1669.
- [59] R. Kühnau, *Die Spanne von Gebieten bei quasikonformen Abbildungen*, Arch. Ration. Mech. Anal. **65** (1977), 299–303.
- [60] R. Kühnau, *Maxima beim konformen Radius einfach zusammenhängender Gebiete*, Ann. Univ. Mariae Curie-Skłodowska Sec. A **46** 8 (1992), 63–73.
- [61] R. Kühnau, *The conformal module of quadrilaterals and of rings*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, Elsevier, Amsterdam (2005), 99–129 (this volume).
- [62] J. Kuttler and V.G. Sigillito, *Lower bounds for Stekloff and free membrane eigenvalues*, SIAM Rev. **10** (1968), 368–370.
- [63] J. Kuttler and V.G. Sigillito, *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl. **23** (1968), 148–160.
- [64] J. Kuttler and V.G. Sigillito, *An inequality for a Stekloff eigenvalue by the method of defect*, Proc. Amer. Math. Soc. **20** (1969), 357–360.
- [65] R.S. Laugesen, *Eigenvalues of Laplacians with mixed boundary conditions, under conformal mapping*, Illinois J. Math. **42** (1998), 19–39.
- [66] R.S. Laugesen, *Eigenvalues of the Laplacian on inhomogeneous membranes*, Amer. J. Math. **120** (1998), 305–344.
- [67] R.S. Laugesen and C. Morpurgo, *Extremals for eigenvalues of Laplacians under conformal mapping*, J. Funct. Anal. **155** (1998), 64–108.
- [68] J.M. Luttinger, *Generalized isoperimetric inequalities*, J. Math. Phys. **14** (1973), 586–593, 1444–1447, 1448–1450.

- [69] E. Makai, *A lower estimation of the principal frequencies of simply connected membranes*, Acta Math. Acad. Sci. Hungar. **16** (1965), 319–323.
- [70] N. Nadirashvili, *Conformal maps and isoperimetric inequalities for eigenvalues of the Neumann problem*, Israel Math. Conf. Proc. **11** (1997), 197–201.
- [71] Z. Nehari, *On the principal frequency of a membrane*, Pacific J. Math. **8** (1958), 285–293.
- [72] R. Osserman, *A note on Hayman's theorem on the bass note of a drum*, Comment. Math. Helv. **52** (1977) 545–555.
- [73] R. Osserman, *Isoperimetric inequalities and eigenvalues of the Laplacian*, Proc. Internat. Congress of Mathematicians, Helsinki, 1978, Vol. 1 (1980), 435–442.
- [74] L.E. Payne, *Isoperimetric inequalities and their applications*, SIAM Review **9** (1967), 453–488.
- [75] L.E. Payne, *Some isoperimetric inequalities for harmonic functions*, SIAM J. Math. Anal. **3** (1970), 354–359.
- [76] L.E. Payne, *Some comments on the past fifty years of isoperimetric inequalities*, Lecture Notes in Pure and Appl. Math. **129** (1991), 53–79.
- [77] L.E. Payne and G.A. Philippin, *Some overdetermined boundary value problems for harmonic functions*, J. Appl. Math. Phys. **42** (1991), 864–873.
- [78] L.E. Payne and H.F. Weinberger, *New bounds for solutions of second order elliptic partial differential equations*, Pacific J. Math. **8** (1958), 551–573.
- [79] J. Peetre, *A generalization of Courant's nodal line theorem*, Math. Scand. **5** (1957), 15–20.
- [80] G. Pólya, *Patterns of Plausible Inference*, Princeton Univ. Press, Princeton, NJ (1954).
- [81] G. Pólya, *On the characteristic frequencies of a symmetric membrane*, Math. Z. **63** (1955), 331–337.
- [82] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc. **11** (1961), 419–433.
- [83] G. Pólya and M. Schiffer, *Convexity of functionals by transplantation*, J. Anal. Math. **3** (1954), 245–345.
- [84] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, Princeton, NJ (1951).
- [85] M.H. Protter, *Lower bounds for the first eigenvalue of elliptic equations*, Ann. of Math. **71** (1960), 423–444.
- [86] M.H. Protter, *Can one hear the shape of a drum? Revisited*, SIAM Rev. **29** (1987), 185–197.
- [87] M.M. Schiffer, *The Fredholm eigen values of plane domains*, Pacific J. Math. **7** (1957), 1187–1225.
- [88] F.J. Schnitzer, *A few remarks on isoperimetric inequalities*, Grazer Math. Ber. **321** (1993), 1–58.
- [89] B. Schwarz, *Bounds for the principal frequency of the nonhomogeneous membrane and for the generalized Dirichlet integral*, Pacific J. Math. **7** (1957), 1653–1676.
- [90] B. Schwarz, *Bounds for sums of reciprocals of eigenvalues*, Bull. Res. Council. Israel **8F** (1959), 91–102.
- [91] J.B. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.
- [92] S.E. Shamma, *Asymptotic behavior of Stekloff eigenvalues and eigenfunctions*, SIAM J. Appl. Math. **20** (1971), 482–490.
- [93] M.W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole Norm. Sup. **19** (1902), 455–490.
- [94] G. Szegő, *Über eine Verallgemeinerung des Dirichletschen Integrals*, Math. Z. **52** (1950), 676–685.
- [95] G. Szegő, *Inequalities for certain eigenvalues of a membrane of given area*, J. Ration. Mech. Anal. **3** (1954), 343–356.
- [96] B.A. Troesch, *Elliptical membranes with smallest second eigenvalue*, Math. Comp. **27** (1973), 767–772.
- [97] H.F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Ration. Mech. Anal. **5** (1956), 633–636.
- [98] H.F. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Ration. Mech. Anal. **43** (1971), 319–320.
- [99] R. Weinstein and A. Stenger, *Methods of Intermediate Problems for Eigenvalues*, Academic Press, New York (1972).
- [100] R. Weinstock, *Inequalities for a classical eigenvalue problem*, J. Ration. Mech. Anal. **3** (1954), 745–448.
- [101] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), 441–479.

CHAPTER 17

Foundations of Quasiconformal Mappings

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Introduction

Quasiconformal (qc) and quasiregular (qr) mappings have been introduced by Grötzsch in 1928 as generalization of conformal mappings and of complex analytic functions respectively [Gr2].

They form now an important field of complex analysis, especially due to their applications in Geometric Function Theory (e.g., in various extremal problems, in Teichmüller spaces, Kleinian groups) as well as in nonlinear partial differential equations, nonlinear potential theory, calculus of variations, harmonic analysis, complex dynamics, differential geometry, topology, but also in fluid mechanics or nonlinear elasticity, in mathematical physics (e.g., electrostatics).

Their theory is presented in monographs like those by Volkovyskij (1954) [Vo3], Künzi (1960) [Kün3], Lavrent'ev (1962) [La7], Andreian Cazacu (1965) [AC8], (1966) [AC9], Lehto and Virtanen (1965, 2nd edition in English 1973) [LV3], Ahlfors (1966) [A12], Belinskij (1974) [B8], Krushkal (1974) [Kr2] and (1975, extended English translation 1979) [Kr3], Krushkal and Kühnau (1983, Russian edition in 1984) [KrKü], Ławrynowicz in cooperation with Krzyż (1983) [Law2], Lehto (1986) [L11] for the two-dimensional case; Caraman (1968, English edition in 1974) [C4], Väisälä (1971) [Vä5], Reshetnyak (1982, English edition in 1989) [R7], Sychev (1983) [Sy], Vuorinen (1988) [Vu2], Rickman (1993) [Ri4], Heinonen, Kilpeläinen and Martio (1993) [HKM] for higher-dimensional case; Heinonen (2001) [He1] for metric spaces; Iwaniec and Martin (2001) [IM1] for mappings of finite distortion.

The development of the theory can be also easily followed from the numerous surveys presented at International Congresses of Mathematicians: e.g., Edinburgh, 1958, by Bers [B11], Stockholm, 1962, by Ahlfors [A9] (and e.g., communication by Lavrent'ev [La8]), Moscow, 1966, by Lehto [L4] and by Gehring [G8], Vancouver, 1974, by Gehring [G11], Lelong-Ferrand [L-F4] and Strebel [Str4], Helsinki, 1978, by Ahlfors [A16] and by Väisälä [Vä7], Warsaw, 1983, by Iwaniec [I1], Berkeley, 1986, by Gehring [G14], Zürich, 1994, by Tukia [Tu], Berlin, 1998, by Astala [As2], Beijing, 2002, by Eremenko [Er] and Heinonen [He2], at other congresses and conferences or in collections of papers: e.g., by Ahlfors in *Lectures on Modern Mathematics* (1964) [A11], by Bers in *Bull. London Math. Soc.* **4** (1972) [B19] and *Bull. Amer. Math. Soc.* **83** (1977) [B20], by Lehto in *Discrete Groups and Automorphic Functions* (1977) [L9], in the volume: *Zum Werk Leonhard Eulers* (1984) [L10], by Rickman in *Ann. Acad. Sci. Fenn.* **13** (1988), the volume dedicated to Ahlfors [Ri2], or in Vuorinen's collection *Quasiconformal Space Mappings 1960–1990* (1992) [Vu3] (Gehring [G15], Martio [Ma2], Rickman [Ri3], Väisälä [Vä8]). There are also other surveys, e.g., by Strebel (1986) [Str8], Krushkal (1989) [Kr4], Kühnau (1992) [Kü13], Zajac (1999) [Za], Srebro (2000) [Sr2].

In this chapter we shall expose the evolution of the quasiconformality ($qcty$) and quasiregularity ($qrty$) in the two-dimensional case. However first let us underline some aspects of the situation in complex analysis at the moment when Grötzsch defined his class of mappings.

The beginning of 20th century was a period of great achievements in function theory: general uniformization theorem by Poincaré and Koebe, boundary behavior of conformal mappings by Carathéodory, value distribution theory for meromorphic functions by

Rolf Nevanlinna. New methods, new tools were waited – and one of them was Grötzsch's quasiconformal mappings (cf. [A11, p. 152]). In the same time, it was a general trend in mathematics to clear up the foundations of the concepts. Axiomatic algebra, topology, measure theory, functional analysis developed intensively in this sense.

As an example of this trend to generalization let us remind Pompeiu's work: after he established the now called Cauchy–Pompeiu formula and defined the areolar derivative (nothing else in regular points (i.e., points of differentiability with Jacobian $J > 0$) than the $\bar{\partial}$ or $\partial_{\bar{z}}$ derivative) (1912, 1913) [Pom1, Pom2], he studied classes of nonanalytic functions, e.g., functions (α) -monogenic which admit such areolar derivative or (α) -holomorphic having continuous areolar derivative [Pom1–Pom4]. This direction was followed by Hayashi (1912) [Hay], Looman (1924) [Lo], Kasner (1927) [Ka] who introduced the name of polygenic functions for functions in the class C^1 , Nicolescu and Călugăreanu both defending in 1928 doctoral thesis in Paris [Ni1, Ni2], [Că1–Că3] and others: Ghermanesco (1927) [Gh1, Gh2], Ciorănescu (1936) [Ci], Moisil (1930, 1931) [M1–M3], Theodoresco (1930) [Th1–Th4] (see references [Th4] and [MŞ]).

Brouwer's problem of characterizing topologically the analytic functions was another important research subject and Szillard's thesis at Göttingen (1927) brought results on this problem [Sz]. Onicescu's holotop functions represented other attempt to solve it [On1–On3].

Finally Brouwer's problem was solved by Stoilow (1927, 1928) [St1–St3, St5]. He introduced the *interior transformations* as mappings $f: G \rightarrow G'$, G and G' domains in $\mathbb{C}$, which verify one of the three equivalent conditions:

- 1°. f is continuous, open and light (or – again equivalently in this two-dimensional case – diskrete);
- 2°. f is a local (l.) homeomorphism, except for a diskrete set $E \subset G$, where f is locally topologically equivalent to $w = z^n$;
- 3°. f satisfies Stoilow's decomposition theorem, i.e., can be written in the form $f = F \circ h$ with h a homeomorphism: $G \rightarrow h(G)$ and $F: h(G) \rightarrow G'$ an analytic function [St5, B9].

In the same period length–area proofs have been used in function theory by Bohr (1918) [Boh], Gross (1918) [Gro], Faber (1922) [Fa], Courant (1922) [HC, in 4th edn III, 6, Section 4, Boundary correspondence, pp. 407–412]. “In its simplest form this method involves the possibility of giving geometrically certain estimates on length of curves and the area of some region swept out by them together with an application of Schwarz's inequality” (quotation from Jenkins [Je4, p. 7]). Grötzsch essentially improved this proof idea creating his “*method of strips*” based on the modules, conformal invariants of quadrilaterals and doubly connected domains, and exposed it in the two lemmas of his first paper: *Über einige Extremalprobleme der konformen Abbildung I* presented by Koebe in the session of 23 July, 1928 of the Science Academy of Saxonia in Leipzig [Gr1].

This method called also the *Grötzsch principle* (see Golusin [Go, IV, Section 5]) became a powerful tool in the study of conformal mappings and univalent (analytic) functions. By it Grötzsch obtained in an outstanding series of papers most of the results already known and many new ones, for simply- or finitely-connected domains and even for infinitely-connected ones.

Besides the 2nd part of this paper in the session at 10 December, 1928, Koebe presented a 3rd paper: *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes* [Gr2], no less important for the mathematical history since it constitutes the birth-date of the qc and qr mappings.

Indeed in this third paper Grötzsch introduced his $\mathcal{A}_Q$ class of mappings.

A bijective mapping of a domain G in the z -plane onto a surface stretched as a Riemann surface over a w -plane is called a $\mathcal{A}_Q$ mapping of bounded infinitesimal distortion Q , if the following conditions are fulfilled:

1. The mapping is continuous everywhere.
2. In general, i.e., excepting at most countably many points which accumulate only to the boundary of G , the mapping is univalent in the neighborhood of each point.
3. At an ordinary point, where the mapping is univalent, an infinitesimal circle is taken over an infinitesimal ellipse with the quotient of the axes a/b verifying the inequalities $1/Q \leq a/b \leq Q$ for the same constant Q at all the ordinary points.
4. At the exceptional points in 2. the mapping is also continuous and a neighborhood of such a point z is mapped onto an n -sheeted surface piece (n finite) so that z corresponds to the ramification point.
5. If $z = x + iy$, $w = f(z) = u + iv$, then in the neighborhood of every ordinary point the partial derivatives u_x, u_y, v_x, v_y are also continuous. For $Q = 1$ the mapping is conformal.

We quoted this definition from [Gr2] to illustrate author's accuracy but also the epoch style.

One sees that an $\mathcal{A}_Q$ mapping is an interior transformation after their definition 2° from above (Grötzsch did not know Stoilow's concept, but he quoted Szillard's thesis) of class C^1 at the ordinary points with the metric constraint 3.

In the following $p = a/b$, denoted by D in [LV3, English edn, p. 17], will be called the *dilatation quotient* and will play a main role.

We shall call an $\mathcal{A}_Q$ mapping a *Grötzsch Q -qr mapping* and if it is a homeomorphism a *Grötzsch Q -qc mapping* (in [LV3] they are called *regular Q -qc function*, respectively *mapping*, and Q is replaced by K).

Among the multitude of proposed generalizations of complex analytic functions, that given by Grötzsch joining the topological properties of these functions with the metric condition 3 (which relaxes the conformal invariance of infinitesimal circles) was the most natural and proved to be the most important one by its numerous, sometimes surprising applications.

Section 1 of this chapter presents some properties concerning the l. behavior of differentiable mappings. On this occasion some l. distortion functions (or characteristics) of the mapping will be introduced.

Section 2 deals with another tool of $qcty$, the module of some configurations (quadrilaterals, ring domains) and the module or extremal length of curve families.

Section 3 is dedicated to the qc and qr mappings in the sense of Grötzsch with emphasis on the work of Grötzsch, Ahlfors, Lavrent'ev and Teichmüller and corresponds to the development of the theory till about 1950.

Nevertheless there have appeared some difficulties working in C^1 : e.g., the solution of some extremal problems presented exceptional points, even arcs, where it was no more differentiable (Grötzsch's definition admitted such points), and more important the Grötzsch's class was not closed with respect to locally uniform (l.u.) convergence.

Section 4 exposes the introduction of the actual general class of qc and qr mappings, starting with the geometric definition by Pfluger (1951) [Pf4] and Ahlfors (1954) [A4] based on the Grötzsch module inequality and continuing with the proof of regularity properties, among which the differentiability a.e., and the connection with Beltrami equation by Mori (1957) [Mo2], Bers (1957) [B9], Pfluger (1959) [Pf7], Gehring and Lehto (1959) [GL] and others, till the analytic definition was established. Other definitions are also discussed and the chapter reflects the stage of the theory till about 1960 when it was already crystallized in the two-dimensional case.

Ulterior the theory developed in two main directions:

(I) extension of the frame from $\mathbb{R}^2$ to $\mathbb{R}^n$, $n \geq 3$, and further till the actual researches in metric spaces, and

(II) relaxing of the constraints imposed to the distortion, instead of a dilatation bounded by a constant a finite dilatation bounded by a function with certain growth properties. This idea was not new, results on mappings with a dilatation bounded on compacts or depending on points were obtained in the 30s in the type problem by Lavrent'ev (1935) [La2], in extremal problems by Teichmüller (1938) [T3], and later by Kühnau beginning from 1964 [Kü2], see also [KrKü] and [Kü13], Lehto from 1966 [L3,L6], and others.

However it was especially David's thesis [Da] that determined a significant increase of the interest in studying this more general case which was connected also to applications, e.g., in nonlinear elasticity. As a now classical result in this direction we mention Stoilow's type factorization for mappings with integrable dilatation due to Iwaniec and Šverak (1993) [IS]. As an other example, we quote the mappings with the distortion bounded by a BMO -function [Rei,ReiRy,RSY,ACS1–ACS3,Sta1,Sta2]. Now mappings of finite distortion are intensively studied.

In the whole chapter (and also in References) the two-dimensional case remains the main theme. The chapter is far from being exhaustive, e.g., we do not treat of problems about Teichmüller spaces and Kleinian groups, nor connections with partial differential equations, which are only mentioned Section 4.4.2. Even before 1960 many papers extended to qc or qr mappings different properties of analytic functions (e.g., concerning value distribution, distortion, boundary behavior, convergence) but only some of them (especially on type problem) could be discussed. References complete partially the information. For instance, References include apart from the papers on the type problem discussed in the text those by Nevanlinna [Ne1,Ne3], Wittich [Wi1,Wi3], Volkovyskij [Vo1,Vo2], Theim [Thi], and the important cycles of Lehto's papers dedicated to Beltrami equation [L3,L6,L7,L9] and to quasiconformal extension of univalent functions [L5,L8,L11]. They also contain the first papers concerning mappings in space by Lavrent'ev (1938) [La3], Markušević (1940) [Mar], Kreines (1941) [Kre], Nevanlinna (1955) [Ne5], Zimmermann [Zim], and after Loewner's paper (1959) [Loe] on conformal capacity, the papers which founded in 1960–1961 the systematic study of qc mappings in space: by Gehring [G1–G4], Väisälä [Vä2,GV1], Caraman [C1–C3] and the monographs [C4,Vä5]. The theory of qr mapping in space began by Reshetnyak's work. References include his first papers on $qrty$ from 1966–1968 [R1–R6] and his monograph [R7] (where one finds a complete list of papers). Further in 1968–1971 Martio, Rickman and Väisälä [MRV1–MRV3] constructed the whole theory now exposed in [Vu2] and [Ri4].

We remark that usually for proofs or more details we shall refer to the English edition of the book by Lehto and Virtanen [LV3]. Instead of Q the *qcty* constant will be also denoted by K as in [LV3].

If the dilatation quotient p is bounded by a constant which is not indicated the mapping is simply called *qc* or *qr*.

The name *qc* appeared in 1935 in Ahlfors' famous memoir [A4] and that of *qr* disseminated when the theory extended in early 1970s through the papers by Martio, Rickman and Väisälä [MRV1–MRV3]. Before the *qr* mappings have been called pseudo-analytic and pseudo-regular or in [LV3] quasiconformal functions and Reshetnyak preserved the name of mappings of bounded distortion (1989) [R7].

1. Differentiability properties

1.1. Definitions of differentiable mappings

Let G be a domain in the plane $z = x + iy$ and $w = w(z)$ a continuous mapping into the plane $w = u + iv$.

The mapping $w(z)$ is *differentiable* at the point $z_0 \in G$ if it admits finite partial derivatives at z_0 : $w_x(z_0)$ and $w_y(z_0)$, and can be written

$$w(z) = w(z_0) + w_x(z_0)(x - x_0) + w_y(z_0)(y - y_0) + \varepsilon(z, z_0)(z - z_0), \quad (1.1)$$

where $\varepsilon(z, z_0) \rightarrow 0$ as $z \rightarrow z_0$. The affine transformation associate to $w(z)$ at z_0 will be denoted by

$$Aw(z_0)(z) = w_0 + w_x(z_0)(x - x_0) + w_y(z_0)(y - y_0) \quad (1.2)$$

and the differential interpreted as a linear mapping by

$$Dw(z_0)(z) = w_x(z_0)x + w_y(z_0)y. \quad (1.3)$$

It is very useful in *qcty* to consider also the complex derivation which Poincaré introduced by means of the relations $z = x + iy$, $\bar{z} = x - iy$, namely

$$Aw(z_0)(z) = w(z_0) + w_z(z_0)(z - z_0) + w_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0) \quad (1.2')$$

and

$$Dw(z_0)(z) = w_z(z_0)z + w_{\bar{z}}(z_0)\bar{z}, \quad (1.3')$$

where $w_z = \frac{1}{2}(w_x - iw_y)$ is the mean of the derivatives in x -axis, respectively, in y -axis direction and $w_{\bar{z}} = \frac{1}{2}(w_x + iw_y)$ has the geometrical significance given by Pompeiu [Pom1, Pom2] as areolar derivative

$$\lim_{r \rightarrow 0} \left(\frac{1}{2\pi i} \int_c w(z) dz \right) / \left(\frac{1}{\pi} \iint_\delta dx dy \right),$$

where δ is a Jordan domain included in $B(z_0, r) = \{z: |z - z_0| < r\}$ and with rectifiable boundary curve c .

The mapping $w(z)$ is conformal at z_0 if it verifies Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \iff w_x = -iw_y \iff w_{\bar{z}} = 0.$$

As we shall consider only sense-preserving (s.-p.) mappings (see [LV3, I, 1.4–1.6]) the Jacobian of $w(z)$ at z_0

$$J(z_0) = u_x v_y - u_y v_x = |w_z|^2 - |w_{\bar{z}}|^2 \geq 0. \quad (1.4)$$

We remind: z_0 is called a *regular point* of $w(z)$ if $w(z)$ is differentiable at z_0 and $J(z_0) > 0$.

We shall also work with the derivative of $w(z)$ in the direction $\varphi \in [0, 2\pi)$:

$$\begin{aligned} \partial_\varphi w(z_0) &= \lim_{r \rightarrow 0} \frac{w(z_0 + r e^{i\varphi}) - w(z_0)}{r e^{i\varphi}} \\ &= (w_x(z_0) \cos \varphi + w_y(z_0) \sin \varphi) e^{-i\varphi} \\ &= w_z(z_0) + w_{\bar{z}}(z_0) e^{-2i\varphi}, \end{aligned} \quad (1.5)$$

last formula appearing by Călugăreanu [Că1].

If γ is a Jordan arc with the origin at z_0 and tangent at z_0 of argument φ , then $|\partial_\varphi w(z_0)|$, called the *dilatation in the direction* φ , will give the quotient of the line elements ds'/ds on $\gamma' = w(\gamma)$ and γ at w_0 and z_0 , respectively, and if $\partial_\varphi w(z_0) \neq 0$, then γ' will have a tangent at w_0 of argument φ' with $\varphi' - \varphi = \arg \partial_\varphi w(z_0) \pmod{2\pi}$.

1.2. Dilatation quotient

Let z_0 be a regular point of $w(z)$. The associate affine transformation $Aw(z_0)$ maps circles with center z_0 onto homothetic ellipses with center w_0 . Having in view of geometrical interpretations, let us consider however the ellipses

$$E(x - x_0)^2 + 2F(x - x_0)(y - y_0) + G(y - y_0)^2 = \rho^2 \quad (1.6)$$

transformed by $Aw(z_0)$ into circles $|w - w_0| = \rho$, where $E = u_x^2 + v_x^2 = |w_x|^2$, $F = u_x v_x + u_y v_y$ and $G = u_y^2 + v_y^2$ at z_0 . Denoting by $a(z_0)$ and $b(z_0)$ the axes of these ellipses, their family is characterized after Lavrent'ev [La2] by $p(z_0) = a(z_0)/b(z_0)$ and if $p(z_0) \neq 1$ (i.e., $w(z)$ is not conformal at z_0) by $\theta(z_0) \in [0, \pi)$ the angle of the great axis and the positive x -axis:

$$\begin{aligned} p &= \frac{E + G + \sqrt{(E + G)^2 - 4J^2}}{2J}, \quad J^2 = EG - F^2, \text{ and} \\ \operatorname{tg} \theta &= \frac{E - G + \sqrt{(E - G)^2 + 4F^2}}{-2F}, \quad -F \operatorname{tg} \theta \geq 0. \end{aligned} \quad (1.7)$$

Details on these properties are given, e.g., in [Vo3, Sections 1–3], [Kün3, 2.1 and 2.2.], [A12, I, A]. The ellipse with the small axis b is denoted by Volkovskij $\mathcal{E}_b(z_0, p, \theta)$, p and θ being called the *characteristics* of the ellipse, of $Aw(z_0)$ and of $w(z)$ at z_0 .

If $Aw(z_0)$ carries $\mathcal{E}_b(z_0, p, \theta)$ (1.6) into the circle $|w - w_0| = \rho$, $w(z)$ transforms it into a Jordan curve which tends to this circle as $b \rightarrow 0$. In classical notation $w(z)$ maps infinitesimal ellipses $E dx^2 + 2F dx dy + G dy^2 = d\rho^2$ onto infinitesimal circles $du^2 + dv^2 = d\rho^2$ and Lavrent'ev [La2] formulated this property by the condition

$$\lim_{b \rightarrow 0} \frac{\max_{z \in \mathcal{E}_b} |w(z) - w_0|}{\min_{z \in \mathcal{E}_b} |w(z) - w_0|} = 1. \quad (1.8)$$

As $abJ = \rho^2$, if $p \neq 1$

$$\begin{aligned} \min_{\varphi} |\partial_{\varphi} w| &= \frac{\rho}{a} = \sqrt{\frac{J}{p}} \text{ for } \varphi = \theta \pmod{\pi}, \\ \max_{\varphi} |\partial_{\varphi} w| &= \frac{\rho}{b} = \sqrt{pJ} \text{ for } \varphi = \theta + \frac{\pi}{2} \pmod{\pi}, \end{aligned} \quad (1.9)$$

or with complex derivatives

$$\min_{\varphi} |\partial_{\varphi} w| = |w_z| - |w_{\bar{z}}| \text{ and } \max_{\varphi} |\partial_{\varphi} w| = |w_z| + |w_{\bar{z}}|. \quad (1.9')$$

Hence, denoting (as in [A12, I (11), p. 4] and [LV3, I, 3.3, p. 17]) by

$$D = \frac{\max_{\varphi} |\partial_{\varphi} w|}{\min_{\varphi} |\partial_{\varphi} w|} \quad (1.10)$$

the *dilatation quotient*, one has as $J > 0 \Leftrightarrow |w_z| > |w_{\bar{z}}|$,

$$D = p = \frac{|w_z| + |w_{\bar{z}}|}{|w_z| - |w_{\bar{z}}|}$$

and from (1.9), (1.10')

$$\arg w_{\bar{z}} - \arg w_z = 2\theta + \pi \pmod{2\pi}.$$

1.3. Complex dilatation and Beltrami equation

Ahlfors [A4] introduced the *complex dilatation*

$$\mu(z_0) = \frac{w_{\bar{z}}}{w_z}, \quad (1.11)$$

a new characteristic for $w(z)$, which verifies

$$\mu = -\frac{p-1}{p+1}e^{2i\theta} \iff p = \frac{1+|\mu|}{1-|\mu|}, \quad \arg \mu = 2\theta + \pi \pmod{2\pi}. \quad (1.11')$$

As $p \geq 1$, it follows $|\mu| \in [0, 1)$ and the conformal case $p = 1$ corresponds to $\mu = 0$. Further

$$\mu = \frac{E - G + 2iF}{E + G + 2J}. \quad (1.11'')$$

This characteristic μ became very important by its connection with the *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \quad (1.12)$$

verified by $w(z)$ at z_0 , or in real variables the Beltrami system of equations generalizing the Cauchy–Riemann one:

$$\begin{aligned} \alpha u_x + \beta u_y &= v_y, \\ \beta u_x + \gamma u_y &= -v_x, \end{aligned} \quad (1.12')$$

where $\alpha = G/J$, $\beta = -F/J$ and $\gamma = E/J$, $\alpha\gamma - \beta^2 = 1$. These coefficients are also given by the relations

$$\begin{aligned} \alpha &= p \cos^2 \theta + \frac{1}{p} \sin^2 \theta, \\ \beta &= \left(p - \frac{1}{p}\right) \cos \theta \sin \theta, \\ \gamma &= p \sin^2 \theta + \frac{1}{p} \cos^2 \theta, \end{aligned} \quad (1.12'')$$

the conformal case corresponding to $\alpha = \gamma = 1$ and $\beta = 0 \Leftrightarrow E = G$ and $F = 0$.

Concerning the angle distortion at z_0 , Juve [Ju] established that if the direction φ is transformed into φ'

$$\frac{1}{p} \leq \frac{d\varphi'}{d\varphi} = \frac{1}{|\partial_\varphi w|^2} J \leq p, \quad (1.13)$$

with equality at the left-hand side for $\varphi = \theta + \frac{\pi}{2}$ and at the right-hand side for $\varphi = \theta$.

Let G , G' , G^* be domains in the plane, $f: G \rightarrow G'$, $f(z) = w$ and $g: G' \rightarrow G^*$, $g(w) = \zeta$ be continuous mappings with $z_0 \in G$ and $w_0 = f(z_0)$ regular points for f and g , respectively. Then by the chain rule one easily obtains in z_0 :

$$\mu_{g \circ f} = \frac{\mu_f + (\mu_g \circ f)e^{-2i \arg f_z}}{1 + \mu_f(\mu_g \circ f)e^{-2i \arg f_{\bar{z}}}} \quad (1.14)$$

or in the Ahlfors form [A12, I (6), p. 9], which reminds the Moebius transformations of the unit disk,

$$\mu_g \circ f = \frac{f_z}{\bar{f}_z} \frac{\mu_{g \circ f} - \mu_f}{1 - \bar{\mu}_f \mu_{g \circ f}}. \quad (1.14')$$

One sees that $|\mu|$ is a conformal invariant, more exactly if g is conformal at w_0 then $\mu_{g \circ f} = \mu_f$ and if f is conformal at z_0 then $\mu_g \circ f = (f'/|f'|)^2 \mu_{g \circ f}$.

Further if f is a homeomorphism

$$\mu_{f^{-1}} = -\frac{f_z}{\bar{f}_z} (\mu_f \circ f^{-1}) = -v_f \circ f^{-1}, \quad (1.15)$$

where

$$v_f = \frac{f_z}{\bar{f}_z} = \left(\frac{f_z}{|f_z|} \right)^2 \mu_f \quad (1.15')$$

is a *second complex dilatation* which gives the characteristics of the ellipse at w_0 image by $Af(z_0)$ of a circle at z_0 :

$$p_{f^{-1}} = p_f \quad \text{and} \quad \theta_{f^{-1}} = \theta_f + \arg f_z + \frac{\pi}{2} \pmod{\pi} \quad (1.16)$$

in corresponding points z_0 and w_0 . If φ and φ' are corresponding directions under $Af(z_0)$, then one has $\partial_\varphi f(z_0) = (\partial_{\varphi'} f^{-1}(w_0))^{-1}$.

The characteristic v determines another equation verified by f at the regular point z_0 :

$$w_{\bar{z}} = v \bar{w}_{\bar{z}}. \quad (1.17)$$

2. Modules and extremal length

2.1. Module of a quadrilateral and of a ring domain

Conformal invariants play an important role in *qcty*. As Lehto underlined [L10, p. 205], even the notion of *qcty* is obtained assuming the quasiinvariance of a conformal invariant, and firstly of the module of quadrilaterals and of ring domains.

A quadrilateral $Q(z_1, z_2, z_3, z_4)$ is a Jordan domain Q with a sequence of four points z_1, z_2, z_3, z_4 on the boundary ∂Q : the vertices, which determine a positive orientation of ∂Q with respect to Q . These points divide ∂Q into four arcs: the sides (z_1, z_2) , (z_2, z_3) , (z_3, z_4) and (z_4, z_1) .

By conformal mapping Q can be represented onto a “canonical” rectangle $R = \{w = u + iv: 0 < u < a, 0 < v < b\}$ such that the points z_1, z_2, z_3, z_4 correspond to the vertices $0, a, a + ib, ib$, respectively. The sides (z_1, z_2) and (z_3, z_4) will be called the a -sides, while (z_2, z_3) and (z_4, z_1) the b -sides.

The number $\frac{a}{b}$ does not depend on the conformal mapping used $f: Q \rightarrow R$, it is a conformal invariant and is called *the (conformal) module* of $Q(z_1, z_2, z_3, z_4)$ with respect to the a -sides: $M(Q(z_1, z_2, z_3, z_4))$, the mapping f being uniquely determined up to similarity transformations (see, e.g., [LV3, I 2.4, p. 15]). Evidently, $M(Q(z_2, z_3, z_4, z_1)) = 1/M(Q(z_1, z_2, z_3, z_4))$. (Other authors prefer to define $M(Q(z_1, z_2, z_3, z_4)) = b/a$ [KrKü, p. 96].)

A ring domain B is a doubly-connected domain, whose boundary components are continua. It is conformally equivalent with an annulus $C = \{w: 0 < r < |w| < R < \infty\}$. By cutting radially this annulus, e.g., along $\arg w = 0$, one obtains a quadrilateral which is represented by $\zeta = \log w$ onto the rectangle $\{\zeta = \xi + i\eta: \log r < \xi < \log R, 0 < \eta < 2\pi\}$. The module of B is defined as $\frac{1}{2\pi} \log \frac{R}{r}$. (Other authors define $M(B) = \log \frac{R}{r}$, see [LV3, I, Section 6, p. 31].)

Grötzsch used these modules together with the length–area method from beginning of his work [Gr1] (1928). Thus let us consider the conformal mapping $f: Q \rightarrow R$ from above, denote $g = f^{-1}: R \rightarrow Q$ and call *module line* each arc $\gamma_u = g(\{w: u = ct, 0 < v < b\})$, $0 < u < a$. Then the length of γ_u : $L(\gamma_u) = \int_0^b |g'(w)| dv$ and the area of Q : $A(Q) = \iint_R |g'(w)|^2 du dv$. By Fubini theorem and Schwarz inequality

$$A = A(Q) = \int_0^a du \int_0^b |g'(w)|^2 dv \geq \frac{1}{b} \int_0^a L(\gamma_u)^2 du \geq \frac{a}{b} L^2, \quad (2.1)$$

where $L = \inf_{\gamma_u} L(\gamma_u)$, and it follows for $M = M(Q)$

$$M \leq \frac{A}{L^2}, \quad (2.2)$$

the *basic inequality of the Grötzsch method of strips* [Kü15, p. 486].

This method was formulated by Grötzsch for ring domains in two lemmas [Gr1].

LEMMA 1 (Principle of Grötzsch [Go, p. 140], [LV3, I, 6.8, Lemma 6.5, p. 37]). *Let be in the annulus $C = \{z: 0 < r < |z| < R < \infty\}$ a finite or infinite number of nonoverlapping strips S_k , $k = 1, 2, \dots$, having at most common boundary arcs. Every S_k is the conformal image of a rectangle R_k so that to the vertices of R_k correspond in a continuous sense the points P_k, P'_k on $|z| = r$ and P''_k, P'''_k on $|z| = R$, the mappings of the boundaries being regular. Let a_k be the length of the side of R_k corresponding to $P_k P'_k$ and b_k that of the other pair of sides. Then*

$$\sum_k \frac{a_k}{b_k} \leq \frac{2\pi}{\log \frac{R}{r}}. \quad (2.3)$$

Equality holds iff all S_k are obtained by radial sections of C and fill C completely.

Golusin proves this lemma for a finite number of strips and emphasizes that each a -side of S_k must contain a nondegenerate arc on $|z| = r$, respectively, on $|z| = R$.

Clearly (2.3) signifies $\sum_k M(S_k) \leq \frac{1}{M(C)}$.

LEMMA 2 [LV3, I, 6.6, Lemma 6.3, p. 35]. *If one considers (instead of S_k from Lemma 1) strips S_k which round $|z| = r$, so that by cutting them to be represented onto rectangles R_k with module $\frac{a_k}{b_k}$ how small, one obtains*

$$\sum_k \frac{a_k}{b_k} \leq \frac{\log \frac{R}{r}}{2\pi} \quad (2.4)$$

and the maximal subdivision of C is given by concentric circles (i.e., equality holds iff every S_k is an annulus and $\sum_k A(S_k) = A(C)$).

Again evidently (2.4) means $\sum_k M(S_k) \leq M(C)$.

Grötzsch remarked in a footnote that Koebe knew such a result but had not published it.

A similar lemma holds for quadrilaterals, and is contained in Grötzsch's dissertation [Gr3] (1929). Golusin proves it in his book [Go, p. 142], together with other results by Lavrent'ev and Shepelev, by himself, by Bermant.

The strip method was also applied by Rengel in his dissertation (1932) [R1] from which we quote the so called *Rengel's inequalities* for a quadrilateral:

$$\frac{s_b^2}{A(Q)} \leq M(Q(z_1, z_2, z_3, z_4)) \leq \frac{A(Q)}{s_a^2}, \quad (2.5)$$

where s_a and s_b are the Euclidean distances between the a -sides and, respectively, the b -sides, with equality iff $Q(z_1, z_2, z_3, z_4)$ is a rectangle (see [Thi, p. 22]). In fact, (2.5) and (2.2) coincide. In [R1] Rengel proved also by Grötzsch's strip method the Lavrent'ev and Shepelev result from above announced in 1930 [LaShe].

2.2. Extremal length and module of curve family

Many other results were obtained by procedures derived from length–area method, called also the method of the extremal metric, by Beurling (1934), Teichmüller (1937, 1938 and later), Spencer (1940) (see Jenkins' book [Je4, I, 1.6, pp. 7–9, and references therein]). Gradually a new method, that of the *extremal length* was conceived by Beurling, presumably in 1943–1944 writes Ahlfors in [A15, p. 81], first presented at the Scandinavian Congress of Mathematicians in Copenhagen 1946 in parallel papers by Beurling and Ahlfors [ABeu1] and published in a systematic account in [ABeu2].

In contrast with the length–area method which works with Euclidean length and area “the extremal length method derives its higher degree of flexibility and usefulness from more general ways of measuring” writes Ahlfors in his book *Conformal Invariants* [A15, p. 50].

In a domain G of the z -plane, let Γ be a family of l. rectifiable paths: curves or arcs γ ; more generally γ could be a finite union of such paths. The aim is to define a conformal invariant $\lambda_G(\Gamma)$, i.e., a number such that if $f: G \rightarrow G'$ is a conformal mapping and Γ' the image family of Γ , then $\lambda_{G'}(\Gamma') = \lambda_G(\Gamma)$. One considers the Riemannian metrics

$ds = \rho|dz|$ attached to G in a conformally invariant way: $\rho|dz| \rightarrow \rho'|dz'|$, $\rho' = \rho|\frac{dz}{dz'}|$, where $z' = f(z)$ and ρ is a nonnegative Borel measurable function in G . Then each γ will have a well-defined length $L(\gamma, \rho) = \int_{\gamma} \rho|dz| = L(\gamma', \rho')$, $\gamma' = f \circ \gamma$ and $A(G, \rho) = \iint_G \rho^2 dx dy = A(G', \rho')$. One introduces also a conformal invariant which depends on the whole family Γ , namely

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho).$$

By definition, the *extremal length* of Γ with respect to G is the conformal invariant

$$\lambda_G(\Gamma) = \sup_{\rho} \frac{L(\Gamma, \rho)^2}{A(G, \rho)}, \quad (2.6)$$

where ρ is subject to the condition $0 < A(G, \rho) < \infty$, and the *module* of Γ is

$$M_G(\Gamma) = \frac{1}{\lambda_G(\Gamma)}. \quad (2.7)$$

One easily sees that the module depends of Γ and not of the domain G . Indeed if $G^* \supset G$, any function ρ considered for $M_G(\Gamma)$ can be extended to G^* by $\rho^* = \rho$ on G and $\rho^* = 0$ on $G^* \setminus G$ so that $M_G(\Gamma) = M_{G^*}(\Gamma)$. Therefore we shall omit the index G in the notation.

There are several variants of this definition. In his lectures at Harvard University in the Spring of 1947, Ahlfors presented many calculations and results about the extremal length of curve families. They were never published in journals and only some of them are contained in the Oklahoma mimeographed lectures [A3] together with variants (normalizations) of this definition [A3, p. 70]. We shall use the following variant (see Hersch [H1], Pfluger [Pf5] and monographs by Pfluger [Pf6, 43.2, p. 211], Jenkins [Je4, II, p. 14], Künzi [Kün3, 1.14, p. 12], Ohtsuka [O2, II], Lehto and Virtanen [LV3, II, Section 4]).

Let Γ be a family of paths (arcs, curves) γ in the domain G . Then

$$M(\Gamma) = \lambda^{-1}(\Gamma) = \inf_{\rho \in \text{Adm}(\Gamma)} A(G, \rho), \quad (2.8)$$

where $\text{Adm}(\Gamma)$ (the set of functions ρ admissible for Γ) consists in all nonnegative Borel measurable functions ρ which satisfy the condition

$$\int_{\gamma} \rho|dz| \geq 1 \quad (2.9)$$

for each l . rectifiable γ in Γ . If $\text{Adm}(\Gamma) = \emptyset$, then $\lambda(\Gamma) = 0$ by definition.

If $\rho_0 \in \text{Adm}(\Gamma)$ satisfies the relation $M(\Gamma) = A(G, \rho_0)$, then the metric $\rho_0(z)|dz|$ is called *extremal* and if such a metric exists it is uniquely determined a.e. in G . By a conformal mapping an extremal metric is transformed in an extremal one.

As an application of the module of a curve family one can define the module of a quadrilateral without using the conformal mappings and deduce also the conformal invariance of this module:

THEOREM. *The module $M(Q(z_1, z_2, z_3, z_4))$ with respect to the a -sides is equal to the module of the family Γ_a of arcs which join in Q the a -sides, and to the extremal length of the family Γ_b which join in Q the b -sides, i.e., separates in Q the a -sides.*

The proof repeats that of (2.2): Let Q be first the rectangle $R = \{z: 0 < x < a, 0 < y < b\}$. As the length of every rectifiable curve in Γ_a is greater than or equal to b , $\rho = 1/b \in \text{Adm}(\Gamma_a)$ and $M(\Gamma_a) \leq a/b$. Take now an arbitrary $\rho \in \text{Adm}(\Gamma_a)$. Since each vertical segment $R_x = \{z: 0 \leq y \leq b\}$, x fixed, $0 < x < a$, belongs to Γ_a , $1 \leq \int_0^b \rho(x + iy) dy$. By Schwarz inequality, $1 \leq b \int_0^b \rho^2(z) dy$, hence by integrating with respect to x : $a/b \leq \int_0^a dx \int_0^b \rho^2(z) dy = A(R, \rho)$, what proves $a/b \leq M(\Gamma_a)$. (One sees that $(1/b)|dz|$ is extremal metric and $M(\{R_x\}_{0 < x < a}) = a/b$.) As the module of a curve family is a conformal invariant, the Theorem is proved for every quadrilateral Q .

AHLFORS' LEMMA. *If a quadrilateral Q is included in a horizontal strip between parallel lines with width β and if the b -sides are separated by a vertical strip of width α , then $M(Q) \geq \alpha/\beta$.*

Similarly the module $M(B)$ of a ring domain B with the boundary consisting of two continua K_1 and K_2 is equal to the module of the family of curves which separate in B the two continua, and the extremal length of the curve family which join them in B .

A generalization is given by the *extremal distance* $d_G(E_1, E_2)$ of two sets E_1 and E_2 in $\overline{G}$, G – a domain as before, which is defined as the extremal length $\lambda(\Gamma)$ of the family Γ of connected arcs which join E_1 and E_2 in G (having one end point in E_1 and one in E_2 , and all the other points in G). The *conjugate extremal distance* $d_G^*(E_1, E_2)$ is defined as $\lambda(\Gamma^*)$, where Γ^* consists of all curves in G which separate E_1 and E_2 [A15, pp. 52–53].

2.3. p -module

The extremal length was further generalized in 1957 by Fuglede [F] and became a basic tool in higher-dimensional *qcty*, see, e.g., the books by Väisälä [Vä5, I, 6], Caraman [C4, Parts 1 and 5], Vuorinen [Vu2, II], Rickman [Ri4, II, 1], Heinonen [He1, 7].

Fuglede introduced the *module of order* $p \geq 1$ (p -module) of a curve family and of a q -dimensional surface family Γ in $\mathbb{R}^n$, $q < n$. More general, he defined the module of a class of measures in an abstract space. In our two-dimensional case G , Γ and $\text{Adm}(\Gamma)$ being as before, the p -module of Γ is

$$M_p(\Gamma) = \inf_{\rho \in \text{Adm}(\Gamma)} \iint_G \rho^p dx dy, \quad (2.10)$$

the usual module $M(\Gamma)$ corresponding to $p = 2$.

M_p is an outer measure in the space of all curves in G :

- (1) $M_p(\emptyset) = 0$,
- (2) $\Gamma_1 \subset \Gamma_2 \Rightarrow M_p(\Gamma_1) \leq M_p(\Gamma_2)$,
- (3) $M_p(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$.

Among other properties of the module let us remind:

If every $\gamma \in \Gamma_2$ has a subcurve $\gamma_1 \in \Gamma_1$, then $M_p(\Gamma_2) \leq M_p(\Gamma_1)$.

If for the families Γ_i , $i = 1, 2, \dots$, there exist disjoint Borel sets $E_i \subset \mathbb{C}$ with $\int_{\gamma} \chi_{\mathbb{C} \setminus E_i} |dz| = 0$, for all $\gamma \in \Gamma_i$ ($\chi_{\mathbb{C} \setminus E_i}$ – the characteristic function of $\mathbb{C} \setminus E_i$), and if every $\gamma \in \Gamma_i$ contains a subcurve in Γ , then $\sum_{i=1}^{\infty} M_p(\Gamma_i) \leq M_p(\Gamma)$, with equality when $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$.

If the curves in Γ have the length greater than or equal to L and A denotes the area of G , then

$$M_p(\Gamma) \leq \frac{A}{L^p}. \quad (2.11)$$

2.4. Module with weight, generalizations and length–area dilatation

Another generalization was given by the *module* (or *extremal length*) with *weight* of Ohtsuka [O1].

Let π be a nonnegative measurable function defined in G , Γ a curve family in G and $\rho \in \text{Adm}(\Gamma)$ as before. Then the *module with weight* π of Γ is by definition

$$M_{\pi}(\Gamma) = \inf_{\rho \in \text{Adm}(\Gamma)} A_{\pi}(G, \rho), \quad (2.12)$$

where

$$A_{\pi}(G, \rho) = \iint_G \pi \rho^2 dx dy. \quad (2.13)$$

Again $\lambda_{\pi}(\Gamma) = M_{\pi}^{-1}(\Gamma)$.

The module with weight found important applications in *qcty*, especially in extremal problems [Kü2, Kü3, Kü9, Kü13, Kü15, KrKü, AC4, AC6, AC7, AC10, AC11, AC13, AC16], elliptic systems of partial differential equations, and thus hydrodynamics [La7], gas dynamics [B10], elasticity [P7], see [Kü13, Kü15]. The connection with the elliptic system, which we write denoting the weight with p instead of π to preserve the tradition

$$u_x = p v_y, \quad u_y = -p v_x \quad (2.14)$$

was established by Duffin in 1962 [Du].

Kühnau extended further the module with weight in the nonlinear sense [Kü9, Kü10]. Namely, the weight p can vary being submitted to a condition $\iint_G \Phi(p(x, y)) dx dy \leq C$, with a fixed (sufficiently large) constant C and a fixed convex function $\Phi(p)$. The new module is

$$M'(\Gamma) = \sup_p \left\{ \inf_{\rho \in \text{Adm}(\Gamma)} \iint_G p \rho^2 dx dy \right\} \quad (2.15)$$

and it is connected with the system (2.14), where p is a function of $u_x^2 + u_y^2$ obtained from Φ [Kü9]. In particular, the case $\Phi(p) = \frac{1}{p} + \frac{p}{\alpha}$, with α – a constant, corresponds to the minimal surface equation [Kü9, Kü10, KrKü, II. Teil, V, p. 150].

Even in 1935 by Ahlfors [A2] and Lavrent'ev [La2], later Kakutani [Kak], Teichmüller [T2], integrals are associated to curve families depending on one parameter (e.g., concentric circles) in order to obtain criteria of regular exhaustibility respectively of the parabolic type for simply connected Riemann surfaces. (Such a surface is of parabolic or hyperbolic type as it can be mapped conformally onto $\mathbb{C}$ or onto $\{z: |z| < 1\}$.) A majorant of the module of an annulus by such an integral was given even by Rengel [R1] and completed with another by Akaza and Kuroda [AK]. Among the numerous researches on the type problem, those by Volkovyskij [Vo2], his hyperbolic criterion, contain integrals and inequalities of the same kind, giving a “continuous form to Grötzsch's principle”, since the integral can be interpreted as a limit case of module sums.

In the same way, this time with the extremal length and by using also the module of elements of the ideal boundary of an open Riemann surface, integral criteria were obtained taking into account not only the dilatation quotient but also the orientation of the characteristic infinitesimal ellipses by Andreian Cazacu [AC1, AC2]. Extremal length extends directly on Riemann surfaces, where the theory of the Green function and harmonic measure ([Ne4, X], [Pf6, Section 43], [As1, IV, Sections 3 and 4]) and the module of boundary components (Jurchescu [J1, J2]) completing their capacity (Sario [Sa3]) were already accomplished.

In order to solve an extremal problem proposed by Volkovyskij [Vo4] (and before by Teichmüller [T4, p. 15]) for qc mapping with dilatation quotient given or bounded by a given function (“mit ortsabhängiger Dilatationsbeschränkung”), Kühnau established an important formula for *the module of a family of curves depending on one parameter* [Kü2]:

Let $S = \{C(t): t' < t < t''\}$ be a curve family in the domain $G \subset \mathbb{C}$, the curves $C(t)$ being defined by a diffeomorphism $z = z(\tau, t): g \rightarrow G$ of a domain $g = \{\tau + it: \tau_1(t) < \tau < \tau_2(t), t' < t < t''\}$ onto G , with $\tau_1(t), \tau_2(t)$ continuous functions of t ,

$$|\partial z / \partial \tau| \neq 0, \quad a = |\partial z / \partial t| \cdot |\sin \alpha| \neq 0, \quad \alpha = \arg(\partial z / \partial t) - \arg(\partial z / \partial \tau) \quad (2.16)$$

and $\int_{C(t)} a^{-1} ds$ a finite, continuous function of t (ds – the line element on $C(t)$).

Then the *module of the curve family* S is

$$M(S) = \int_{t'}^{t''} \frac{dt}{\int_{C(t)} a^{-1} ds}. \quad (2.17)$$

Formula (2.17) has numerous applications. In the same paper Kühnau derived from it inequalities for the module of the image curve family by a “differential geometric” mapping (diffeomorphism as $z = z(\tau, t)$ from above) $w = w(z)$ with the quotient dilatation $p(z) \leq p_0(z)$ – a given function ≥ 1 , continuous, some times of class C^1 , bounded in G ,

$$\int_{t'}^{t''} \left(\int_{C(t)} a^{-1} p ds \right)^{-1} dt \leq M(S^*) \leq \int_{t'}^{t''} \left(\int_{C(t)} a^{-1} p^{-1} ds \right)^{-1} dt, \quad (2.18)$$

where S^* is the family of the image curves $C^*(t) = w(C(t))$.

With (2.18) he solved extremal problems (among which that of Teichmüller–Volkovyskij) finding extremal mappings (for which (2.18) inequalities become equality), the orientation θ of their characteristic ellipses (Section 1, (1.7)), the differential equations verified by them. As particular case in [Kü2, Section 5] other module estimations are deduced when $\mathcal{S}$ consists in parallel to x - or y -axis segments. These estimations contain results by af Hällström [Hä1], Hersch [H1], Volkovyskij [Vo2].

Further in [Kü2] the theory is extended to $\mathbb{R}^3$, the formula for the module of a one-parameter surface family is given (similarly, one obtains the module formula for two-parameter curve families) and extremal problems are solved. Here again work by Nevanlinna [Ne5], Šabat [Ša4] and Kühnau [Kü1] are continued.

In the same period 1963, starting from the same Teichmüller–Volkovyskij's problem (mentioned by Šabat in the Preface to the volume [AB2]) Andreian Cazacu established independently *the module with weight formula* for curve families of the same type but defined as the level lines of a module function [J2] a harmonic function giving the module of a configuration (e.g., a quadrilateral, a ring domain, a part of the ideal boundary of a Riemann surface; we called such family modular) or a quasiconformal image of a modular family (called quasimodular) [AC4, AC6]. If the curve family is denoted $\mathcal{C} = \{C_t: t' < t < t''\}$ and the weight is π ,

$$M_\pi(\mathcal{C}) = \int_{t'}^{t''} L_{\pi^{-1}}(t)^{-1} dt \quad (2.19)$$

with

$$L_{\pi^{-1}}(t) = \int_{C_t} \pi^{-1} \left| \frac{dt}{dn} \right| ds, \quad (2.20)$$

where if C_t is given by $t = t(z)$ (z – a local parameter, if we work on a Riemann surface), $\frac{dt}{dn} = |\text{grad } t|$, module of the derivative of t in the normal direction to C_t , and ds is the line element of C_t . One easily verifies that $\frac{dt}{dn} = a^{-1}$ with a in (2.16).

Essential for (2.19) is that through a.e. point $z \in G$ passes one and only one curve of the family $\mathcal{C}$, which has a tangent a.e., is l. rectifiable for a.e. t , and that Fubini theorem works for $\mathcal{C}$ [AC13].

A *transformation formula* for the weighted module of $\mathcal{C}$ results from (2.19) with respect to a qc mapping $f: G \rightarrow G^*$, which we write in l. parameters $z^* = z^*(z)$ if G and G^* are domains of Riemann surfaces [AC4, AC6, AC13, AC21]. Formula was to be applied in $qcty$, so that we supposed that f is a qc mapping (see definition and properties of general qc mappings in Section 4). In order that this formula holds, f may be an s.-p. homeomorphism, differentiable a.e. in G , l. bimeasurable and l. AC on C_t for any t with the possible exception of a $\lambda_{\pi d^{-1}}$ -negligible subfamily of $\mathcal{C}$, f^{-1} having the same properties with respect to $\mathcal{C}^* = f(\mathcal{C})$ and $\pi^* = \pi \circ f^{-1}$. Then

$$M_{\pi^*}(\mathcal{C}^*) = M_{\pi d^{-1}}(\mathcal{C}), \quad (2.21)$$

where d is a *length–area dilatation* of f which already we considered in the integral criteria [AC1,AC2].

Namely, if z is a regular point of f in G where the curve C_t has tangent and $\alpha \in [0, \pi)$ means now the angle between the tangent direction to this curve at z and the major axis of the characteristic ellipse of f at z while p is the dilatation quotient of f at z ,

$$d = d_{f,C_t} = \frac{\cos^2 \alpha}{p} + p \sin^2 \alpha = \left(\frac{ds^*}{ds} \right)^2 \frac{1}{J}. \quad (2.22)$$

Here ds^* is the line element of $C_t^* = f(C_t)$ at $z^* = z^*(z)$, corresponding to ds of C_t at z , and ds^*/ds the dilatation of f in the direction tangent to C_t at z , i.e., the module of the derivative of f along this direction.

A curve family is λ_π -negligible if its extremal length with weight π is zero. If the exceptional family from above lies in a set of measure null in G , then (2.21) holds.

One sees that the connection with a in (2.16) is given by $d = a^{-1} |\partial z / \partial \tau|$. Further $p^{-1} \leq d \leq p$ and $d_{f,C_t}(z) = d_{f^{-1},C_t^*}^{-1}(z^*)$.

Formula (2.21) shows that the module with weight is not only a generalization of the module but is connected in a natural way with it (take $\pi = 1$ in (2.21)).

As a consequence of (2.21) we have a monotony property: Let be $\mathcal{C}$, π and the homeomorphisms $f_k : G \rightarrow G^*$, $k = 1, 2$, C_k^* , π_k^* , d_k as in the transformation formula for $f = f_k$. If $d_1 \leq d_2$ a.e. in G , then $\lambda_{\pi_1^*}(C_1^*) \leq \lambda_{\pi_2^*}(C_2^*)$.

If $d_o \leq d_{f,C} \leq d^\circ$ a.e., with d_o and d° defined a.e. in $G \rightarrow (0, \infty)$, measurable and with the exceptional subfamily $\lambda_{\pi(d^\circ)^{-1}}$ -negligible, then

$$\begin{aligned} \lambda_{\pi d_o^{-1}}(\mathcal{C}) &\leq \lambda_{\pi^*}(C^*) \leq \lambda_{\pi(d^\circ)^{-1}}(\mathcal{C}) \quad \text{or} \\ M_{\pi(d^\circ)^{-1}}(\mathcal{C}) &\leq M_{\pi^*}(C^*) \leq M_{\pi d_o^{-1}}(\mathcal{C}) \end{aligned} \quad (2.23)$$

and for d° , d_o constants

$$(d^\circ)^{-1} M_\pi(\mathcal{C}) \leq M_{\pi^*}(C^*) \leq d_o^{-1} M_\pi(\mathcal{C}). \quad (2.23')$$

The formulas (2.19), (2.21) also extend to the p -module or p -module with weight [AC13] and give solution of Teichmüller–Volkovyskij’s problem and other extremal problems [AC7,AC10,AC11,AC16]. They extend in $\mathbb{R}^n$ for families of curves and of q -surfaces, $2 \leq q \leq n-1$ [AC14,AC17–AC20] and permit to precise for qr mappings the module inequalities by Martio, Rickman and Väisälä [MRV1, 3.2.], Poleckij [Pol, Theorem 2], Väisälä [Vä6, Section 3], by using multiplicity functions and the module with weight.

The module formulas (2.17), (2.19) has been found again about 10 years later by Rodin [Ro].

2.5. Connections and generalizations

The module has connections with other important notions as Dirichlet integral, capacity, see, e.g., Ahlfors [A12,A15], Ziemer [Zi], Caraman [C5–C15], who dedicated to this sub-

ject many papers, Hesse [Hes], Vuorinen [Vu1], [Vu2, II], Heinonen and Koskela [HK2], Heinonen [He1], see also [Ri4]. There is a vast literature concerning extremal length, modules and capacity among which surveys, e.g., Rodin [Ro], Srebro [Sr2].

In connection with the development of the qc theory in general metric spaces these notions obtained a diskrete form which permits to avoid regularity conditions and analytic methods, so that they continued to be crucial tools, see Cannon [Can] – combinatorial Riemann mapping theorem, Heinonen and Koskela [HK1] – diskrete module, Cristea [Cr4], Srebro [Sr2] – diskrete capacity in the sense of [HK1].

3. Grötzsch qc and qr mappings

3.1. Grötzsch's work

As already mentioned qc and qr mappings have been introduced in 1928 by Grötzsch [Gr2] as mappings of class $\mathcal{A}_Q$. We begin this short history of the qc ty and qr ty outset by reformulating Grötzsch's definition of this class which was in general till 1950 the frame of the theory.

Let again G and G^* be domains in the z - respectively w -plane $\mathbb{C}$ and $K \geq 1$ a constant (Q will be denoted in the following by K).

DEFINITION 1. An s.-p. diffeomorphism of class C^1 (i.e., an s.-p. continuously differentiable homeomorphism with the Jacobian $J > 0$) f or $w = w(z) : G \rightarrow G^*$ is a K - qc mapping in the sense of Grötzsch if its dilatation quotient $p(z) = D(z)$ (Section 1.2) verifies in G the inequality

$$p(z) \leq K, \quad (3.1)$$

which is equivalent to each of the following inequalities

$$\max_{\varphi} |\partial_{\varphi} w| \leq K \min_{\varphi} |\partial_{\varphi} w|, \quad (3.1')$$

$$\max_{\varphi} |\partial_{\varphi} w|^2 \leq K J, \quad (3.1'')$$

$$|w_{\bar{z}}| \leq k |w_z| \quad \text{with } k = \frac{K-1}{K+1}, \quad (3.1''')$$

$$|w_{\bar{z}}|^2 + |w_z|^2 = \frac{1}{2}(|w_x|^2 + |w_y|^2) = \frac{1}{2}(E + G) \leq \frac{1}{2}\left(K + \frac{1}{K}\right)J. \quad (3.1^{iv})$$

The inverse of a Grötzsch K - qc mapping is also a Grötzsch K - qc mapping and the composition $f_2 \circ f_1$ of two such mappings $f_1 : G \rightarrow G^*$ and $f_2 : G^* \rightarrow G^{**}$ with $K = K_j$ for f_j , $j = 1, 2$, is a Grötzsch $K_1 K_2$ - qc mapping; $K = 1$ corresponds to a conformal mapping.

DEFINITION 2. An s.-p. interior transformation f of class C^1 which verifies (3.1) in any point $z \in G$, where f is an l. homeomorphism (so-called ordinary point) is a *Grötzsch K -qr mapping*.

Again the composition of a Grötzsch K_1 -qr mapping with a Grötzsch K_2 -qr mapping is a Grötzsch $K_1 K_2$ -qr mapping, and if $K = 1$ the mapping is analytic.

The definitions extend for mappings between Riemann surfaces by asking that the mapping expressed in l. parameters be K -qc resp. K -qr. In particular, this applies to the Riemann sphere $\widehat{\mathbb{C}}$, where the inversion is used to define the mapping at $z = \infty$ and for the value $w = \infty$. If K is not precised the mapping is simply called Grötzsch qc respectively qr.

With his strip method ([Gr1] and Section 2) which he extends in [Gr2] to injective $\mathcal{A}_Q$ mappings (his K -qc mappings) by geometrical arguments, Grötzsch deduces that “distortion boundedness in infinitesimal small is in connection with distortion boundedness in the large”. He first proves an inequality of type (2.2): for $\varepsilon > 0$ arbitrarily small, a sufficiently thin rectangle $R \subset G$ has the module $M(R)$ verifying

$$\frac{A}{L^2} \geq \frac{1 - \varepsilon^2}{K} M(R), \quad (3.2)$$

where A is the area of $f(R)$ and L the length of the image of the long side of R .

From (3.2) Grötzsch deduces his *celebrated inequalities*:

(a) *For rectangles*: If a rectangle R of sides a and b is represented under an $\mathcal{A}_K$ mapping f with vertices correspondence onto a rectangle of sides a and hb such that the a -sides correspond, then

$$\frac{1}{K} \leq h \leq K. \quad (3.3)$$

Equality takes place for extremal affine transformations.

(b) *For annuli* by means of logarithm function: If f is an injective $\mathcal{A}_K$ mapping of $r < |z| < 1$ onto $r^* < |w| < 1$, then

$$r^K \leq r^* \leq r^{1/K}, \quad (3.4)$$

with equality by certain extremal mappings, any of which obtained from the other by rotation and symmetry.

(c) *General module inequalities, case of a quadrilateral Q* represented by an injective $\mathcal{A}_K$ mapping f with vertices correspondence onto a quadrilateral Q^* :

$$\frac{1}{K} M(Q) \leq M(Q^*) \leq K M(Q), \quad (3.5)$$

where $M(Q)$ and $M(Q^*)$ are the modules of Q resp. $Q^* = f(Q)$.

Let us remark that (3.5) follows directly from the Theorem in Section 2.2. and (2.23').

We shall return to these inequalities at the end of this section.

We gave the particular case (a) since Grötzsch solves there a first *extremal problem* for his K - qc mappings: he establishes how the module of a rectangle can vary under a K - qc mapping and indicates the extremal mappings. The same can be said about ring domains. Here in an implicitly way is proved that $r > 0$ corresponds to $r^* > 0$, hence a boundary component consisting in a point cannot be transformed by K - qc mappings into a continuum, a *type invariance* assertion.

Grötzsch generalized also other results from conformal case to his K - qc mappings, among which the *boundary correspondence theorem*.

The last part of the paper [Gr2] is dedicated to the K - qr mappings for which Grötzsch proves the *Picard theorem*. This theorem was studied by many mathematicians which wanted to extend complex analytic function theory for different classes of mappings, e.g., Bernstein (1910) [Ber], Szillárd (1927) [Sz], Onicescu (1928) [On1], Stoilow (1927) [St1], (1928) [St2, St3]. In the lecture held in Paris (1960) [St7] Stoilow justifies this interest for Picard theorem as this theorem represents “the type of geometric–topological facts in the classic function theory”.

Grötzsch’s proof of Picard theorem considers an $\mathcal{A}_K$ mapping $w = f(z)$ with an essential singularity, e.g., in $z = 0$. Suppose that f does not take three values in $0 < |z| < r$, for some r . Let $\mathcal{R}$ be the covering Riemann surface of the inverse mapping of f and π the projection of $\mathcal{R}$ in the w -plane; f lifts to a homeomorphism $\mathcal{F}: 0 < |z| < r \rightarrow \mathcal{R}$ and by a classic theorem there is a conformal mapping $\mathcal{G}: \mathcal{R} \rightarrow r^* < |Z| < r$. Since the analytic function $g = \pi \circ \mathcal{G}^{-1}$ omits three values, $r^* > 0$. However, then the $\mathcal{A}_K$ mapping $\mathcal{G} \circ \mathcal{F}$ would represent $0 < |z| < r$ onto $r^* < |Z| < r$, what is a contradiction by Lemma 2, Section 2 and the module inequality, or by the type invariance mentioned above.

Grötzsch continued to study and solve extremal problems for conformal and in [Gr4] K - qc mappings, for simply but also multiply connected domains.

For example, in the family of all K - qc mappings $w = f(z)$ of $|z| < 1$ into itself, by fixing two points z_1 and z_2 in $|z| < 1$, one asks to find the maximum of the hyperbolic distance between $w_j = f(z_j)$, $j = 1, 2$. Grötzsch proves that this maximal hyperbolic distance is attained for the extremal mapping with the dilatation quotient $p(z) \equiv K$ and the direction of the maximal distortion given by the family of confocal ellipses in the hyperbolic geometry corresponding to w_1 and w_2 ; for this extremal mapping $f(|z| < 1)$ coincides with $|z| < 1$. The same holds if one works with the family of all K - qr mappings of $|z| < 1$ into $|w| < 1$, and represents a *generalization of Schwarz–Pick lemma*.

Another result with very important consequences is that the maximal distortion directions of the extremal mapping are given by a quadratic differential. The trajectories of the quadratic differential, say $Q(w)dw^2$, characterized by the inequality $Q(w)dw^2 > 0$, determine the $\arg dw \pmod{\pi}$, hence a direction through the point w , which is just the main distortion direction corresponding to w . The differential inequality is also satisfied on the image boundary.

The appearance of the quadratic differentials in the solution of this problem constitutes another connection between $qcty$ and classical function theory (see [Kü13, pp. 144–145]).

In 1932, Grötzsch introduced another kind of extremal problems [Gr5]: given two domains G and G^* topologically but not conformally equivalent, does there exist a qc mapping $f: G \rightarrow f(G) = G^*$ and in affirmative case, does exist one whose maximal dilatation is a minimum?

(The maximal dilatation of f is

$$D_f = \sup_{z \in G} D(z), \quad (3.6)$$

where $D(z) = p(z)$ as in Section 1, (1.10), (1.10').)

Such a mapping was called by Grötzsch *most nearly conformal*, and he solved the problem in many significant cases. For example, in the simplest one, when R and R^* are canonical rectangles (Section 2.1) with sides a, b resp. a^*, b^* and $(a/b) \leq (a^*/b^*)$, for every qc mapping (preserving vertices and sending a -sides into a^* -sides): $R \rightarrow R^*$

$$\frac{a^*}{b^*} : \frac{a}{b} \leq D_f$$

with equality only for the affine mapping $w = (a^*/a)x + i(b^*/b)y$ which is the most nearly conformal mapping. If $(a/b) \geq (a^*/b^*)$ one changes R and R^* . (See also [A12, pp. 6–7] and [Kü13, p. 153].) These examples and results of Grötzsch opened the way to Teichmüller's theory of extremal qc mappings.

The Grötzsch inequalities have a large application field. Even the general notions of qc and qr mappings are geometrically defined by using them (see Section 4.1).

The inequalities (2.18) by Kühnau and (2.23), (2.23') by Andreian Cazacu are their generalizations for the module of curve families.

They continue to inspire new research. As example we quote the new version of Grötzsch principle and Reich–Strebel inequality [ReStr] by Mateljević and his school, see Marković and Mateljević [MM1, MM2]. To remark that on this occasion new applications of the length–area dilatation d (2.22) appear. For instance T_μ in [MM1, MM2] is d on the vertical direction ($x = \text{const}$) or on the tangent to geodesics of the quadratic differential and in [GuMa] D_μ (resp. $D_{-\mu}$) is d on the tangent direction to the circles $|z| = \text{const}$ (resp. on the radial direction).

3.2. Quasiconformal mappings in Ahlfors' value distribution

The year 1935 marks a stage in the $qcty$ history by two papers.

One of them [A2] contains the Ahlfors metric-topological theory of value distribution corresponding to the Nevanlinna theory for meromorphic functions in the plane $\mathbb{C}$ or disk $|z| < R < \infty$, and obtained in 1936 at the Oslo International Congress of Mathematicians one of the two for the first time awarded Fields medals.

In this paper Ahlfors used *quasiconformal mappings*, term which appeared then for the first time. Let W^* be a simply connected Riemann surface and f a homeomorphism defined on the z -plane $\mathbb{C}$ with values onto W^* such that the metric W^* be given by a positively defined differential form $ds^2 = E(z) dx^2 + 2F(z) dx dy + G(z) dy^2$, with sufficiently regular coefficients E, F, G , the length of a curve C (resp. the area of a domain D on W^*) being equal with $\int_{f^{-1}(C)} \sqrt{E dx^2 + 2F dx dy + G dy^2}$ (resp. $\iint_{f^{-1}(D)} \sqrt{EG - F^2} dx dy$). The mapping f is quasiconformal if there is a constant $K \geq 1$ such that

$$E + G \leq 2K\sqrt{EG - F^2}, \quad (3.7)$$

the case when f is conformal corresponding to $\mathcal{K} = 1$ (f is a Grötzsch K -qc mapping with $\frac{1}{2}(K + \frac{1}{K}) = \mathcal{K}$ (see (3.1^{iv})).

The disks $D_r = \{z: |z| < r\}$ define an exhaustion of W^* by $W_r^* = f(D_r)$ with the area $A(r) = \iint_{D_r} \sqrt{EG - F^2} dx dy$, and the length of the boundary $L(r) = \int_{C_r} ds = \int_{C_r} \sqrt{E dx^2 + 2F dx dy + G dy^2}$, $C_r = \{z: |z| = r\}$.

By length-area method,

$$\frac{dr}{r} \leq 4\pi \mathcal{K} \frac{dA(r)}{L(r)^2}.$$

By integrating it follows that the total logarithmic measure of the intervals in $\mathbb{R}_+$, where $L(r) \geq A(r)^{\frac{1}{2}+\varepsilon}$ is finite, so that there exist sequences r_n increasing to ∞ with $\lim_{n \rightarrow \infty} \frac{L(r_n)}{A(r_n)} = 0$, characteristic condition for $W_{r_n}^*$ to form a regular exhaustion of W^* .

A similar device is applied for a K -qc mapping f of a disk $\{z: |z| < R\}$ onto W^* and it follows that W^* is regularly exhaustible if $\lim_{r \rightarrow R} (R - r)A(r) = \infty$.

In both cases, W^* being regularly exhaustible all results of Ahlfors' theory on covering surfaces can be applied to it.

Ahlfors formulates these results in the following theorem.

MAPPING THEOREM. *Let W^* be a simply connected Riemann surface. If there is a conformal or qc mapping of W^* onto $\mathbb{C}$ (respectively onto a disk $|z| < R$), then there is also a regular exhaustion of W^* (in the disk case with the additional condition $\lim_{r \rightarrow R} (R - r)A(r) = \infty$).*

The paper [A2] contains other considerations on the type problem and as a consequence a type criterion.

Ahlfors remained deeply interested in *qcty* and brought later an essential contribution to this field (see Sections 3.4 and 4.1).

3.3. Lavrent'ev's almost analytic mappings

The other important paper from 1935 is due to Lavrent'ev [La1, La2], who introduced a larger class of mappings, called *almost analytic* (a.anal.), renouncing to the differentiability and to the condition that p is bounded by a constant (3.1).

DEFINITION 3. The mapping $w = f(z): G \rightarrow G^*$ is *a.anal.* if the following conditions are fulfilled:

- (1) f is continuous.
- (2) f is an s.-p. l. homeomorphism, except for a countable set of points.
- (3) There are two real functions $p(z) \geq 1$ and $\theta(z)$ (the *characteristic functions* of f) such that:

- (a) Except for a set $E \subset G$ consisting in a finite number of analytic arcs, p is continuous and θ is continuous at any z where $p(z) \neq 1$.

- (b) p is u. continuous in every domain $\Delta \subset G$ with $\Delta \cap E = \emptyset$ and $\partial\Delta$ – a Jordan analytic curve; if in addition $p(z) \neq 1$ on $\overline{\Delta} \subset G$, θ is also u. continuous in Δ .
- (c) If $z_0 \in G \setminus E$ and $\mathcal{E}$ is an ellipse with the center z_0 and the characteristics $p(z_0) = \frac{a}{b} \geq 1$, $\theta(z_0)$ (Section 1.2), then

$$\lim_{a \rightarrow 0} \frac{\max_{z \in \mathcal{E}} |f(z) - f(z_0)|}{\min_{z \in \mathcal{E}} |f(z) - f(z_0)|} = 1. \quad (3.8)$$

(See Section 1, (1.8), where the mapping was differentiable at z_0 .)

At once after Lavrent'ev published his note on a.anal. mappings in *C. R. Acad. Sci. Paris* [La1], Stoilow proved in the same journal [St4] that the first two conditions in the definition of a.anal. mappings imply that they are interior transformations, hence they have all the topological properties of the analytic functions.

To obtain results on a.anal. mappings Lavrent'ev approximates them by a.anal. mappings of class C^1 , often with bounded p (in fact Grötzsch qr mappings) and thoroughly studies the mappings of the unit disc. Several lemmas offer distortion results, e.g., he proved by length–area method:

LEMMA 1 IN [La2]. *Let f be a homeomorphism of $|z| \leq 1$ onto $|w| \leq 1$, with $f(0) = 0$ and a.anal. of class C^1 on $0 < |z| < 1$. For $q(r) = \max_{|z|=r} p(z)$,*

$$|f(z)| < \pi \sqrt{2 \int_{|z|}^1 \frac{dr}{r q(r)}}. \quad (3.9)$$

CONSEQUENCE OF LEMMAS 2 AND 3 IN [La2]. If $1 \leq p(z) \leq 1 + \varepsilon$, $\varepsilon > 0$, the family of homeomorphisms f_ε of $|z| \leq 1$ onto $|w| \leq 1$, a.anal. of class C^1 in $|z| < 1$, with $f_\varepsilon(0) = 0$ and $f_\varepsilon(1) = 1$, is equicontinuous in $|z| < 1$ and tends uniformly to z as $\varepsilon \rightarrow 0$.

The main result of the paper [La2] is the following theorem.

EXISTENCE THEOREM. *Let G be a simply connected domain conformally equivalent to a disc, p continuous and bounded in G and θ continuous for $p(z) \neq 1$. Then there is an a.anal. homeomorphism f mapping G onto $|w| < 1$ with the characteristics p and θ .*

By a conformal mapping the theorem reduces to the case of the unit disc. A larger class of homeomorphisms $\mathcal{P}_\varepsilon$ is considered which verify instead of (3.8) the condition

$$\lim_{a \rightarrow 0} \frac{\max_{z \in \mathcal{E}} |f(z) - f(z_0)|}{\min_{z \in \mathcal{E}} |f(z) - f(z_0)|} < 1 + \varepsilon. \quad (3.8')$$

For an arbitrary z_0 , $|z_0| < 1$, an affine mapping which transforms the ellipse of characteristics $p(z_0)$, $\theta(z_0)$ in a circle is of class $\mathcal{P}_\varepsilon$ in a sufficiently small neighborhood of z_0 . A sewing theorem gives then a $\mathcal{P}_\varepsilon$ homeomorphism for the whole disk and by letting $\varepsilon \rightarrow 0$ one obtains the required a.anal. mapping.

The boundedness condition for p is replaced by the divergence of an integral which appears also in (3.9) and in the parabolic type criteria presented by Lavrent'ev as geometric applications of the Existence theorem. Thus he proves:

If p is continuous in $0 < |z| \leq 1$ and θ is continuous for $p \neq 1$, the divergence of the integral

$$\int_0^1 \frac{dr}{r q(r)}, \quad q(r) = \max_{|z|=r} p(z), \quad (3.10)$$

assures the existence of an a.anal. homeomorphism f of $|z| \leq 1$ onto $|w| \leq 1$ with $f(0) = 0$, $f(1) = 1$ and the given characteristics p and θ .

Another important result is the following theorem.

DECOMPOSITION THEOREM. *Every a.anal. mapping F of $|z| < 1$ with characteristics p , θ as in the Existence theorem may be written $F = \Phi \circ f$ with f the a.anal. homeomorphism from the Existence theorem and Φ an analytic function, and conversely starting with f , for every analytic function Φ in $|w| \leq 1$, $F = \Phi \circ f$ is a.anal. with the same characteristics as f .*

With this result the way to extend properties of analytic functions is open. Lavrent'ev proves a *Montel type theorem* for families of a.anal. mappings with the same characteristics p and θ , mappings bounded by the same constant; the *unicity theorem* (the a.anal. homeomorphism given by the existence theorem is uniquely determined by conformal normalization, e.g., if $f: |z| \leq 1 \rightarrow |w| \leq 1$ by $f(0) = 0$ and $f(1) = 1$) and the *identity theorem*; *Picard theorem* for a.anal. mappings in $0 < |z| < 1$ with unbounded p but divergent integral (3.10).

Today the Existence theorem is interpreted in terms of solution existence of the associate Beltrami equation by passing from p, θ to the complex dilatation μ ((1.11) and (1.12)). If for a.anal. mappings of class C^1 the connection is immediate, in general one has to enlarge the classical concept of solution according to the distribution theory. Even in 1938, Morrey [Mor] solved the problem, but his result remained in the frame of partial differential equations till 1956 when Bers discovered his relevancy to *qcty* [B9]. We shall return to this subject in Section 4.2.

Lavrent'ev applied his existence theorem to solve the classic problem of conformal mapping of a smooth orientable surface in $\mathbb{R}^3$ into the plane, e.g., of a "two-dimensional Riemannian space", given by a quadratic differential form

$$ds^2 = E dx^2 + 2F dx dy + G dy^2, \quad EG - F^2 > 0, \quad (3.11)$$

with E, F, G defined and continuous on $|z| \leq 1$, onto $|w| \leq 1$.

There is a beautiful exposition of the subject and its history in Lehto's paper [L10]. The problem appeared in cartography (see Kühnau [Kü13, p. 142]) and was solved by Gauss in 1825 [Ga].

Let S be a smooth orientable surface in $\mathbb{R}^3$, $P \in S$, $f = (f_1, f_2, f_3)$ – a diffeomorphism of a domain in z -plane onto a neighborhood of P (where f^{-1} defines the l. parameter z). Then $E = \sum_{i=1}^3 (\frac{\partial f_i}{\partial x})^2$, $F = \sum_{i=1}^3 \frac{\partial f_i}{\partial x} \cdot \frac{\partial f_i}{\partial y}$ and $G = \sum_{i=1}^3 (\frac{\partial f_i}{\partial y})^2$. The form (3.11) is independent of the l. representation of S .

By calculus Gauss proved that f is conformal iff $E = G$ and $F = 0$, what in the complex representation, with z and $\bar{z}$ instead of x and y (not used by Gauss but simpler)

$$ds = \lambda |dz + \mu d\bar{z}|, \quad (3.12)$$

where $\lambda^2 = \frac{1}{4}(E + G + 2J)$ and μ as in Section 1, (1.11''), means $\mu = 0$, hence $ds = \lambda |dz|$.

An l. coordinate z with f conformal is called *isothermal* and Gauss sought a diffeomorphism $z \rightarrow w$ such that w be isothermal.

If w is a diffeomorphic solution of Beltrami equation $w_{\bar{z}} = \mu w_z$ with μ as in (3.12), then w is an isothermal coordinate since $ds = (\lambda/|w_z|)|dw|$. Gauss solved this equation and thus the problem to find an l. conformal mapping of a smooth orientable surface into the plane.

Lehto deduces in [L10] in the light of actual theory several important interpretations of Gauss' result:

1. The complex dilatation of a qc mapping can be prescribed.
2. A smooth orientable surface in $\mathbb{R}^3$ (more general an abstract surface with a Riemannian metric) can be made into a Riemann surface by means of the l. isothermal coordinates (since if w_1 and w_2 are diffeomorphic solutions of the same Beltrami equation, $w_1 \circ w_2^{-1}$ is a conformal mapping, fact underlined also in the decomposition theorem from above).
3. From l. solutions one can obtain global solutions: Let D be a simply connected plane domain and suppose that there are l. injectively solutions of the equation $w_{\bar{z}} = \mu w_z$. These define on D a structure of Riemann surface which we denote by $\mathcal{D}$, and by the general uniformization theorem there is a conformal mapping F of $\mathcal{D}$ onto a plane domain D^* . Then F is a global solution of $w_{\bar{z}} = \mu w_z$ from D to D^* .

Around 1915, Lichtenstein and independently Korn solved Beltrami equation for μ Hölder continuous and showed that the injective solutions are diffeomorphisms. However there are diffeomorphisms with the complex dilatation continuous but not Hölder continuous, and a continuous μ does not imply a solution of class C^1 . In any case, the paper of Lavrent'ev dealing with continuous μ represented a significant progress.

At the end of this paper, Lavrent'ev proposed to study a more general mapping class, where the characteristic functions are no more given. These mappings have the same topological properties (1) and (2) in Definition 3 of the a.anal. mappings and a bounded distortion of circles in the sense that

$$\sigma(z) = \overline{\lim}_{r \rightarrow 0} \frac{|f(z + re^{i\varphi}) - f(z)|}{|f(z + re^{i\psi}) - f(z)|} \leq M, \quad (3.13)$$

where $0 \leq \varphi, \psi < 2\pi$, M is a constant and f is defined as in the almost whole paper in $|z| < 1$.

Lavrent'ev called this class Θ and remarked as a consequence of his results, that if f_n is an l.u. convergent sequence of a.anal. mappings in $|z| < 1$ with the corresponding p_n bounded by M , then the limit supposed nonconstant belongs to Θ . He set the problem if any $f \in \Theta$ is the limit of such a sequence.

In fact, the definition (3.13) will lead to the metric definition of the general qc respectively qr mappings, whose equivalence with the other geometric and analytic definitions was established by Gehring in 1960 [G1] (see Section 4.3).

In the following decennia the connection between Lavrent'ev's work and PDEs will increase. He, his school and other mathematicians will develop a vast theory of mappings defined as solutions of elliptic systems of PDEs with important applications in mechanics, physics and technics (see a short account in Section 4.2).

3.4. Teichmüller's results

Next years the research continued on the way opened by Grötzsch, Ahlfors, Lavrent'ev, especially in the type problem. The apparition in 1936 of Rolf Nevanlinna's famous book *Eindeutige analytische Funktionen* [Ne2] provided the Geometric Function Theory with new tools from harmonic function and potential theory, function theoretic majorant principles, relations between non-Euclidean and Euclidean metrics, to a new complete account of Nevanlinna's value distribution theory, to Riemann surfaces of univalent functions and Ahlfors' theory of covering surfaces. Qc mappings were also presented at the end of Ahlfors' results (under the name of *mappings with bounded excentricity*, changed into qc mappings in the 2nd edn (1953)), as an extension of the conformal case (Chapter XIII, Section 8). The type problem was emphasized from the *Introduction*, were Nevanlinna considered it as "central problem in the general theory of conformal mapping, an interesting and complicated question, left open by the classical uniformization theory", and the entire Chapter XII was dedicated to it, what stimulated the work in this direction. Thus in 1937 appeared a paper by Kakutani [Kak] and another by Teichmüller [T2] applying again qc mappings to the type problem (see also Kobayashi [Kob1, Kob2]).

The last one [T2] marked the beginning of a brilliant cycle of papers by Teichmüller (1937–1944) thanks whom " qc mappings rose in the forefront of complex analysis" (quotation from Lehto [L10, p. 210]).

After a Ph.D. Thesis on functional analysis in Göttingen (1935) under supervision of Hasse [T1] and papers on algebra, domains in which he will continue to publish, Teichmüller's research field enlarged including also geometric function theory. To mention that in 1936–1937 Rolf Nevanlinna was a visiting professor in Göttingen and probably introduced Teichmüller in value distribution theory [Ab2].

In [T2] Teichmüller deals with a simply connected Riemann surface $\mathcal{W}$ over the extended complex plane (w), which is ramified only over a finite number of points $w = a_1, \dots, a_q$. By a Jordan curve L through $a_1, \dots, a_q$ (in this order), (w) is decomposed in two domains $\mathcal{J}$ rounded by L in direct sense and $\mathcal{A}$ in opposite sense. The points a_j decompose L into arcs s_k and $\mathcal{W}$ is obtained by linking together a finite or infinite number of $\mathcal{J}$ and $\mathcal{A}$ copies along some of s_k . Thus $\mathcal{W}$ is represented by a line complex [Ne2, XI, Section 2].

Teichmüller proves in two steps that two surfaces $\mathcal{W}$ with the same line complex have the same type: first, that they can be *qcly* mapped one onto the other and then, that *qc* mappings invari the type. The last result have been already proved by Grötzsch [Gr2] (see Section 3.1) with the same length–area method and by reading the proofs Teichmüller added this information he received from Wittich when the paper [T2] was already in the press.

The following Teichmüller paper on this field *Untersuchungen über konforme und quasikonforme Abbildung* was presented as Habilitation writing in Berlin (1937) under the supervision of Bieberbach and appeared in 1938 [T3]. It is a vast work containing many significant results, quoted and some of them included by Nevanlinna in the 2nd edition of his book (1953) as well as in the 3rd English edition *Analytic Functions* (1970) [Ne2], and in the books by Küenzi [Kün3], Lehto and Virtanen [LV3], Ahlfors [A12,A15].

By this work Teichmüller succeeded to prove that *qc* mappings are not a simple generalization of conformal mappings but a useful tool, necessary for solving complex analytic problems. They have to be adequate to the problems: the dilatation quotient denoted by D has not to be bounded by a constant but to satisfy specific conditions, e.g., not to increase very quickly at the boundary.

Teichmüller's papers are written in a warm direct style. He emphasizes aims, ideas, difficulties, clearly explains notions, methods, and compares his results with others concerning similar problems.

He aimed to develop Grötzsch's work in order to obtain by passing to the limit information about asymptotic behavior of conformal or *qc* mappings but was also interested in Ahlfors' distortion theorem [A1], [Ne2, IV, Section 4] and combined Grötzsch's with Ahlfors' method from 1930 [A1].

The paper begins with a thorough study of the module of a ring domain G , defined as

$$M = \log \frac{R}{r} \quad (3.14)$$

if G is conformally equivalent to $\{w: r < |w| < R\}$, $0 < r < R < \infty$ (see Section 2.1), and the reduced module of a simply connected domain (see Küenzi [Kün3, 1.7, 1.8]). Parallel to results about the module Teichmüller always proves the corresponding results for the reduced module. Different inequalities are established, among which the relation between module and logarithmic area (similar to (2.2)).

The extremal problem of the nearest boundary point was solved by the theorem.

GRÖTZSCH THEOREM. *If G is a ring domain with a boundary component $|z| = 1$ and separating $|z| = 1$ from ∞ , and if G does not contain a point z_0 with $|z_0| = P > 1$, then the module of G*

$$M(G) \leq \log \Phi(P), \quad (3.15)$$

where $\log \Phi(P)$ is the module of the Grötzsch extremal (or normal) domain G_P : the exterior of the unit disk cut along the real axis from P till ∞ . Equality in (3.15) occurs iff $G = G_P$.

After proving this theorem, properties of the function Φ are given; there are calculation formulas for $\log \Phi(P)$ by means of elliptic integrals (see [Kün3, 1.9, p. 7]). By a conformal mapping the Grötzsch extremal domain can be chosen as the unit disk cut from 0 to r , $0 < r < 1$, along the positive real axis. Among the ring domains that separate the points 0 and r from $|z| = 1$, the module of this domain is maximum [Gr1], [LV3, II, Section 1, p. 53].

Then Teichmüller generalizes the problem renouncing to the hypothesis that one of the boundary components is a circle and solves the problem of the nearest and farthest boundary points:

TEICHMÜLLER THEOREM. *Let G be a ring domain separating 0 of ∞ , ρ be $\max |z|$ on the complementary component of G which contains 0 and P be $\min |z|$ on the other complementary component. Then*

$$M(G) \leq \log \Psi\left(\frac{P}{\rho}\right), \quad (3.16)$$

where $\log \Psi\left(\frac{P}{\rho}\right)$ is the module of the Teichmüller extremal (normal) domain $G_{\rho, P}$: the extended complex plane (z) cut along the real axis between $[-\rho, 0]$ and $[P, +\infty]$. Equality holds only for $G = G_{\rho, P}$ (see also [LV3, II, Section 1]).

Relations between the functions Φ and Ψ are given.

As a consequence Teichmüller obtains:

If $G = \{z: r < |z| < R\}$ with $\frac{R}{r} > e^\pi$ and f is a conformal mapping of G onto a domain $f(G)$ separating 0 from ∞ , then there is a circle with center 0 included in $f(G)$; e^π cannot be replaced by a smaller constant.

Extremal domains of Grötzsch, Teichmüller (and Mori [Mo1], see [Kün3, 1.11], [LV3, II, 1.5]) have many applications.

Further Teichmüller deals with the module of quadrilaterals. By his previous results on ring domains and ingenious conformal mappings he succeeds to give a new geometric proof of Ahlfors' distortion theorem with the best possible estimate, improving that of Ahlfors (see [Ne2, IV, Section 4, p. 92]).

Coming again to ring domains he establishes the module and the reduced module theorems:

It was known from the Grötzsch principle that if G' and G'' are two disjoint ring domains in $G = \{z: r < |z| < R\}$ that separate 0 from ∞ , one has for their modules $M' + M'' \leq M$ with equality only if G' and G'' are obtained by cutting G along a circle $|z| = \rho$, $r < \rho < R$. Suppose that G' separates G'' from 0.

TEICHMÜLLER MODULE THEOREM. *For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ (independent of r and R), such that if $M' + M'' \geq M - \delta$, every point z separated from 0 by G' and from ∞ by G'' belongs to the annulus*

$$\log r + M' - \varepsilon \leq \log |z| \leq \log R - M'' + \varepsilon. \quad (3.17)$$

If $\varepsilon \rightarrow 0$, then $\delta \rightarrow 0$ and $\log |z| = \log r + M' = \log R - M''$. An asymptotic estimate for $\delta(\varepsilon)$ is given (see [Kün3, 1.16, p. 16]).

As an application (important, e.g., in the theory of Riemann surfaces when the curves C_λ can correspond to an exhaustion) Teichmüller studies a family of Jordan curves C_λ , where λ runs in a set $\mathcal{M}$ of real numbers which accumulate to ∞ , 0 is in the interior of each C_λ , $C_\lambda \cap C_\mu = \emptyset$ for any pair $\lambda, \mu \in \mathcal{M}$ and C_μ separates C_λ from ∞ for $\lambda < \mu$. Conditions that C_λ be almost a circle as $\lambda \rightarrow \infty$ are determined by means of the ring domains bounded by C_λ and C_μ and their modules.

Then Teichmüller concentrates on Grötzsch qc mappings with the dilatation quotient $D(z)$ estimated from above in function of the treated problem.

First an important example of qc mappings is analyzed, namely $w = w(z)$, where

$$z = re^{i\varphi} \rightarrow w = \rho(r)e^{i\varphi}, \quad 0 \leq r < \infty, \quad (3.18)$$

and $\rho(r)$ is an increasing continuous, piecewise partially continuously differentiable function.

Further the following module inequalities are established:

If the annulus $r_1 < |z| < r_2$ is qc ly mapped onto the annulus $\rho_1 < |w| < \rho_2$ and the dilatation quotient verifies the inequality $D(z) \leq C(|z|)$ then

$$\int_{r_1}^{r_2} \frac{1}{C(r)} \frac{dr}{r} \leq \log \frac{\rho_2}{\rho_1} \leq \int_{r_1}^{r_2} C(r) \frac{dr}{r} \quad (3.19)$$

with equality only for mappings as in the previous example (3.18).

The inequalities (3.19) reduce for $C(|z|) \leq K$ – a constant to Grötzsch's inequalities (3.4).

They imply type criteria: The divergence of the integral $\int^\infty \frac{dr}{C(r)r}$ is sufficient for the parabolic type (result already known from Lavrent'ev [La2] (see (3.10)) and the convergence of $\int^1 C(r) \frac{dr}{r}$, when the qc mapping is defined on $|z| < 1$, for the hyperbolic type.

Stoilow especially appreciated Teichmüller's work "so rich in new and fecund ideas". In his lecture on qc mappings (Paris, May 1960) [St7] he presented Teichmüller's inequalities (3.19) and derived from them a short elegant proof of Picard's theorem, in essence on the same way of Grötzsch but using his decomposition theorem.

From his module theorems and their consequences Teichmüller obtains an important theorem on the asymptotic behavior of qc mappings:

If the z -plane is qc ly mapped onto the w -plane with

$$D(z) \leq C(|z|) \quad \text{and} \quad \int^\infty (C(r) - 1) \frac{dr}{r} < \infty, \quad (3.20)$$

then there exists a constant $\gamma > 0$ such that $|w| \simeq \gamma|z|$ when $z \rightarrow \infty$.

Estimate for γ is also given. Conditions (3.20) may be weakened, by asking only that

$$\iint_{|z|>M} (D(z) - 1) \frac{dx dy}{|z|^2} < \infty. \quad (3.21)$$

This theorem, writes Teichmüller, was one of the motivations of the paper [T3]. In the summer 1936, Teichmüller and Wittich tried without success to prove it with Ahlfors' method [A1]. In 1948 Wittich gave an analytic proof [Wi2] besides the geometric one by Teichmüller, the theorem being called Teichmüller–Wittich in Künzi's book [Kün3, 2.7], where both proofs are exposed, or in Volkovyskij's [Vo3].

It was Belinskij who succeeded the first to improve this theorem. He obtained that under the hypothesis (3.21) there exists $\lim_{z \rightarrow \infty} \frac{w}{z} \neq 0, \infty$ [Be1, Be8], see also, e.g., [Kün3, 2.8, p. 34]. Other generalizations were given by Lehto [L1], Shabat [Ša1].

Finally the paper [T3] contains a hyperbolic type criterion for a subclass of the Riemann surfaces $\mathcal{W}$ studied in [T2], criterion based on a measure of the ramification of the surface defined with Speiser's tree.

For all richness of results contained in [T3] the most celebrated paper of Teichmüller remains *Extremale quasikonforme Abbildungen und quadratische Differentiale* published in 1939 [T4].

Starting from Grötzsch's examples of extremal problems concerning most nearly conformal mappings and extremal problems considered by himself, Teichmüller succeeded to draw out a general principle about extremal problems in the vast frame of Riemann surfaces, based on a fascinating discovery: the connection of these problems with the quadratic differentials on these surfaces.

He indicates from the beginning the aim of the paper: the study of the behavior of the conformal invariants under qc mappings, namely the finding of the most nearly conformal mappings by certain conditions. He will give the solution of this problem without a complete proof but founds it by examples and heuristic arguments, such that any doubt be excluded and to induce new proof attempts. As much as possible he will render the idea succession which led to this solution.

The work utilizes notions and methods not only from function theory but from topology, differential and algebraic geometry, infinitesimal calculus, PDEs, Galois theory, thus he reminds and sometimes even proves some of them. In the same time the paper represents an ample research program.

First Teichmüller defines the *main domains* (Hauptbereiche) with which he will work: closed Riemann surfaces orientable or not, bordered or without border (i.e., compact Kleinian surfaces [AllGr]) with interior or boundary points distinguished (punctures, i.e., interior removed points, are replaced by distinguished points). They are characterized by the genus $g \geq 0$ – the number of handles in orientable case and $2g + \gamma$, $\gamma > 0$ – the number of cross-caps in the nonorientable case, $n \geq 0$ – the number of borders, $h \geq 0$ and $k \geq 0$ – the numbers of distinguished interior respectively boundary points. A mapping between main domains transforms distinguished points in distinguished points.

Let ρ be the parameter number of the continuous group of conformal automorphisms of a main domain.

Without proof one admits that by identification of conformally equivalent main domains of a fixed topological type the obtained classes form a topological manifold $\mathcal{R}^\sigma$, which is locally homeomorphic with the $\sigma (\geq 0)$ -dimensional Euclidean space, where

$$\sigma - \rho = -6 + 6g + 3\gamma + 3n + 2h + k. \quad (3.22)$$

This dimension formula will be later established as well as its connection with the Riemann–Roch formula.

The conformal invariants of the main domains are functions in $\mathcal{R}^\sigma$.

Teichmüller sets the following three problems:

1. Given a conformal invariant J as a function in $\mathcal{R}^\sigma$, a main domain $\mathcal{H}$ and a number $C \geq 1$, what are the values taken by J for the main domains $\mathcal{H}'$ images of $\mathcal{H}$ under qc mappings with the dilatation quotient $\leq C$.

This is obvious the extension of the Grötzsch problem, however Teichmüller proposes even more: to majorize the dilatation quotient not by a constant C but by a function so that for every point $p \in \mathcal{H}$ to have $D(p) \leq C(p)$, condition much studied the last time. Nevertheless in this paper he will consider the case $C(p) = C -$ a constant.

2. Let P be a point in $\mathcal{R}^\sigma$, $\mathcal{H}_P$ a corresponding main domain and C a number > 1 . One asks about the sets $\overline{\mathcal{U}}_C(P)$ and $\mathcal{U}_C(P)$ of all points $Q \in \mathcal{R}^\sigma$ such that there exists a qc mapping: $\mathcal{H}_P \rightarrow \mathcal{H}_Q$ with $D(p) \leq C$ (resp. $< C$) for all $p \in \mathcal{H}_P$.

These sets become closed respectively open neighborhoods of P by organizing $\mathcal{R}^\sigma$ as a *metric space* with the between two points P and Q in $\mathcal{R}^\sigma$ defined by:

$$[P, Q] = \log \inf_f \sup_p D_f(p), \quad (3.23)$$

where f is an arbitrary qc mapping of $\mathcal{H}_P$ onto $\mathcal{H}_Q$, p an arbitrary point in $\mathcal{H}_P$ and $D_f(p)$ the dilatation quotient of f at p (hence, $\sup_p D_f(p) = D_f$ is the maximal dilatation of f , see (3.6)).

Teichmüller proves that $[P, Q]$ is a distance once the existence of a most nearly conformal mapping $\mathcal{H}_P \rightarrow \mathcal{H}_Q$ established. He admits this and the compatibility of the $\mathcal{U}_C$ neighborhoods with the topology of $\mathcal{R}^\sigma$, and proposes the study of this metric, its geodesics and the conjecture that $\mathcal{R}^\sigma$ is a Finsler space with respect to it.

3. Given two main domains represented by the points P and Q in $\mathcal{R}^\sigma$, Teichmüller asks about $[P, Q]$ and about the totality of extremal qc mappings $\mathcal{H}_P \rightarrow \mathcal{H}_Q$ for which overall the dilatation quotient $D \leq e^{[P, Q]}$.

He conjectures that there exists always an extremal qc mapping unique till conformal automorphisms of $\mathcal{H}_P$ and $\mathcal{H}_Q$.

In the Grötzsch examples and other cases for plane domains extremal qc mappings were given by a conformal mapping followed by an affine mapping and by another conformal mapping.

Teichmüller conjectures that extremal qc mappings have constant dilatation quotient K .

In general for a qc mapping if the dilatation quotient is greater than 1, the diameters of the infinitesimal circles mapped onto the great axes of the corresponding infinitesimal ellipses determine a direction field and thus a family of curves tangent to these directions of maximal dilatation.

Teichmüller conjectures: for an extremal qc mapping this direction field is given by a quadratic differential $d\zeta^2$ which in l. parameter z is of the form $\varphi(z) dz^2$ with φ a meromorphic function (a holomorphic one in the case of closed Riemann surfaces), namely by the horizontal trajectories of $d\zeta^2$, where $d\zeta^2 > 0$, i.e., $\arg dz = -\frac{1}{2} \arg \varphi(z) \pmod{\pi}$.

In the following we shortly present extremal problems only in the case of the closed Riemann surfaces (without distinguished points), denote such a surface by R instead of $\mathcal{H}$ or $\mathcal{H}_P$ and omit details on zeros of differentials.

If $p \in R$, z is an l. parameter with $z(p) = 0$ and $\varphi(z)$ has a zero of order m at $z = 0$, then by definition the order of $\varphi(z) dz^2$ at p is m ; if $m = 0$,

$$\zeta = \int_0^z \sqrt{\varphi(z)} dz \quad (3.24)$$

is a *natural parameter* associated to $\varphi(z) dz^2$ at p .

A *Teichmüller mapping* between two Riemann surfaces R and R' is a conformal mapping or a qc mapping determined by a constant K and a quadratic differential $\varphi(z) dz^2 \neq 0$, i.e., with the complex dilatation (see (1.11))

$$\mu(z) = k \frac{\bar{\varphi}(z)}{|\varphi(z)|}, \quad k = \frac{K-1}{K+1}. \quad (3.25)$$

$(\mu(z) d\bar{z}/dz)$ is also invariant by change of l. parameters and is called a Beltrami differential of type $(-1, 1)$ on R .)

Extremal qc mappings coincide with the Teichmüller mappings.

If $f: R \rightarrow R'$ is a Teichmüller mapping determined by k and $d\zeta^2 = \varphi(z) dz^2 \neq 0$, then there exists a unique quadratic differential $d\zeta'^2 = \psi(z') dz'^2$ on R' with the properties:

- (i) The order of $\psi(z') dz'^2$ at $f(p)$ is equal to the order of $\varphi(z) dz^2$ at p .
- (ii) If ζ is the natural parameter of $\varphi(z) dz^2$ at $p \in R$ where $\varphi(z(p)) \neq 0$, and ζ' the natural parameter of $\psi(z') dz'^2$ at $f(p)$, then in the neighborhood of p , f verifies

$$\zeta' = \frac{\zeta + k\bar{\zeta}}{1-k} = K\xi + i\eta \quad (3.26)$$

(hence f is a conformal mapping followed by an affine one and followed by a conformal mapping as in the known examples).

- (iii) The inverse mapping $f^{-1}: R' \rightarrow R$ is also a Teichmüller mapping defined by k and $-\psi(z') dz'^2$.

Teichmüller treated the following extremal problem: Given a homeomorphism $f: R \rightarrow R'$ to determine in the class of qc mappings $R \rightarrow R'$ homotopic to f the extremal

one minimizing the maximal dilatation. *Teichmüller theorem* asserts the existence and the unicity of the extremal mapping, which is a Teichmüller mapping.

He proved the unicity of the extremal mapping by means of uniformization and length–area method, and in 1943 [T8] the existence based on a complicated continuity method.

This proof was “rather hard to read”, and since “the foundations of the theory were not commensurate with the loftiness of Teichmüller’s vision”, Ahlfors has retaken the basic concepts in the context of general qc mappings (by means of the new geometric definition which avoided all differentiability conditions, see Section 4.1). He gave a complete proof of the uniqueness part of Teichmüller’s theorem modeled on Teichmüller’s own proof and a variational proof of the existence part. This Ahlfors paper published in 1954 [A4] has been “very influential and has led to a resurgence of interest in qc mappings and Teichmüller theory”. In 1957 Bers obtained a “very clear version of Teichmüller’s proof” [B12] and in 1969 Hamilton [Ha] gave “an amazingly short and direct proof of the existence theorem” (quotations from Ahlfors’ commentary in his *Collected Papers* [A17, p. 319]). There are now several proofs for the existence theorem, see also [Str8].

However the paper [T4] contains also another essential contribution of Teichmüller: the spaces called today with his name.

In the memoir *Theorie der Abelschen Funktionen* (1857) [Rie], Riemann had showed that the classes of conformally equivalent compact Riemann surfaces of genus g depend on $m(g)$ complex parameters called *modules*: $m(g) = 0, 1$ or $3g - 3$ as $g = 0, 1$ resp. $g > 1$ and form a *continuum* described by these parameters. Teichmüller applied his results to *Riemann’s module problem* to study this space of classes of conformally equivalent Riemann surfaces, denoted in [T4] by $\mathcal{R}^\sigma$ with $\sigma = m(g)$ and usually now by M_g – the *space of modules*, and to endow it with a complex analytic structure. With this aim he introduced a covering space of $\mathcal{R}^\sigma$ now denoted T_g and called the *Teichmüller space*, much easier to handle than the module space M_g and which gradually permitted to solve the module problem.

By fixing a Riemann surface R_0 (all surfaces we consider here are compact of the fixed genus g) Teichmüller defines the triplet (R_0, f, R) with R another Riemann surface and f an s.-p. homeomorphism $R_0 \rightarrow R$.

Two triplets (R_0, f, R) and $(R_0, \tilde{f}, \tilde{R})$ are equivalent if there is a conformal mapping $\chi : R \rightarrow \tilde{R}$ such that $\tilde{f}^{-1} \circ \chi \circ f$ be homotop with the identity mapping I_{R_0} . These equivalence classes are the points of $T_g(R_0)$ and an evident bijection

$$\beta_{R_0 R_1}^h : T_g(R_0) \rightarrow T_g(R_1), (R_0, f, R) \mapsto (R_1, f \circ h, R),$$

with h an s.-p. homeomorphism: $R_1 \rightarrow R_0$, permits to identify in a natural way $T_g(R_0)$ and $T_g(R_1)$ obtaining T_g . (There are also other ways to define T_g : instead of triplets, canonical systems of generators of the fundamental group of the Riemann surface [A6] or normalized sets of generators of the cover transformation group relative to the universal covering of the Riemann surface [B12].) The projection $\Pi_g : T_g \rightarrow M_g$ is based on the mapping $(R_0, f, R) \mapsto R$ and $M_g = T_g / \mathcal{M}_g$, where $\mathcal{M}_g$ is the *modular group*,

$$\mathcal{M}_g(R_0) = \{\beta_{R_0 R_0}^{f_0} : f_0 - \text{an s.-p. homeomorphism } R_0 \rightarrow R_0\}.$$

Teichmüller proved that T_g is *homeomorphic to the Euclidean space* $\mathbb{R}^{m(g)}$ (as a topological theorem rediscovery of a classical result of Fricke [FK]) and defines the *Teichmüller distance* (3.23) in T_g : In $T_g(R_0)$ the distance between two classes of equivalent triplets represented by (R_0, f, R) and (R_0, f', R') is equal to $\log K(F)$, where $F: R \rightarrow R'$ is the Teichmüller mapping corresponding to the class of the triplet $(R, f' \circ f^{-1}, R')$. There is a bijection between the points of $T_g(R_0)$ – classes of triplets (R_0, f, R) and the corresponding Teichmüller mappings, hence the pairs (k, φ) consisting in a constant k , $0 < k < 1$, and a quadratic differential φdz^2 on R_0 determined till to a positive constant factor. As the quadratic differentials on a compact Riemann surface of genus g form a $\mathbb{C}$ -linear space of dimension $m(g)$, the space $T_g(R_0)$ can be parameterized such that a bijection be established between $T_g(R_0)$ and the unit ball in $\mathbb{R}^{m(g)}$ (see, e.g., [A6, p. 49]).

The *module problem* requires to find a complex structure on M_g . In [T4] and ulterior [T11] Teichmüller dealt with this problem. Following his way Ahlfors endowed T_g with the complex structure of an analytic manifold [A6]. The solution was also obtained by Rauch [Ra1–Ra3], Bers (by means of qc mappings interpreted as solution of Beltrami equation) [B11, B13, B15], Kodaira and Spencer [KS] and Weil [We]. The module space M_g is a complex analytic space (a manifold with singularities) of dimension $m(g)$. Weil organized T_g as a Riemannian, Kählerian manifold and geometric properties concerning the curvature has been obtained by Ahlfors [A7].

For $g = 0$, T_0 reduces to a point. The case $g = 1$ was solved in detail already by Teichmüller [T4], T_1 being represented by the superior half-plane with the non-Euclidean hyperbolic metric.

In [T4] and other papers Teichmüller treated many cases of extremal problems among which for pentagons, hexagons [T6] and also another extremal problem type: one gives a sufficiently smooth homeomorphism of $|z| = 1$ onto itself and asks about the extremal qc mapping which minimizes the maximal dilatation among the qc mappings of $|z| \leq 1$ onto itself with the given boundary correspondence. Interesting that for this type of problems the extremal mappings is no more always unique (Strebel [Str8]).

Teichmüller's work determined important researches and the theory of Teichmüller spaces developed intensively by Ahlfors, Bers their schools and others, and in the same time developed the theory of Kleinian groups. Mathematicians as Grauert, Grothendieck, Poénaru, Thurston, Donaldson, Earle, Eells, Douady established important connections with various mathematical fields. Unexpected applications in classical function theory appeared as in the univalent functions with qc extension (Lehto [L8], Kühnau [Kü6], Lehto's monograph [L11]). A vast literature is dedicated to Teichmüller spaces: surveys by Ahlfors [A11, A16], Bers [B18–B20], Earle [E], Krushkal [Kr4], Kühnau [Kü13], papers and monographs by Ahlfors [A12], Abikoff [Ab1], Krushkal [Kr3], Gardiner [Gar], Nag [N] and Imayoshi and Taniguchi [IT].

4. General qc and qr mappings

4.1. Geometric definition

In spite of the numerous results obtained in $qcty$ and its applications, about 1950 the necessity to enlarge the class of regular qc mappings became ever clearer.

Thus in January 1951 *Annales de l'Institut Fourier* received the paper *Quasikonforme Abbildungen und logarithmische Kapazität* by Pfluger [Pf4], where results by Grötzsch and Teichmüller concerning doubly connected domains were extended to domains G bounded by $n \geq 2$ Jordan curves $\gamma_1, \dots, \gamma_n$. Pfluger decomposed $\Gamma = \partial G$ in two groups of curves $\Gamma_0 = \{\gamma_1, \dots, \gamma_k\}$ and $\Gamma_1 = \{\gamma_{k+1}, \dots, \gamma_n\}$, denoted the obtained configuration by (Γ_0, Γ_1) and defined its *module*

$$M = M(\Gamma_0, \Gamma_1) = \inf_u \frac{1}{2\pi} \mathcal{D}(u), \quad (4.1)$$

where u is a harmonic function in G with $\int_{\Gamma_0} \frac{\partial u}{\partial n} ds = 2\pi$ (n – the inner normal), $u|_{\Gamma_0} = 0$, $u|_{\Gamma_1} = \text{const}$ and $\mathcal{D}(u) = \iint_G |\text{grad } u|^2 dx dy$ – the Dirichlet integral. The extremal function u exists and its value on Γ_1 is equal to M . If one regards (Γ_0, Γ_1) as a condenser, then $2\pi/M$ is its *capacity*. We denote by Γ_λ the level line $u = \lambda \in [0, M]$ (thus the boundary part Γ_1 is now denoted Γ_M). Pfluger proved:

Let f be a qc mapping of the configuration (Γ_0, Γ_M) onto a configuration $(\Gamma'_0, \Gamma'_{M'})$ (of module M') with the dilatation quotient D bounded by a constant K or with $K_\lambda = \max\{D(z): z \in \Gamma_\lambda\}$. Then $K^{-1}M \leq M' \leq KM$ (Grötzsch's inequalities (3.5)) respectively $\int_0^M \frac{d\lambda}{K_\lambda} \leq M'$ (Teichmüller's inequality (3.19)).

Much more, for an arbitrary domain G exhausted by a sequence of subdomains $G_0 \subset G_1 \subset \dots \subset G_n \subset \dots$ with $\Gamma_n = \partial G_n$ consisting in a finite number of Jordan curves, $M(\Gamma_0, \Gamma_n)$, $n = 1, 2, \dots$, form a monotonically increasing sequence and as its limit is finite or infinite, $\Gamma = \partial G$ is of *positive* respectively of *null logarithmic capacity*.

It follows that the boundary property to be of null logarithmic capacity is invariant under qc mappings and also under mappings to which, for an exhaustion G_n and an evident notation, an unbounded sequence $\int_0^{M_n} d\lambda / K_\lambda^{(n)}$ corresponds. (The theory was later extended to the ideal boundary of Riemann surfaces [Pf6, Section 43] and further to parts or elements of it [J2].)

Pfluger's paper contains many other results concerning the behavior under qc mappings of the reduced module, the Robin and the capacity constants, and also distortion theorems, generalizing theorems from univalent function theory like the Koebe–Bieberbach one.

All the paper is written for Grötzsch qc mappings but in a note at the end Pfluger asserts:

(1) The results remain valid for a much more general class of mappings, which preserves the topological properties and verifies the Grötzsch inequalities (3.5) for all quadrilaterals Q with $\bar{Q} \subset G$.

(2) It suffices that these inequalities be verified only for small quadrilaterals.

(3) As dilatation quotient at z is defined $\inf K$ taken over the constants K in the Grötzsch inequalities for all quadrilaterals in the neighborhood of z .

In this note Pfluger introduced the *class of general qc and qr mappings* (which we shall call further simply *qc* respectively *qr mappings*) by their *geometric definition*, which we formulate first only in the *qc* case:

DEFINITION 4. An s.-p. homeomorphism $f: G \rightarrow G'$ is K -qc if, for any quadrilateral Q with $\bar{Q} \subset G$, module $M(Q)$ and image $Q' = f(Q)$ with module $M(Q')$,

$$\frac{1}{K}M(Q) \leq M(Q') \leq KM(Q) \quad (4.2)$$

for a certain constant K (evidently ≥ 1).

As we already saw, it is sufficient to ask instead of (4.2) only

$$M(Q') \leq KM(Q). \quad (4.2')$$

It was Ahlfors who introduced in 1953 (see his memoir [A4] already quoted in Section 3.4) the same class of qc mappings by the same geometric definition and founded their theory by proving their fundamental properties. The paper essentially contributed to the spreading of this qc mapping class.

After Ahlfors [A4] one defines the *maximal dilatation* of an s.-p. homeomorphism $f: G \rightarrow G'$ as

$$K(G) = \sup_Q \frac{M(Q')}{M(Q)} \quad (4.3)$$

for all quadrilaterals Q with $\bar{Q} \subset G$, and Definition 1 has the equivalent form:

DEFINITION 4'. An s.-p. homeomorphism $f: G \rightarrow G'$ is a qc mapping if its maximal dilatation $K(G)$ is finite. In this case f is K -qc for any $K \geq K(G)$.

A conformal mapping is 1-qc and all Grötzsch qc mappings are qc. The maximal dilatation is a conformal invariant. If f is K -qc, then f^{-1} is also K -qc; if $f_1: G \rightarrow G'$ is K_1 -qc and $f_2: G' \rightarrow G''$ is K_2 -qc, then $f_2 \circ f_1$ is K_1K_2 -qc.

By using the expression of the module with the extremal length and other properties of the module (Theorem in Section 2.2, decomposition of a quadrilateral by a cross-cut joining the a -sides), Ahlfors proves a vast series of theorems:

- $K(G) = \sup K(G_j)$ if $G = \bigcup G_j$, what justifies Pfluger's assertion (2): f is K -qc iff, for every point $z \in G$, there is a neighborhood, where f is K -qc.
- The maximal dilatation does not change by removing an analytic arc from the domain.
- Grötzsch's inequalities for K -qc mappings between ring domains.
- K -qc mappings of the unit disk onto itself are Hölder continuous with exponent $1/K$.
- They extend to a homeomorphism of the closed unit disk onto itself (result obtained also by Tôki and Shibata [TSh]).
- These mappings form a normal family and the limits are either in the family or a constant of module 1.
- Deletion of an analytic arc does not change the maximal dilatation and as a consequence the reflection principle holds (see also [LV3, I, 8.2–8.4, and V, Section 3]).

Definition 4 and many properties hold on Riemann surfaces because they can be locally formulated, by means of the parametric neighborhoods.

Concerning Pfluger's assertion (3) the name used of dilatation quotient is now replaced by that of *maximal dilatation of the mapping at the considered point*. According to Lehto and Virtanen [LV3, I, Section 9], $f: G \rightarrow G'$ being an s.-p. homeomorphism and z a point in G , the *maximal dilatation of f at z* is

$$F_f(z) = F(z) = \inf_{\mathcal{U}_z} K(\mathcal{U}_z), \quad (4.4)$$

for every neighborhood $\mathcal{U}_z \subset G$ and $K(\mathcal{U}_z)$ defined by (4.3).

Evidently, $F(z)$ is a conformal invariant, $F_f(z) = F_{f^{-1}}(f(z))$, and $F(z) < \infty$ iff f is *qc* in a neighborhood of z .

Since

$$K(G) = \sup_{z \in G} F(z), \quad (4.5)$$

one obtains a local characterization of the *qcty*:

THEOREM. *An s.-p. homeomorphism $f: G \rightarrow G'$ is K -qc iff $F(z) \leq K$ for all $z \in G$ [LV3, I, Theorem 9.1].*

By definition F is upper semicontinuous. Moreover, $F(z) = \overline{\lim}_{\zeta \rightarrow z} F(\zeta)$, even if $F(z) = \infty$, [LV3, I, Theorem 9.2].

THEOREM. *At any differentiability point z of a qc mapping f ,*

$$\max_{\alpha} |\partial_{\alpha} f(z)| \leq F(z) \min_{\alpha} |\partial_{\alpha} f(z)|, \quad (4.6)$$

hence, if z is a regular point of f

$$D(z) \leq F(z), \quad (4.7)$$

with $D(z) = p(z)$ the dilatation quotient (1.10), (1.10').

For a Grötzsch *qc* mapping

$$D(z) = F(z) \quad \forall z \in G. \quad (4.8)$$

One proves that if a homeomorphism f is differentiable at a point z_0 and *qc* in a neighborhood of z_0 , then the following three assertions are equivalent: $J(z_0) > 0$, $\min_{\alpha} |\partial_{\alpha} f(z_0)| > 0$ and $|f_z(z_0)| > 0$ (see [LV3, I, 9.5–9.6]).

There is also another important l. characteristic of the *qc* mappings: $H(z)$ called the *circular*, the *linear* or the *metric dilatation* which indicates the distortion of small circles

centered at z . It was proposed by Lavrent'ev (see (3.13)) in order to obtain a class of mappings larger than the a.anal. ones and will be used in Section 4.3.

Let be again $f: G \rightarrow G'$ an s.-p. homeomorphism, z_0 a point of G and $r > 0$ such that $\{|z - z_0| \leq r\} \subset G$. If one denotes

$$\begin{aligned} l(z_0, r) &= \min_{|z - z_0| = r} |f(z) - f(z_0)|, \\ L(z_0, r) &= \max_{|z - z_0| = r} |f(z) - f(z_0)| \end{aligned} \quad (4.9)$$

and

$$H_f(z_0, r) = H(z_0, r) = \frac{L(z_0, r)}{l(z_0, r)}, \quad (4.9')$$

then the *circular dilatation* of f at z_0 is given by

$$H_f(z_0) = H(z_0) = \overline{\lim}_{r \rightarrow 0} H(z_0, r). \quad (4.9'')$$

The definition extends by inversion if G and $G' \subset \widehat{\mathbb{C}}$ and $z_0 = \infty$ or $f(z_0) = \infty$: $H_f(\infty) = H_{\tilde{f}}(0)$, where $\tilde{f}(\zeta) = f(1/\zeta)$ in the neighborhood of $\zeta = 0$ and, if $f(z_0) = \infty$, $H_f(z_0) = H_{1/f}(z_0)$.

At a regular point z of f

$$H(z) = D(z), \quad (4.10)$$

hence for a Grötzsch qc mapping with only regular points as in Definition 1, Section 3.1,

$$H(z) = D(z) = F(z) \quad \forall z \in G.$$

On the contrary, at nonregular points it is possible that $H(z)$ be finite while $F(z) = \infty$, as in the example $f: \mathbb{C} \rightarrow \{|w| < 1\}$

$$f(z) = e^{-\frac{1}{|z|} + i \arg z} \quad \text{for } z \neq 0; \quad f(0) = 0.$$

Mori [Mo2] proved that if f is K - qc , then H is bounded by a number depending only on K , namely $H(z) \leq e^{\pi K}$ by means of (4.13) and Lehto, Virtanen and Väisälä [LVVä] gave a lower bound for $H(z)$ (see [LV3, II, Section 9, Theorems 1–3, and other results in V, Section 2]).

As qc mappings have been studied simultaneously with the qr ones (called pseudo-analytic in that time) let us return to the end of Pfluger's paper [Pf4] and formulate the *geometric definition for general qr mappings*, i.e., to associate the topological properties of an interior transformation with the module inequality expressing the bounded distortion. It was exactly what Hersch and Pfluger made in 1952 [HPf] obtaining Definition 5' from

below, but we shall first give the following equivalent form based on the interior transformation decomposition theorem:

DEFINITION 5. A K - qr mapping f of the domain G into $\widehat{\mathbb{C}}$ is an interior mapping of G which permits a representation

$$f = g \circ h, \quad (4.11)$$

where $h: G \rightarrow G'$ is a K - qc mapping (i.e., K - qc homeomorphism) and g a nonconstant analytic function in G' .

This formulation is general since any definition can be used for the K - $qcty$ of h and thus a corresponding definition for K - $qrty$ is obtained.

DEFINITION 5'. A K - qr mapping f of G is a continuous mapping of G , l. homeomorphic and s.-p. up to a set $E \subset G$ of isolated points (i.e., an interior mapping), which verifies the module condition

$$M(Q') \leqslant K M(Q), \quad Q' = f(Q), \quad (4.12)$$

for every quadrilateral Q such that $\overline{Q} \subset G$ and f maps $\overline{Q}$ topologically.

In this definition it is possible to ask only that f maps topologically Q onto a quadrilateral Q' , that f satisfies locally a condition (4.12) or to replace (4.12) by

$$\sup_{z \in G \setminus E} F(z) \leqslant K$$

with F from (4.4) see [LV3, VI, 1.7].

Starting with Definition 5', Hersch and Pfluger [HPf] extended to qr mappings many properties of the analytic functions: Schwarz lemma, Pick-Schwarz lemma on the hyperbolic distance between points in a simply-connected domain, Jensen's inequality and Blaschke's theorem of zeros. They generalized also the variation of the harmonic measure under K - qc mappings.

Later Hersch [H1,H2] thoroughly studied the extremal length and its applications to K - qc and K - qr mappings, proved distortion results for harmonic measure, hyperbolic distance, Green function, generalized Schwarz lemma and Phragmén-Lindelöf theorem.

Among the many papers dedicated in that period to qc and qr mappings that of Mori [Mo2] distinguishes by a rich succession of lemmas and theorems, which retakes and continues (sometimes in a stronger form) Ahlfors' work [A4] to construct the theory of these mappings. Very useful in applications are the following two results:

MORI'S LEMMA [Mo2, Lemma 1]. Let $Q_n(A_n, B_n, C_n, D_n)$, $n = 1, 2, \dots$, be a sequence of quadrilaterals and $Q(A, B, C, D)$ a quadrilateral. If the boundary arcs $A_n B_n, \dots, D_n A_n$ of Q_n converge in Fréchet's sense to the boundary arcs $AB, \dots, DA$ of Q , respectively (i.e., there are homeomorphisms $\tau_n: \partial Q_n \rightarrow \partial Q$ with $\tau_n(A_n) = A$, $\tau_n(B_n) = B$, $\tau_n(C_n) = C$ and $\tau_n(D_n) = D$, and for any $\varepsilon > 0$, if n is sufficiently large,

$|\tau_n(P) - P| < \varepsilon$ for every $P \in \partial Q_n$, then the modules $M(Q_n)$ converge to $M(Q)$.

LEMMA 4 [Mo2]. Let $w = f(z)$ be a K -qc mapping of G onto G' , and $\{|z - z_0| \leq r\} \subset G$. Then for $m(r) = l(z_0, r)$ and $M(r) = L(z_0, r)$ in (4.9)

$$M(r) \leq e^{\pi K} m(r). \quad (4.13)$$

Another important result (Theorem 1, see below) establishes analytic properties of the K -qc mappings in geometric sense, and was essential in proving the equivalence between geometric and analytic definitions.

THEOREM 1 [Mo2]. Let $w = f(z)$ be a K -qc mapping of G onto G' . Then:

- (i) f is differentiable a.e. in G ,
- (ii) at every point $z \in G$, where f is differentiable,

$$\max_{\varphi} |\partial_{\varphi} f(z)|^2 \leq K J(z), \quad \text{and} \quad (4.14)$$

(iii) for almost all $y = y_0$, $f(x, y_0)$ is AC (absolutely continuous) in x on any closed interval contained in the intersection of $y = y_0$ and G .

“In virtue of these properties” wrote Mori, “we can extend almost all known results on continuously differentiable qc mappings to the class of K -qc mappings, by simple recapitulation of the original proofs”.

Further Mori gives properties for K -qc and K -qr mappings in $|z| < 1$ with values onto or into $|w| < 1$. As an example the Hölder continuity, precisising the results of Ahlfors [A4], Yûjôbô [Yû1], which we present with the best possible constant 16 as in other paper of Mori [Mo1] instead of 48 in [Mo2]; see also Ahlfors [A12, p. 47]:

THEOREM. Let f be a K -qc mapping of $|z| < 1$ onto $|w| < 1$, normalized by $f(0) = 0$. Then for any two points z_1 and z_2 in $\{|z| \leq 1\}$,

$$16^{-K} |z_1 - z_2|^K \leq |f(z_1) - f(z_2)| \leq 16 |z_1 - z_2|^{1/K}. \quad (4.15)$$

The mapping $w = z|z|^{\frac{1}{K}-1}$ shows that the exponent $\frac{1}{K}$ cannot in general be replaced by any greater number [LV3, pp. 70–71].

An immediate consequence of Mori’s lemma asserts that the limit of an l.u. convergent sequence of K -qc mappings in G is a K -qc mapping or a constant. Mori completes the result for a sequence of K -qr mappings: if the limit is nonconstant then it is a K -qr mapping too. He also proves that the family of K -qc mappings f of $|z| < 1$ onto $|w| < 1$ normalized by $f(0) = 0$ is normal, more precisely any sequence of the family contains a subsequence which converges uniformly on the closed disk $|z| \leq 1$ to a K -qc mapping of $|z| < 1$ onto $|w| < 1$. These results express the compactness of K -qc and K -qr mapping classes, already emphasized by Ahlfors [A4] and which was one important ground to extend the class of Grötzsch mappings. Contributions in this direction brought Tôki

and Shibata [TSh], Shibata [Shi1], Väisälä [Vä1] generalizing Lehto and Virtanen [LV1] to qr ty, and others. Let us quote a Shibata's:

THEOREM. *Given a K -qc mapping f of $|z| < 1$ onto $|w| < 1$, there is a sequence f_n of Grötzsch K -qc mappings of $|z| < 1$ onto $|w| < 1$, which $u.$ converges on $|z| < 1$ to f .*

Bers [B9] extends the theorem for K -qr mappings solving thus affirmatively another question raised by Mori [Mo2]: the class of Grötzsch K -qr mappings is dense in that of the K -qr mappings.

An excellent exposition of the numerous and beautiful results which constituted around 1960 the theory of qc and qr mappings are contained in Küenzi's monograph [Kün3], especially in Chapters 4 and 5.

However summary was this presentation, it pointed out the efficiency of the geometric definition what determined its choice as starting point and frame for the first part of Lehto and Virtanen's book [LV3].

4.2. Analytic definition

Another main definition for qc and qr mappings, called *analytic*, has its roots in the initial Grötzsch definition combining the topological properties with some regularity conditions and one of the equivalent dilatation conditions (3.1)–(3.1^{iv}). The connection with Beltrami equation led to a new formulation due to Bers. Notions and tools of real analysis are essential in this point of view, so that the $qcty$ monographs dedicated a special chapter to them (see Volkovyskij [Vo3], Lehto and Virtanen [LV3], Väisälä [Vä5] and Caraman [C4]). We limit ourselves to enunciate only a few definitions or results, see a systematic exposition in [LV3, III].

The analytic definition has now the form, in a certain sense the weakest possible, given by Gehring and Lehto [GL], [LV3, IV, Section 2]:

DEFINITION 6. An s.-p. homeomorphism $f : G \rightarrow G'$ is K -qc if

- (1) f is ACL (absolutely continuous on lines) in G and
- (2) f satisfies the dilatation condition:

$$\max_{\varphi} |\partial_{\varphi} f(z)| \leq K \min_{\varphi} |\partial_{\varphi} f(z)|. \quad (4.16)$$

A continuous function defined in G is ACL in G if in every rectangle $R = \{z: a < x < b, c < y < d\}$, $R \subset G$, f is absolutely continuous as a function of x on a. all segments $I_y = \{z: a \leq x \leq b\}$ and as a function of y on a. all segments $I_x = \{z: c \leq y \leq d\}$. It follows that these functions have finite partial derivatives a.e. in G [LV3, III, Section 3, pp. 127–128].

The analytic definition for qr mappings is deduced from Definitions 5 and 6.

The first contribution to the analytic approach was due to Morrey in 1938 [Mor], who studied the mappings $w = f(z) : G \rightarrow \mathbb{C}$ as generalized solutions of an elliptic partial

differential equation system

$$\begin{aligned}v_x &= -(b_2 u_x + c u_y + e), \\v_y &= a u_x + b_1 u_y + d\end{aligned}\tag{4.17}$$

with $4ac - (b_1 + b_2)^2 \geq m > 0$ and with uniformly bounded and measurable coefficients, in particular of a Beltrami system (1.12).

The solutions are ACT with partial derivatives in L^2_{loc} . (ACT, i.e., AC in Tonelli's sense, means ACL with partial derivatives in L^1_{loc} [LV3, III, Section 6, p. 143].)

Morrey proved many important properties of the solutions: theorems of unicity, existence, compacity, Hölder inequality, different integration formulas.

Concerning the Beltrami system Morrey established:

(α) if f_1 and f_2 are two solutions in the same domain G and f_1 is a homeomorphism, then $f_2 \circ f_1^{-1}$ is analytic;

(β) if f is a solution, so is $g \circ f$ for g analytic;

(γ) in any domain there exists a homeomorphic solution;

(δ) if f is an injective solution in a domain G , the Jacobian $J > 0$ a.e., and for any measurable set $E \subset G$, the image $f(E)$ is also measurable with the Lebesgue measure

$$m(f(E)) = \iint_E J dx dy,$$

i.e., the set function $\Phi: E \rightarrow m(f(E))$ associated to f is AC (one says that f is measurable and also AC).

Research was continued by Bers and Nirenberg [BN]; Bers [B9] proved that the derivatives of a solution belong to L^2_{loc} and Bojarski [Bo1] to L^p_{loc} to a power $p > 2$ depending only on K . More exactly,

$$p < p(K) \leq \frac{2K}{K-1}$$

[LV3, V, 5.4, p. 215]; ulterior, in 1994 Astala [As1] proved that

$$p(K) = \frac{2K}{K-1}\tag{4.18}$$

as a consequence of his theorem of area distortion, which solved a problem opened by Bojarski in 1957 and formulated by Gehring and Reich in 1966 [GR]. Astala's theorem asserts for the unit disk $B(0, 1)$:

If $f: B(0, 1) \rightarrow B(0, 1)$ is a K -qc mapping with $f(0) = 0$, then

$$m(f(E)) \leq M m(E)^{1/K}$$

for all Borel measurable sets $E \subset B(0, 1)$ with a constant M depending only on K , $M = 1 + O(K - 1)$.

It follows that for each K -qc mapping $f: G \rightarrow G'$ the derivatives belong to $L^p_{\text{loc}}(G)$ if $p < \frac{2K}{K-1}$, and $w = |z|^{1/K} \frac{z}{|z|}$ shows that this is no more true for $p \geq \frac{2K}{K-1}$. (For other consequences see [As1, As2].)

Morrey's paper, due to its connection with PDEs, was almost 20 years ignored by the mathematicians working in *qcty*, since in 1956 when Bers discovered it, recognized its great significance for *qcty* and spread it. It was the moment when Bers was charged to edit the posthumous paper of Mori, which we discussed before [Mo2]. Among the problems raised by Mori in this paper there was the question if *qc* mappings are measurable. Bers showed that the affirmative solution was already given by Morrey in [Mor], assertion (δ).

In 1940–1950s a lot of work was done by studying mappings defined as solutions of PDE systems, solutions called *pseudo-analytic functions*, and *pseudo-conformal mappings* if they were injective. A significant contribution in this sense, with many applications in mechanics and physics, is due to Lavrent'ev and his school. Šapiro in 1941 [Š], Šabat in 1945 and later [Ša1–Ša3] considered the so-called *mappings with two pairs of characteristics* which transform infinitesimal ellipses into infinitesimal ellipses and Lavrent'ev himself dealt with general elliptic PDE systems creating a theory with rich consequences [La4–La6], see his monograph [La7]; Položij introduced the *p-analytic functions* (1946–1948), see also Section 2, (2.14), and *(p, q)-analytic functions* (1953–1957) and thoroughly studied them [P1–P6], see the monograph [P7]; Vekua and his school: Daniljuk [D], Bojarski [Bo1, Bo2] and others constructed the theory of the *generalized analytic functions* presented in the monograph [Ve]. In the same period another branch of Lavrent'ev's school treated *qc* mappings from the Geometric Function Theory point of view: the Volkovyskij school in Lwow represented by Belinskij, Gol'dberg, Pesin, Mihalchuk. However, returning to the PDEs direction we have to mention for the same period the work on pseudo-analytic functions by Bers [B1–B8], Nirenberg [Nir1, Nir2], Finn [Fi1, Fi2], Bers and Nirenberg [BN], Finn and Serrin [FiS]. Research continued and we shall quote the work by Renelt dedicated especially of the PDE system $f_{\bar{z}} = \nu f_z + \mu \overline{f_z}$, with ν and μ measurable functions and $\|\nu\| + \|\mu\|_\infty < 1$, and synthesized in [Ren1, Ren2].

Concerning the analytic definition two directions developed: one represented by Caccioppoli [Ca1–Ca4], Yûjôbô [Yû1–Yû4], Bers and Nirenberg [BN], who searched to obtain the best form of this definition by analytic means, and other Ahlfors [A4], Pfluger [Pf4], Mori [Mo2], starting with the geometric Definition 4, studied analytic properties of the corresponding mappings.

Thus Strebel proved in 1955 [Str1] that the geometric K -qc mappings are ACL and Mori's Theorem 1 from above contained the fact that geometric Definition 4 implies the analytic Definition 6. But in that time the analytic definition had still a more complicated form. By editing Mori's paper [Mo2], Bers could only write that with Mori's Theorem 1 one obtains “in a few lines” the equivalence of the two definitions generally accepted as “natural” the geometric respectively the analytic, and completed Theorem 1 by proving that a geometric K -qc mapping has partial derivatives in L^2_{loc} .

Bers published a proof of this equivalence [B9] in the same issue of *Transactions of American Mathematical Society* **84** (1957) immediately after Mori's paper. First he presented the analytic definition and the concepts used:

DEFINITION 7 (Morrey, Caccioppoli, Bers–Nirenberg). A mapping $f : G \rightarrow \mathbb{C}$ is K - qr if f has L^2 -derivatives that satisfy a.e. in G

$$\max_{\varphi} |\partial_{\varphi} f|^2 \leqslant KJ. \quad (4.19)$$

After recalling the concept of L^2 -derivatives in the sense of Sobolev [So] and Friedrichs [Fr], Bers gives the following definitions which we present as in [LV3, III, Section 6, p. 143], respectively [LV3, IV, Section 5, p. 184]. Let be $p \geqslant 1$. Then:

- f has L^p -derivatives if it is ACL in G and its partial derivatives f_x, f_y belong to L^p_{loc} .
- f is a generalized L^p -solution of a Beltrami equation

$$w_{\bar{z}} = \mu w_z \quad (4.20)$$

with μ measurable function and $\|\mu\|_{\infty} < 1$, if it has L^p -derivatives which verify (4.20) a.e. in G .

Bers considered $p = 2$ and continuous solutions but indicates that continuity follows if the solution is measurable and belongs to L^2_{loc} together with its derivatives.

Definition 7 is equivalent to the following one.

DEFINITION 7'. A mapping $f : G \rightarrow \mathbb{C}$ is K - qr if it is a generalized L^2 -solution of a Beltrami equation with $\|\mu\|_{\infty} \leqslant k < 1$, $k = (K - 1)/(K + 1)$.

Both Definitions 7 and 7' have now more general form, see [LV3, VI, Section 2].

Coming again to Mori's Theorem 1 and Definition 7, Bers proved easily that the derivatives of a K - qc mapping in geometric sense are l. square integrable, hence that Definition 4 implies Definition 7, and with (β) from Morrey, this remains valid for K - qr mappings too. Conversely, with (δ) it is possible to repeat Grötzsch's proof for the module inequality (4.2) in the case of a qc -homeomorphism after Definition 7' and obtain that it verifies Definition 4. Further with (γ) and (α) the same is true for qrr .

In 1959 Pfluger brought a new simplification to the analytic definition [Pf7]. He succeeded to renounce the condition on L^2 -derivatives working with the following analytic definition.

DEFINITION 8. An s.-p. homeomorphism $f : G \rightarrow G'$ is K - qc if

- (1) f is ACT (i.e., ACL and has L^1_{loc} -derivatives) and
- (2) inequality (4.19) is verified a.e. in G .

Pfluger proved the equivalence between Definitions 4 and 8, without using the Rademacher–Stepanoff theorem to obtain that a K - qc mapping in geometric sense is differentiable a.e.

Nevertheless Definition 6, the definitive form of the analytic definition for K - qc mappings, could be obtained only in 1959, when Gehring and Lehto proved the beautiful

THEOREM ([GL], [LV3, III, Section 3, p. 128]). Let f be a continuous open mapping (in particular, a homeomorphism) of the domain G into $\mathbb{C}$ with finite partial derivatives a.e. in G . Then f is differentiable a.e. in G .

The condition ACL in Definition 6 is necessary as shows a counterexample of an s.-p. homeomorphism, conformal a.e. but not qc in geometric sense [LV3, IV, 2.2, p. 167].

Studies on qc definitions continued. Thus in 1962 Bers [B16] gave a new elegant, short proof of the equivalence between Definitions 4 and 7, namely he proved the equality

$$K_a(G, f) = K_g(G, f),$$

where $K_a(G, f) = \min\{K: |f_x|^2 + |f_y|^2 \leq (K + \frac{1}{K})J \text{ a.e. in } G\}$ and $K_g(G, f) = K(G)$ for f , see (4.3). In the proof Bers applied two theorems fundamental for the $qcty$ in the plane:

THE MEASURABLE RIEMANN MAPPING THEOREM (Ahlfors and Bers (1960) [AB1]). *For a Beltrami equation with the coefficient μ measurable and $\|\mu\|_\infty < 1$ one has not only existence and unicity of a normalized homeomorphic solution f^μ (e.g., if μ is defined in $\mathbb{C}$ a homeomorphism $f^\mu: \mathbb{C} \rightarrow \mathbb{C}$ with $f^\mu(0) = 0$, $f^\mu(1) = 1$) but also a “nice” dependence of f^μ on μ (e.g., if $\mu(z, t_1, \dots, t_h)$ depends analytically on real or complex parameters t_j , the same is true of f^μ).*

BEURLING–AHLFORS THEOREM (1956 [BeuA]). *There is a K - qc mapping f of the superior half-plane $\{z: y > 0\}$ onto itself with the boundary correspondence $x \rightarrow \varphi(x)$ iff*

$$\frac{1}{\rho} \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq \rho \quad (4.21)$$

for a certain constant ρ and arbitrary $x \in \mathbb{R}$, $t \in \mathbb{R} \setminus \{0\}$.

More exactly Beurling and Ahlfors construct a Grötzsch K - qc mapping with the boundary correspondence φ and with $K \leq \rho^2$; for any K - qc mapping with the boundary correspondence φ , $K \geq 1 + A \ln \rho$, where A is a constant.

A function φ verifying (4.21) is called ρ -*quasisymmetric*, and in the last time the concept was extended to higher dimensions and metric spaces; see Section 4.3 and, e.g., [Sr2].

4.3. Metric definition

A third main definition of the $qcty$ and $qrtv$ was proposed by Lavrent'ev [La2] (see Section 3.3 from above) and is based on the circular dilatation $H(z)$ (4.9)–(4.9''). Gehring [G1] gave it the following now classical form, also included in [LV3, IV, Section 4, pp. 177–181].

DEFINITION 9. An s.-p. homeomorphism $f: G \rightarrow G'$ is K - qc if

- (1) $H(z)$ is bounded in G and
- (2) $H(z) \leq K$ a.e. in G .

Gehring proved in [G1] the equivalence of Definition 9 with the analytic Definition 6. Much more he proved that (1) can be replaced by the weaker condition

(1') $H(z)$ is finite in G outside a set of σ -finite linear measure (as Yûjôbô formulated without proof in [Yû4]).

Condition (1) can be replaced also by:

(1*) f satisfies (NL) (i.e., Luzin's condition N: for every horizontal rectangle $R = \{z: a < x < b, c < y < d\}$, $\bar{R} \subset G$, and a.e. $y_0 \in (c, d)$ if E is a set of linear measure 0 on the segment $\{y = y_0\} \cap R$, $f(E)$ is of linear measure 0, and the same holds for x instead of y).

Examples prove that condition (2) is not sufficient for K -qc (e.g., there is a non- qc homeomorphism with $H(z) = 1$ a.e. in G , see [Thi, IV, 4.2, p. 177]) and illustrate different inequalities between $H(z)$ and K (e.g., there is K -qc homeomorphism of $\mathbb{C}$ such that $H(z) = K$ a.e. but $H(z) = K^2 > K$ on a set of Hausdorff dimension 2, hence it is not possible to ask that condition (2) holds overall [G1]).

Evidently, f is qc iff H is bounded.

One obtains a metric definition for K -qr mappings by means of Definitions 5 and 9, as well as by asking in Definition 5' instead of condition (4.12) that $H(z)$ is finite in $G \setminus E$ and $\leq K$ a.e. in $G \setminus E$.

The metric Definition 9 had to be very important by the possibility to be extended directly in higher dimensions and, in the last time, to metric spaces.

To underline only two aspects of this progress we denote by x, y points in the definition domain, by $|x - y|$ the corresponding distance and extend the definitions in (4.9)–(4.9'') (for details and references, see, e.g., Srebro [Sr2]).

- In 1995 Heinonen and Koskela [HK1] succeeded to replace in Definition 9, $H(x) = \overline{\lim}_{r \rightarrow 0} H(x, r)$ (4.9'') by $\underline{\lim}_{r \rightarrow 0} H(x, r)$; see also Cristea [Cr4].
- Besides the qc mappings one considered *quasisymmetric mappings* for which there is some constant $H \in [1, \infty]$ such that $H(x, r) \leq H$ for all x in the definition domain and $r > 0$ (Tukia and Väisälä [TuVä]). In one-dimensional case one obtains the quasisymmetric functions of Beurling–Ahlfors (4.21).

To the older history of the metric definition belongs also Jenkins paper (1957) [Je3] which gives a sufficient criterion for $qcty$:

Let f be an s.-p. homeomorphism: $G \rightarrow G'$. If for any z_0 in G both $m(z_0) = \underline{\lim}_{r \rightarrow 0} (l(z_0, r)r^{-1})$ and $M(z_0) = \overline{\lim}_{r \rightarrow 0} (L(z_0, r)r^{-1})$ (with the notations in (4.9)), are positive and finite, and if there is a constant $K \geq 1$ such that $(M(z_0)/m(z_0)) \leq K$, then f is K^2 - qc .

The condition is only sufficient since, e.g., $w = z|z|^{K-1}$ is a K - qc mapping of $\mathbb{C}$ but $m(0) = M(0) = 0$.

Another extension of Lavrent'ev's definition is due to Pesin (1955) [Pe1, Pe2], who replaced in the dilatation condition the family of concentric circles by a topological image of such a family not too much distorted. He worked with s.-p. homeomorphisms of the unit disk onto itself.

The family $C(z, k)$ of curves C_α , $0 < \alpha < 1$, is called, $1 \leq k < \infty$, with respect to the point z if the curves C_α are the images of the circles $|t| = \alpha$ by a homeomorphism of $|t| < 1$ onto a neighborhood of z , so that $t = 0$ corresponds to z and

$$\lim_{\alpha \rightarrow 0} \frac{\max_{z' \in C_\alpha} |z' - z|}{\min_{z' \in C_\alpha} |z' - z|} = k \quad (4.22)$$

(in fact, $k = H(0)$, (4.9'') for the mapping $z' = z'(t)$).

An s.-p. homeomorphism $w = f(z) : |z| < 1 \rightarrow |w| < 1$ is called *regular at z* if it maps a certain regular family $C(z, k')$ in a regular family $C(f(z), k'')$ and the number $q(z) = \inf k' k''$ for all possible families $C(z, k')$ is called the *characteristic of the mapping f* .

A mapping f *regular* (i.e., regular at every point z in $|z| < 1$) is called a *generalized K -qc mapping* if $\text{ess sup}_{|z| \leq 1} q(z) \leq K$.

Pesin established many results for this mapping class, some of them with Belinskij, [Pe1–Pe3, BeP], among which the absolute continuity (for the associate set function), the compacity with respect to the 1. uniform convergence, the behavior relative to Ahlfors–Beurling null-sets [ABeu2]. The class is the closure of the Lavrent'ev class and coincides with Pfluger's [Pf4], i.e., with that of *qc mappings*. It contains as subclass that of mappings with two pairs of characteristics by Šabat [Ša1] (see also Näättänen [Nä], [LV3, pp. 181–182]) and was extended by Caraman to $n(\geq 2)$ -dimension under the name of Markuševič–Pesin [C4, pp. 127, 286] and used by Reshetnyak for defining the *qerty* in space [R3]. Belinskij extended for these mappings the existence and unicity theorem of Lavrent'ev [Be8, II].

4.4. Other definitions

There are many other characterizations of *qc mappings* which can be interpreted as definitions. In 1967 Gehring [G9] published a survey on some of them (see also [LV3, IV, Section 3, pp. 170–177], [C4, Part 2]).

In this section f is always an s.-p. homeomorphism $f : G \rightarrow G'$.

For instance in the geometric Definition 4 it is sufficient to consider instead of all quadrilaterals only the rectangles, and even only the rectangles with horizontal a -sides if one works with both inequalities (4.2) (or equivalently with horizontal and vertical a -sides and a single inequality (4.2')). The mapping will be a $(K + \sqrt{K^2 - 1})$ -qc one, see Gehring and Väisälä (1961) [GV1]. Further it is sufficient to consider only rectangles with the a -sides on two fixed directions; if the angle between them is 2τ ($0 < \tau \leq \pi/4$), the mapping will be $((K + \sqrt{K^2 - \sin^2 2\tau})/(2 \sin^2 \tau))$ -qc [AC5].

The inequality (4.2') verified on a single direction does not imply *qerty* but many analytic properties hold on that direction, thus it gives a way to a class more general of mappings we denoted by $\mathcal{O}$, since in another equivalent approach the orientation of the characteristic ellipses is relevant [AC5, AC12, AC13, AC15].

Further another definition is obtained by asking the module inequality (4.2') for ring domains, or even subclasses of ring domains, see Gehring and Väisälä [GV1], Reich [Re1], Lehto and Virtanen [LV3, I, Section 7].

Kelings [Ke] proved definitions by means of the distortion of harmonic and respectively of hyperbolic measure.

After finding a definition of the measure of the angle formed by two arcs with only extremity common, in 1965 Agard and Gehring [AgG], Agard [Ag] and in 1966 Taari [Ta] gave definitions by the angle distortion.

A very important definition is based on extremal length or module of curve families, the so-called *general module condition*, Väisälä (1961) [Vä2], [LV3, IV, 3.3, p. 171]:

DEFINITION 10. f is K -qc if, for any arc family $\mathcal{C} \subset G$ with image $\mathcal{C}'$, the module (2.8) verifies

$$M(\mathcal{C}) \leqslant K M(\mathcal{C}'). \quad (4.23)$$

The inequality (4.23) can be asked only for certain families, for instance: The qc mappings are the s.-p. homeomorphisms which invari the curve families of extremal length zero, Renggli (1962) [Regli3] (Renggli gave also another geometric definition in [Rgli2]).

In connection with Section 2, (2.21), the extremal length gives a definition of the class $\mathcal{O}$ and other subclasses [AC3, AC4, AC13, AC15].

We finish with the following definition:

DEFINITION 11. Let $\mathcal{Q}_K$ denote the class of K -qc mappings and $\mathcal{Q}_K^*$ that of Grötzsch K -qc mappings. Then $f \in \mathcal{Q}_K$ iff there is a sequence $f_n \in \mathcal{Q}_K^*$, $f_n: G_n \rightarrow G'_n$, $\overline{G}_n \subset G$, $G_1 \subset G_2 \subset \dots$, $\bigcup G_n = G$, l. uniformly convergent to f in G . (If G is simply connected, G_n can be taken equal to G .)

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References

- [Ab1] W. Abikoff, *The Real Analytic Theory of Teichmüller Space*, Lecture Notes in Math., Vol. 820, Springer-Verlag, Berlin–Heidelberg–New York (1980).
- [Ab2] W. Abikoff, *Oswald Teichmüller*, Math. Intelligencer **8** (3) (1986), 8–16, 33.
- [Ag] S.B. Agard, *Topics in the theory of quasiconformal mappings*, Dissertation, Univ. of Michigan (1965); Dissert. Abstr. USA **26** (1965), 2767.
- [AgG] S.B. Agard and F.W. Gehring, *Angles and quasiconformal mappings*, Proc. London Math. Soc. (3) **14A** (1965), 1–21.
- [Agm] S. Agmon, *A property of quasi-conformal mappings*, J. Ration. Mech. Anal. **3** (1954), 763–765.
- [A1] L.V. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen*, Acta Soc. Sci. Fenn. A **19** (1930), 1–40.
- [A2] L.V. Ahlfors, *Zur Theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.
- [A3] L.V. Ahlfors, *Conformal Mapping*, Lectures at Oklahoma Agric. and Mech. College, Dept. of Math., Summer Session (1951).
- [A4] L.V. Ahlfors, *On quasiconformal mappings*, J. Anal. Math. **3** (1954), 1–58, 207–208.
- [A5] L.V. Ahlfors, *Conformality with respect to Riemannian metrics*, Ann. Acad. Sci. Fenn. A **I206** (1955), 1–22.
- [A6] L.V. Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces*, Analytic Functions, Princeton Univ. Press (1960), 45–66.

- [A7] L.V. Ahlfors, *Curvature properties of Teichmüller space*, J. Anal. Math. **9** (1961), 161–176.
- [A8] L.V. Ahlfors, *Some remarks on Teichmüller spaces of Riemann surfaces*, Ann. of Math. **74** (1961), 171–191.
- [A9] L.V. Ahlfors, *Teichmüller spaces*, Proc. ICM Stockholm, 15–22 August, 1962, Inst. Mittag-Leffler, Djursholm (1963), 3–9.
- [A10] L.V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
- [A11] L.V. Ahlfors, *Quasiconformal mappings and their applications*, Lectures on Modern Mathematics, Vol. II, T.L. Saaty, ed., Wiley, New York (1964), 151–164.
- [A12] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, NJ (1966).
- [A13] L.V. Ahlfors, *The structure of a finitely generated Kleinian group*, Acta Math. **122** (1969), 1–17.
- [A14] L.V. Ahlfors, *Two lectures on Kleinian groups*, Proc. Romanian–Finnish Seminar on Teichmüller Space and Quasiconformal Mappings Braşov, 26–30 August, 1969, Publ. House Acad. Romania, Bucureşti (1971), 49–64.
- [A15] L.V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
- [A16] L.V. Ahlfors, *Quasiconformal mappings, Teichmüller spaces and Kleinian groups*, Proc. ICM Helsinki 1978, August 15–23, Acad. Sci. Fenn. (1980), 71–84.
- [A17] L.V. Ahlfors, *Collected Papers*, Birkhäuser, Boston–Basel–Stuttgart (1982).
- [AB1] L. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 385–404.
- [AB2] L. Ahlfors and L. Bers, *The Space of Riemann Surfaces and Quasiconformal Mappings*, Izd. Innostr. Lit., Moscow (1961) (in Russian).
- [ABeu1] L.V. Ahlfors and A. Beurling, *Invariants conformes et problèmes extrémaux*, C. R. du 10^e Congrès des Math. Scandinaves, Copenhagen (1946), 341–351.
- [ABeu2] L.V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1/2) (1950), 101–129.
- [ASa] L.V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton (1960).
- [AW] L.V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. **13** (1962), 975–978.
- [AK] T. Akaza and T. Kuroda, *Module of annulus*, Nagoya Math. J. **18** (1961), 37–41.
- [AllGr] N.L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Lecture Notes in Math., Vol. 219, Springer-Verlag, Berlin (1971).
- [AC2] C. Andreian Cazacu, *Sur les transformations pseudo-analytiques*, Rev. Math. Pures Appl. **2** (1957), 383–397.
- [AC3] C. Andreian Cazacu, *On quasiconformal mappings*, Dokl. Akad. Nauk SSSR **126** (2) (1959), 235–238 (in Russian).
- [AC1] C. Andreian Cazacu, *Sur les relations entre les fonctions caractéristiques de la pseudo-analyticité, Lucrările celui de al IV-lea Congres al Matematicienilor Români*, Bucureşti 1956, Ed. Acad. R. P. R., Bucureşti (1960), 82–85.
- [AC4] C. Andreian Cazacu, *Sur l'application de la longueur extrémale dans la théorie des représentations quasi-conformes*, Mathematica (Cluj) **6** (29) (1) (1964), 5–10.
- [AC5] C. Andreian Cazacu, *Sur les inégalités de Rengel et la définition géométrique des représentations quasi-conformes*, Rev. Roumaine Math. Pures Appl. **9** (2) (1964), 141–155.
- [AC6] C. Andreian Cazacu, *Sur un problème de L.I. Volkovyski*, Rev. Roumaine Math. Pures Appl. **10** (1) (1965), 43–63.
- [AC7] C. Andreian Cazacu, *Problèmes extrémaux des représentations quasi-conformes*, Rev. Roumaine Math. Pures Appl. **10** (4) (1965), 409–429.
- [AC8] C. Andreian Cazacu, *Reprezentări cvasiconforme*, Probleme Moderne de Teoria Funcţiilor, Ed. Acad. R. P. R., Bucureşti (1965), 204–309.
- [AC9] C. Andreian Cazacu, *Suprafeţe Riemanniene*, Topologie, Categorii şi Suprafeţe Riemanniene, Ed. Acad. R. P. R., Bucureşti (1966), 241–393.
- [AC10] C. Andreian Cazacu, *Une propriété caractéristique des représentations quasi-conformes extrémales*, Rev. Roumaine Math. Pures Appl. **12** (2) (1967), 167–176.

- [AC11] C. Andreian Cazacu, *Sur les applications quasi-conformes de M.A. Lavrentieff*, Rev. Roumaine Math. Pures Appl. **13** (9) (1968), 1217–1223.
- [AC12] C. Andreian Cazacu, *Bemerkungen über den Begriff der quasikonformen Abbildung*, Math. Nachr. **40** (1–3) (1969), 27–42.
- [AC13] C. Andreian Cazacu, *Influence of the orientation of the characteristic ellipses on the properties of the quasiconformal mappings*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Braşov 1969, Ed. Acad. R. S. R., Bucureşti (1971), 65–85.
- [AC14] C. Andreian Cazacu, *Some formulae on the extremal length in n -dimensional case*, Proc. Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Braşov 1969, Ed. Acad. R. S. R., Bucureşti (1971), 87–102.
- [AC15] C. Andreian Cazacu, *A generalization of the quasiconformality*, Topics in Analysis, Colloquium on Mathematical Analysis, Jyväskylä 1970, Lecture Notes in Math., Vol. 419, Springer-Verlag Berlin-Heidelberg-New York (1974), 4–17.
- [AC16] C. Andreian Cazacu, *Partial differential equations related to extremal problems for quasiconformal mappings*, Ordinary and Partial Differential Equations, Lecture Notes in Math., Vol. 415, Springer-Verlag, Berlin–Heidelberg–New York (1974), 1–14.
- [AC17] C. Andreian Cazacu, *Modules and quasiconformality*, Ist. Naz. Alta Mat. Roma, Sympos. Math., Vol. 18, Academic Press, London and New York (1976), 519–534.
- [AC18] C. Andreian Cazacu, *Module inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. A I Math. **2** (1976), 17–28.
- [AC19] C. Andreian Cazacu, *On extremal quasiconformal mappings*, Rev. Roumaine Math. Pures Appl. **22** (10) (1977), 1359–1365.
- [AC20] C. Andreian Cazacu, *On the Grötzsch and Rengel inequalities*, Complex Analysis, Joensuu 1978, Lecture Notes in Math., Vol. 747, Springer-Verlag, Berlin–Heidelberg–New York (1979), 10–23.
- [AC21] C. Andreian Cazacu, *Quasiconformality and Riemann surfaces with global exhaustion*, Rev. Roumaine Math. Pures Appl. **40** (2) (1995), 149–156.
- [ACS1] C. Andreian Cazacu and V. Stanciu, *Quasiconformal homeomorphisms between Riemann surfaces*, Topics in Complex Analysis, Banach Center Publ., Vol. 31, Inst. of Math. Polish Acad. Sci., Warszawa (1995), 35–43.
- [ACS2] C. Andreian Cazacu and V. Stanciu, *Normal and compact families of BMO - and BMO_{loc} -QC mappings*, Math. Reports **2** (52) (4) (2000), 407–419.
- [ACS3] C. Andreian Cazacu and V. Stanciu, *BMO -Mappings in the plane*, Topics in Analysis and Its Applications, NATO Sci. Ser. II Math. Phys. Chem., Vol. 147, Kluwer, Dordrecht–Boston–London (2004), 11–30.
- [As1] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
- [As2] K. Astala, *Analytic aspects of quasiconformality*, Proc. ICM Berlin, 1998 August 18–27, Vol. 2, Documenta Math., Journal der Deutschen Mathematiker-Vereinigung, Extra Volume ICM 1998, print by Geronimo GmbH, D-83026 Rosenheim, Germany (1998), 617–626.
- [Be2] P.P. Belinskij, *On distortion under quasiconformal mappings*, Dokl. Akad. Nauk SSSR **91** (1953), 997–998 (in Russian).
- [Be3] P.P. Belinskij, *On metric properties of quasiconformal mappings*, Dokl. Akad. Nauk SSSR **93** (1953), 589–590 (in Russian).
- [Be1] P.P. Belinskij, *Behaviour of a quasiconformal mapping at an isolated singular point*, Dokl. Akad. Nauk SSSR **91** (1953), 709–710; Uč. Zap. L'vov. Univ. Ser. Meh.-Mat. **6** (1954), 58–70 (in Russian).
- [Be4] P.P. Belinskij, *On the measure of the area by quasiconformal mappings*, Dokl. Akad. Nauk SSSR **121** (1958), 16–17 (in Russian).
- [Be5] P.P. Belinskij, *On the solution of extremal problems of quasiconformal mappings*, Dokl. Akad. Nauk SSSR **121** (1958), 199–201 (in Russian).
- [Be6] P.P. Belinskij, *On the normality of families of quasiconformal mappings*, Dokl. Akad. Nauk SSSR **128** (1959), 651–652 (in Russian).
- [Be7] P.P. Belinskij, *The use of variational method in solving extremal problems of quasiconformal mappings*, Sibirsk. Mat. J. **1** (1960), 303–330 (in Russian).
- [Be8] P.P. Belinskij, *General Properties of Quasiconformal Mappings*, Nauka, Novosibirsk (1974) (in Russian).

- [BeG] P.P. Belinskij and A.A. Gol'dberg, *Application of a theorem on conformal mappings to invariance problem of the defect of meromorphic functions*, Ukrainian Math. J. **6** (1) (1954), 128–134 (in Russian).
- [BeP] P.P. Belinskij and I.N. Pesin, *On the closure of the class of differentiable quasiconformal mappings*, Dokl. Akad. Nauk SSSR **102** (1955), 865–866 (in Russian).
- [Bel] E. Beltrami, *Delle variabile complesse sopra una superficie qualunque*, Ann. Mat. Pura Appl. (2) **1** (1867–1868), 329–366.
- [Ber] S. Bernstein, *Sur une généralisation des théorèmes de Liouville et de Picard*, C. R. Acad. Sci. Paris **151** (1910) (II), 636–639.
- [B1] L. Bers, *Partial differential equations and generalized analytic functions*, Proc. Nat. Acad. Sci. USA **36** (1950), 130–136; **37** (1951), 42–47.
- [B2] L. Bers, *Theory of Pseudo-Analytic Functions*, Inst. Math. Mech., New York Univ. (1953) Mimeo.
- [B3] L. Bers, *Partial differential equations and pseudo-analytic functions on Riemann surfaces*, Ann. of Math. Stud., Vol. 30, Princeton Univ. Press (1953), 155–167.
- [B4] L. Bers, *Univalent solutions of linear elliptic systems*, Commun. Pure Appl. Math. **6** (1953), 513–526.
- [B5] L. Bers, *Nonlinear elliptic equations without nonlinear entire solutions*, J. Ration. Mech. Anal. **3** (1954), 767–787.
- [B6] L. Bers, *Function-theoretical properties of solutions of partial differential equations of elliptic type*, Ann. of Math. Stud., Vol. 33, Princeton Univ. Press (1954), 69–94.
- [B7] L. Bers, *Local Theory of Pseudo-Analytic Functions*, Lectures on Functions of a Complex Variable. Michigan Univ. Press, Ann Arbor (1955), 214–244.
- [B8] L. Bers, *An outline of the theory of pseudo-analytic functions*, Bull. Amer. Math. Soc. **62** (1956), 291–331.
- [B9] L. Bers, *On a theorem of Mori and the definition of quasiconformality*, Trans. Amer. Math. Soc. **84** (1957), 78–84.
- [B10] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Wiley, New York (1958); Russian transl.: Izd. Inostr. Lit., Moscow (1961).
- [B11] L. Bers, *Spaces of Riemann surfaces*, Proc. ICM Edinburgh 1958, Princeton Univ. Press (1960), 349–361.
- [B12] L. Bers, *Quasiconformal mappings and Teichmüller Theorem*, Analytic Functions, Princeton Univ. Press, Princeton (1960), 89–119.
- [B13] L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97.
- [B14] L. Bers, *Spaces of Riemann surfaces as bounded domains*, Bull. Amer. Math. Soc. **66** (1960), 98–109; Corr.: Bull. Amer. Math. Soc. **67** (1961), 465–466; *Holomorphic differentials as functions of moduli*, Bull. Amer. Math. Soc. **67** (1961), 206–210.
- [B15] L. Bers, *Uniformization by Beltrami equations*, Commun. Pures Appl. Math. **14** (1961), 215–228.
- [B16] L. Bers, *The equivalence of two definitions of quasiconformal mappings*, Comment. Math. Helv. **37** (2) (1962), 148–154.
- [B17] L. Bers, *Automorphic forms and Poincaré series for infinitely generated Fuchsian groups*, Amer. J. Math. **87** (1965), 196–214.
- [B18] L. Bers, *Extremal quasiconformal mappings*, Advances in the Theory of Riemann Surfaces, Proc. Conf. Stony Brook, NY, 1969, Ann. of Math. Stud., Vol. 66, Princeton Univ. Press (1971), 27–52.
- [B19] L. Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.
- [B20] L. Bers, *Quasiconformal mappings with applications to differential equations, function theory and topology*, Bull. Amer. Math. Soc. **83** (1977), 1083–1100.
- [B21] L. Bers, *An application of quasiconformal mappings to topology*, Proc. Romanian–Finnish Seminar on Complex Analysis, Bucharest 1976, Lecture Notes in Math., Vol. 743, Springer-Verlag, Berlin (1979), 57–65.
- [BG1] L. Bers and A. Gelbart, *On a class of differential equations in mechanics of continua*, Quart. Appl. Math. **1** (1943), 168–188.
- [BG2] L. Bers and A. Gelbart, *On a class of functions defined by partial differential equations*, Trans. Amer. Math. Soc. **56** (1944), 67–93.
- [BN] L. Bers and L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications; On linear and nonlinear elliptic boundary value problems in the*

- plane*, Atti del Convegno Internaz. sulle Equazioni alle Derivate Parziali, Agosto, Trieste 1954, Ed. Cremonese, Roma (1955), 111–140; 141–165.
- [BeuA] A. Beurling and L. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.
- [Bob] N. Boboc, *On the invariance of the set of unfolding points under quasi-conformal transformations*, Com. Acad. R. P. R. **12** (1962), 161–165 (in Romanian).
- [Boh] H. Bohr, *Über streckentreue und konforme Abbildung*, Math. Z. **1** (1918), 403–420.
- [Bo1] B.V. Bojarski, *Homeomorphic solutions of Beltrami systems*, Dokl. Akad. Nauk SSSR **102** (1955), 661–664 (in Russian).
- [Bo2] B.V. Bojarski, *Generalized solutions of a system of the first order differential equations of elliptic type with discontinuous coefficients*, Mat. Sb. **43** (1957), 451–503 (in Russian).
- [BoI] B.V. Bojarski and T. Iwaniec, *Another approach to Liouville theorem*, Math. Nachr. **107** (1982), 253–262.
- [BoI2] B.V. Bojarski and T. Iwaniec, *Analytic foundations of the theory of quasi-conformal mappings in $\mathbb{R}^n$* , Ann. Acad. Sci. Fenn. A I Math. **8** (1983), 257–324.
- [Ca1] R. Caccioppoli, *Sur une généralisation des fonctions analytiques et les familles normales*, C. R. Acad. Sci. Paris **235** (1952), 116–118.
- [Ca2] R. Caccioppoli, *Sur une généralisation des fonctions analytiques*, C. R. Acad. Sci. Paris **235** (1952), 228–229.
- [Ca3] R. Caccioppoli, *Fondamenti per una teoria generale delle funzioni pseudo-analitiche di una variabile complessa, I, II*, Atti Accad. Naz. Lincei, Rendiconti **13** (1952), 197–204, 321–329.
- [Ca4] R. Caccioppoli, *Funzioni pseudo-analitiche e rappresentazioni pseudo-conformi delle superficie riemanniane*, Ricerche Mat. **2** (1953), 104–127.
- [Că1] Gh. Călugăreanu, *Sur les fonctions polygènes d'une variable complexe*, Thèse Fac. Sci., Gauthier-Villars, Paris (1928); C. R. Acad. Sci. Paris **186** (1928), 930–932.
- [Că2] Gh. Călugăreanu, *Sur une classe d'équations du second ordre intégrable à l'aide des fonctions polygènes*, C. R. Acad. Sci. Paris **186** (1928), 1406–1407.
- [Că3] Gh. Călugăreanu, *Les fonctions polygènes comme intégrales d'équations différentielles*, Trans. Amer. Math. Soc. **31** (1929), 372–378.
- [Can] J.W. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math. **173** (1994), 155–234.
- [C1] P. Caraman, *La différentiabilité des représentations quasi-conformes tridimensionnelles*, Atti Accad. Naz. Lincei, Rendiconti (8) **29** (1960), 181–185.
- [C2] P. Caraman, *Introduction in the theory of three-dimensional quasiconformal mappings; Theorem of surface Θ and theorem of the topological invariance of the domains in the three-dimensional Euclidean space; Contributions to the study of the three-dimensional quasiconformal mappings*, Acad. R.S.P. Române Fil. Iași, Stud. Cerc. St. Mat. **11** (1960), 63–92; 235–241; 243–269 (in Romanian).
- [C3] P. Caraman, *Contributions à la théorie des représentations quasi-conformes n -dimensionnelles*, Rev. Math. Pures Appl. **6** (1961), 311–356.
- [C4] P. Caraman, *Homeomorfisme Cvasiconforme n -Dimensionale*, Ed. Acad. R. S. R., București (1968); *n -Dimensional Quasiconformal Homeomorphisms*, Ed. Acad. R. S. R., București and Abacus Press, Tunbridge Wells, Kent (1974).
- [C5] P. Caraman, *p -capacity and p -modulus*, Ist. Naz. Alta Mat. Roma, Sympos. Math., Vol. 18, Academic Press, London–New York (1976), 455–484.
- [C6] P. Caraman, *Relations between capacities, h -measures Hausdorff and p -modules*, Rev. Anal. Numér. **7** (1) (1978), 13–49.
- [C7] P. Caraman, *About the equality between the p -module and the p -capacity in $\mathbb{R}^n$* , Proc. Analytic Functions, Blazejewko 1982, Lecture Notes in Math., Vol. 1039, Springer-Verlag, Berlin (1983), 32–83.
- [C8] P. Caraman, *A property of the p -module of the arc families*, C. R. Acad. Sci. Paris Sér. I **299** (18) (1984), 903–905.
- [C9] P. Caraman, *New cases of equality between p -module and p -capacity*, Ann. Polon. Math. **55** (1991), 35–56.
- [C10] P. Caraman, *p -module and p -capacity of a topological cylinder*, Rev. Roumaine Math. Pures Appl. **36** (7/8) (1991), 333–343.

- [C11] P. Caraman, *p-module and p-capacity in R^n* , Rev. Roumaine Math. Pures Appl. **36** (7/8) (1991), 345–353.
- [C12] P. Caraman, *Relations between p-capacity and p-modulus, I, II*, Rev. Roumaine Math. Pures Appl. **39** (6) (1994), 509–553, 555–575.
- [C13] P. Caraman, *About cases of equality between the p-module and the p-capacity*, Analysis and Topology, World Scientific, Singapore (1998), 215–240.
- [C14] P. Caraman, *Remarks on the equality between p-module and p-capacity*, Math. Reports **2** (52) (4) (2000), 441–451.
- [C15] P. Caraman, *New results about the equality between the p-capacity and the p-module in $\mathbb{R}^n$* , Bull. Soc. Sci. Lett. Łódź **52** (2002), 61–75.
- [Ci] N. Ciorănescu, *Sur la définition de la monogénéité et les fonctions monogènes aréolairement*, Mathematica (Cluj) **12** (1936), 26–69.
- [CC] C. Constantinescu und A. Cornea, *Ideale Ränder Riemannscher Flächen*, Ergeb. Math., Bd. 32, Springer-Verlag, Berlin (1963).
- [Cou] R. Courant, *Über eine Eigenschaft der Abbildungsfunktionen bei konformen Abbildungen*, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. (1914), 101–109; *Bemerkungen zu meiner Note “Über eine Eigenschaft der Abbildungsfunktionen bei konformen Abbildungen”*, Nachr. Akad. Wiss. Göttingen Math.-Phys.-Chem. Kl. (1922), 69–70.
- [Cr1] M. Cristea, *Some conditions for quasiconformality*, Ann. Acad. Sci. Fenn. A I Math. **14** (1989), 345–350.
- [Cr2] M. Cristea, *A generalization of a theorem of Zorich*, Bull. Math. Soc. Sci. Math. Roumaine **34** (3) (1990), 207–217.
- [Cr3] M. Cristea, *Some conditions for quasiregularity, I, II*, Rev. Roumaine Math. Pures Appl. **39** (6) (1994), 587–597, 599–569.
- [Cr4] M. Cristea, *Definitions of quasiconformality*, Ann. Acad. Sci. Fenn. A I Math. **25** (2000), 389–404.
- [Cr5] M. Cristea, *A generalization of the argument principle*, Complex Var. **42** (2000), 333–345.
- [Cr6] M. Cristea, *A note on the local injectivity of weakly differentiable mappings having constant Jacobian sign*, Math. Scand. **93** (2003), 91–98.
- [Cr7] M. Cristea, *Quasiregularity in metric spaces*, Preprint 10, Inst. Mat. Acad. Române (2004), 1–17.
- [Cr8] M. Cristea, *Mappings with finite distortion and arbitrary Jacobian sign*, Preprint 11, Inst. Mat. Acad. Române (2004), 1–14.
- [D] I.I. Daniiliuk, *On the mappings corresponding to solutions of elliptic type equations*, Dokl. Akad. Nauk SSSR **120** (1) (1958), 17–20 (in Russian).
- [Da] G. David, *Solutions d’équation de Beltrami avec $\|\mu\|_\infty = 1$* , Ann. Acad. Sci. Fenn. A I Math. **13** (1988), 25–70.
- [DE] A. Douady and C.J. Earle, *Conformally natural extensions of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.
- [Du] R.J. Duffin, *The extremal length of a network*, J. Math. Anal. Appl. **5** (1962), 200–215.
- [E] C.J. Earle, *Teichmüller theory*, Discrete Groups and Automorphic Functions, Academic Press, London (1977), 143–162.
- [EE] C.J. Earle and J. Eells, Jr., *On the differential geometry of Teichmüller spaces*, J. Anal. Math. **19** (1967), 35–52.
- [Er] A. Eremenko, *Value distribution and potential theory*, Proc. ICM Beijing 2002, Vol. 2, Higher Ed. Press, Beijing (2002), 681–690.
- [Fa] G. Faber, *Über den Hauptsatz aus der Theorie der konformen Abbildung*, Sitzgsber. Math.-Phys. Kl. Bayer, Akad. Wiss. München (1922), 91–100.
- [Fi1] R. Finn, *Sur quelques généralisations du théorème de Picard*, C. R. Acad. Sci. Paris **235** (1952), 596–598.
- [Fi2] R. Finn, *On a problem of type with applications to elliptic partial differential equations*, J. Ration. Mech. Anal. **3** (1954), 789–799.
- [FiS] R. Finn and J. Serrin, *On Hölder continuity of quasiconformal and elliptic mappings*, Trans. Amer. Math. Soc. **89** (1958), 1–15.
- [FK] R. Fricke und F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Teubner, Stuttgart (1926).

- [Fr] K.O. Friedrichs, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc. **55** (1944), 132–151.
- [F] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.
- [Gar] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
- [Ga] C.F. Gauss, *Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer andern gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird*, Astronomische Abhandlungen, Altona **3** (1825); Carl Friederich Gauss Werke, Göttingen **4** (1880), 189–216.
- [G1] F.W. Gehring, *The definitions and exceptional sets for quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I **281** (1960), 1–28.
- [G2] F.W. Gehring, *The Liouville theorem in space*, Notices Amer. Math. Soc. **7** (1960), 523–524.
- [G3] F.W. Gehring, *Ring domains in space*, Notices Amer. Math. Soc. **7** (1960), 636.
- [G4] F.W. Gehring, *Quasiconformal mappings in space*, Notices Amer. Math. Soc. **7** (1960), 636–637; Bull. Amer. Math. Soc. **69** (1963), 146–164.
- [G5] F.W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.
- [G6] F.W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.
- [G7] F.W. Gehring, *Extremal length definitions for the conformal capacity of rings in space*, Michigan Math. J. **9** (1962), 137–150.
- [G8] F.W. Gehring, *Extension theorems for quasiconformal mappings in n -space*, Proc. ICM Moscow 1966, Mir, Moskva (1968), 313–318; J. Anal. Math. **19** (1967), 119–169.
- [G9] F.W. Gehring, *Definitions for a class of plane quasiconformal mappings*, Nagoya Math. J. **29** (1967), 175–184.
- [G10] F.W. Gehring, *The L^p -integrability of the partial derivatives quasiconformal mapping*, Acta Math. **130** (1973), 265–277.
- [G11] F.W. Gehring, *Some metric properties of quasiconformal mappings*, Proc. ICM Vancouver 1974, Vol. 2, Canadian Math. Congress Montreal, PQ, Canada (1975), 203–206.
- [G12] F.W. Gehring, *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. **52** (1977), 561–572.
- [G13] F.W. Gehring, *Characteristic Properties of Quasidisks*, Presses Univ. Montréal, PQ, Canada (1982).
- [G14] F.W. Gehring, *Topics in quasiconformal mappings*, Proc. ICM Berkeley, California, 1986, Amer. Math. Soc., Providence, RI (1987), 62–80.
- [G15] F.W. Gehring, *Topics in quasiconformal mappings*, Quasiconformal space mappings. A collection of surveys 1960–1990, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 20–38.
- [GL] F.W. Gehring and O. Lehto, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. A I Math. **272** (1959), 1–9.
- [GR] F.W. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I **388** (1966), 1–15.
- [GV1] F.W. Gehring and J. Väisälä, *On the geometric definition for quasiconformal mappings*, Comment. Math. Helv. **36** (1961), 19–32.
- [GV2] F.W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in spaces*, Acta Math. **114** (1965), 1–70.
- [GV3] F.W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. **6** (2) (1973), 504–512.
- [GeRa] M. Gerstenhaber and H.E. Rauch, *On the extremal quasiconformal mappings, I, II*, Proc. Math. Acad. Sci. USA **40** (1954), 808–812, 991–994.
- [Gh1] M. Ghermănescu, *Sur l'intégrale des fonctions de deux variables complexes*, Bull. Scient. de l'Éc. Polytechn. de Timișoara **1** (1928), 245–248.
- [Gh2] M. Ghermănescu, *Sur une classe de fonctions monogènes*, Bull. Scient. de l'Éc. Polytechn. de Timișoara **2** (1929), 188–194.
- [Go] G.M. Golusin, *Geometrische Funktionentheorie*, Deutscher Verlag Wiss., Berlin (1957); Transl. of Russian edn: *The Geometric Theory of Functions of a Complex Variable*, Gos. Izd. Tehn.-Teor. Liter., Moscow–Leningrad (1952); English edn: Amer. Math. Soc., Providence, RI (1969).
- [Gro] W. Gross, *Über das Flächenmass von Punktmengen*, Monatsh. Math. **29** (1918), 145–176.

- [Gr1] H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung, I, II*, Ber. Verh. Sächs. Akad. Wiss. Math.-Nat. Kl. Leipzig **80** (1928), 367–376, 497–502.
- [Gr2] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Verh. Sächs. Akad. Wiss. Math.-Nat. Kl. Leipzig **80** (1928), 503–507.
- [Gr3] H. Grötzsch, *Über konforme Abbildung unendlich vielfach zusammenhängender schlichter Bereiche mit endlich vielen Häufungsrandkomponenten* (Dissertation, Leipzig), Ber. Verh. Sächs. Akad. Wiss. Math.-Nat. Kl. Leipzig **81** (1929), 51–86.
- [Gr4] H. Grötzsch, *Über die Verzerrung bei nichtkonformen Abbildungen mehrfach zusammenhängender schlichter Bereiche*, Ber. Verh. Sächs. Akad. Math.-Nat. Kl. Leipzig **82** (1930), 69–80.
- [Gr5] H. Grötzsch, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Verh. Sächs. Akad. Wiss. Math.-Nat. Kl. Leipzig **84** (1932), 114–120.
- [GuMa] V. Gutlyanski and O. Martio, *Conformality of a quasiconformal mapping at a point*, J. Anal. Math. **91** (2003), 179–192.
- [Hä1] G. af Hällström, *Eine quasikonforme Abbildung mit Anwendungen auf die Wertverteilungslehre*, Acta Acad. Abo. Math.-Phys. **18** (8) (1952), 1–16.
- [Hä2] G. af Hällström, *Wertverteilungssätze pseudo-meromorpher Funktionen*, Acta Acad. Abo. Math.-Phys. **21** (9) (1958), 1–23.
- [Ha] R.S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. **138** (1969), 399–406.
- [Hay] T. Hayashi, *On areolar holomorphic functions*, Rend. Circ. Mat. Palermo **34** (1912), 220–224.
- [He1] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York (2001).
- [He2] J. Heinonen, *The branch set of a quasiregular mapping*, Proc. ICM Beijing 2002, Vol. 2, Higher Ed. Press, Beijing (2002), 691–700.
- [HKM] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press–Oxford Univ. Press, New York (1993).
- [HK1] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), 61–79.
- [HK2] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.
- [H1] J. Hersch, *Longueurs extrémales et théorie des fonctions*, Comment. Math. Helv. **29** (1955), 301–337.
- [H2] J. Hersch, *Contributions à la théorie des fonctions pseudo-analytiques*, Comment. Math. Helv. **30** (1956), 1–19.
- [HPf] J. Hersch and A. Pfluger, *Généralisation du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques*, C. R. Acad. Sci. Paris **234** (1952), 43–45.
- [Hes] J. Hesse, *A p -extremal length and p -capacity equality*, Ark. Mat. **13** (1975), 131–144.
- [HC] A. Hurwitz und R. Courant, *Funktionentheorie*, Grundlehren Math. Wiss., Bd. 3, Springer-Verlag, Berlin (1922); 4th edn (1964).
- [IT] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo (1992).
- [I1] T. Iwaniec, *Some aspects of partial differential equations and quasiregular mappings*, Proc. ICM August 16–24, 1983, Warszawa, North-Holland, Amsterdam (1984), 1193–1208.
- [I2] T. Iwaniec, *L^p -theory of quasiregular mappings*, Quasiconformal space mappings. A collection of surveys 1960–1990, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 39–64.
- [I3] T. Iwaniec, *Current advances in quasiconformal geometry and nonlinear analysis*, XVI Rolf Nevanlinna Colloq. Joensuu 1995, de Gruyter, Berlin (1996), 59–80.
- [IM1] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford Univ. Press (2001).
- [IM2] T. Iwaniec and G. Martin, *The Beltrami equation*, Preprint, Univ. of Syracuse (2000); to appear in Mem. Amer. Math. Soc.
- [IŠ] T. Iwaniec and V. Šverák, *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. **118** (1) (1993), 181–188.
- [Je1] J.A. Jenkins, *Some results related to extremal length*, Contributions to the Theory of Riemann Surfaces, Ann. of Math. Stud., Vol. 30, Princeton Univ. Press (1953), 87–94.
- [Je2] J.A. Jenkins, *On quasi-conformal mappings*, J. Ration. Mech. Anal. **5** (1956), 343–352.

- [Je3] J.A. Jenkins, *A new criterion for quasiconformal mappings*, Ann. of Math. Ser. II **65** (1957), 208–214.
- [Je4] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Ergeb. Math., Neue Folge, Heft 18, Springer-Verlag, Berlin (1958).
- [J1] M. Jurchescu, *L'invariance K -quasi-conforme de la parabolicité d'un élément frontière*, C. R. Acad. Sci. Paris **246** (1958), 2997–2999.
- [J2] M. Jurchescu, *Modulus of a boundary component*, Pacific J. Math. **8** (4) (1958), 791–809.
- [Ju] Y. Juve, *Über gewisse Verzerrungseigenschaften konformer und quasikonformer Abbildungen*, Ann. Acad. Sci. Fenn. A I **174** (1954), 1–40.
- [K] S. Kahramaner, *Sur le comportement d'une représentation presque-conforme dans le voisinage d'un point singulier*, Rev. Fac. Sci. Univ. Istanbul. Sér. A. **22** (1957), 127–139.
- [Kak] S. Kakutani, *Applications of the theory of pseudo-regular functions to the type problem of Riemann surfaces*, Japan. J. Math. **13** (1937), 375–392.
- [Ka] E. Kasner, *A new theory of polygenic (non-monogenic) functions*, Science **46** (1927), 561–582.
- [Ke] J.A. Kelings, *Characterization of quasiconformal mappings in terms of harmonic and hyperbolic measure*, Ann. Acad. Sci. Fenn. A I Math. **368** (1965), 1–16.
- [Kob1] Z. Kobayashi, *A remark of the type of Riemann surfaces*, Sci. Rep. Tokyo Burnika Daigaku A **3** (62) (1937), 185–193.
- [Kob2] Z. Kobayashi, *On the Kakutani's theory of Riemann surfaces*, Sci. Rep. Tokyo Bunrika Daigaku A **4** (76) (1940), 9–44.
- [KS] K. Kodaira and D.C. Spencer, *Existence of complex structure on a differential family of deformations of compact complex manifolds*, Ann. of Math. (2) **70** (1959), 145–166.
- [Ko] P. Koebe, *Zur konformen Abbildung unendlich-vielfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Nachr. Ges. Wiss. Göttingen (1918), 60–71.
- [Kre] M. Kreines, *Sur une classe de fonctions de plusieurs variables*, Mat. Sb. **9** (51) (1941), 713–720.
- [Kr1] S.L. Krushkal, *Extremal quasiconformal mappings*, Sibirsk. Math. J. **10** (1969), 411–419 (in Russian).
- [Kr2] S.L. Krushkal, *Variational Methods in the Theory of Quasiconformal Mappings*, Novosibirsk State Univ. (1974) (in Russian).
- [Kr3] S.L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Winston, Washington, DC (1979); Russian edn: Nauka, Moscow (1975).
- [Kr4] S.L. Krushkal, *New methods and trends in geometric function theory*, Proc. 2nd Siberian Winter School Algebra and Analysis, Tomsk 1989, Amer. Math. Soc. Transl. Ser. 2, Vol. 151, Amer. Math. Soc., Providence, RI (1992), 81–89.
- [KrKü] S.L. Krushkal and R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*; I. Teil, S.L. Krushkal, *Quasikonforme Abbildungen, invariante Metriken und Teichmüller-sche Räume*, II. Teil, R. Kühnau, *Quasikonforme Abbildungen mit ortsabhängiger Dilatationsbeschränkung*, Teubner-Texte Math., Vol. 54, Teubner, Leipzig (1983); Russian edn: Novosibirsk (1984).
- [Kü1] R. Kühnau, *Elementare Beispiele von möglichst konformen Abbildungen in dreidimensionalen Raum*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **11** (1962), 729–732.
- [Kü2] R. Kühnau, *Über gewisse Extremalprobleme der quasikonformen Abbildung*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Nat. Reihe **13** (1964), 35–39.
- [Kü3] R. Kühnau, *Quasikonforme Abbildungen und Extremalprobleme bei Feldern in inhomogenen Medien, I, II*, J. Reine Angew. Math. **231** (1968), 101–113, **238** (1969), 61–66.
- [Kü4] R. Kühnau, *Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung*, Math. Nachr. **40** (1969), 1–11.
- [Kü5] R. Kühnau, *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen*, Math. Nachr. **48** (1971), 77–105.
- [Kü6] R. Kühnau, *Eine Klasse nichtschlichter konformer Abbildungen mit einer schlichten quasikonformen Fortsetzung*, Math. Nachr. **59** (1974), 261–263.
- [Kü7] R. Kühnau, *Geometrie der konformen Abbildung auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag Wiss., Berlin (1974).
- [Kü8] R. Kühnau, *Zur Methode der Randintegration bei quasikonformen Abbildungen*, Ann. Polon. Math. **31** (1976), 269–289.

- [Kü9] R. Kühnau, *Über Extremalprobleme bei im Mittel quasikonformen Abbildungen*, Complex Analysis – Proc. 5th Romanian–Finnish Seminar, Part 1, Bucharest 1981, Lecture Notes in Math., Vol. 1013, Springer-Verlag, Berlin (1983), 113–124.
- [Kü10] R. Kühnau, *Extremalcharakterisierung von Unterschallgasströmungen durch quasikonforme Abbildungen*, Banach Center Publ. **11** (1983), 199–210.
- [Kü11] R. Kühnau, *Wann sind die Grunskyschen Koeffizientenbedingungen hinreichend für Q -quasikonforme Fortsetzbarkeit?* Comment. Math. Helv. **61** (1986), 290–307.
- [Kü12] R. Kühnau, *Möglichst konforme Spiegelung an einem Jordanbogen auf der Zahlenebene*, Ann. Acad. Sci. Fenn. A I Math. **14** (1989), 357–367.
- [Kü13] R. Kühnau, *Einige neuere Entwicklungen bei quasikonformen Abbildungen*, Jahresber. Deutsch. Math.-Verein. **94** (1992), 141–169.
- [Kü14] R. Kühnau, *Herbert Grötzsch zum Gedächtnis*, Jahresber. Deutsch. Math.-Verein. **99** (1997), 122–145.
- [Kü15] R. Kühnau, *Der konforme Modul von Vierecken*, Analysis and Topology, World Scientific, Singapore (1998), 483–495.
- [Kü16] R. Kühnau, *Drei Funktionale eines Quasikreises*, Ann. Acad. Sci. Fenn. A I Math. **25** (2000), 413–415.
- [Kün1] H.P. Künzi, *Einführung in die Theorie der quasikonformen Abbildung*, Elem. Math. **11** (1956), 121–129.
- [Kün2] H.P. Künzi, *Quasikonforme Abbildungen*, Ann. Acad. Sci. Fenn. A I **249** (2) (1958), 1–24.
- [Kün3] H.P. Künzi, *Quasikonforme Abbildungen*, Ergeb. Math., Neue Folge, Heft 26, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
- [Ku] T. Kuusalo, *Verallgemeinerter Riemannscher Abbildungssatz und quasikonforme Mannigfaltigkeiten*, Ann. Acad. Sci. Fenn. A I **409** (1967), 1–24.
- [La1] M.A. Lavréntieff, *Sur une classe de représentations continues*, C. R. Acad. Sci. Paris **200** (1935), 1010–1013.
- [La2] M.A. Lavréntieff, *Sur une classe de représentations continues*, Mat. Sb. (N.S.) /Recueil Mathématique, Nouvelle Série/ Akad. Nauk SSSR **42** (1935), 407–424.
- [La3] M.A. Lavréntieff, *Sur une classe de transformations quasi-conformes et sur les sillages gazeux*, C. R. Acad. Sci. USSR (N.S.) **20** (1938), 343–346.
- [La4] M.A. Lavrent'ev, *Les représentations quasi-conformes et leurs systèmes dérivés*, C. R. Acad. Sci. USSR (N.S.) **52** (1946), 287–290 (in Russian).
- [La5] M.A. Lavrent'ev, *A general problem of the theory of quasiconformal mappings of plane regions*, Mat. Sb. (N.S.) **21** (63) (1947), 285–320 (in Russian), Amer. Math. Soc. Transl. **46** (1951).
- [La6] M.A. Lavrent'ev, *A fundamental theorem of the theory of quasi-conformal mappings of plane regions*, Izv. Akad. Nauk SSSR **12** (6) (1948), 513–554; Amer. Math. Soc. Transl. **29** (1950) (in Russian).
- [La7] M.A. Lavrent'ev, *Variational Method for Boundary Value Problems for Equation Systems of Elliptic Type*, Akad. Nauk SSSR, Moscow (1962) (in Russian).
- [La8] M.A. Lavrent'ev, *Sur les représentations quasi-conformes*, Proc. ICM Stockholm, Djursholm 15–22 August 1962, ICM Abstracts of Short Communications, Stockholm 1962, Almqvist and Wiksells, Uppsala (1962), p. 90.
- [LaSh] M.A. Lavrent'ev and B.V. Shabat, *Geometrical properties of the solutions of non linear partial differential equation systems*, Dokl. Acad. Nauk SSSR **112** (1957), 810–811 (in Russian).
- [LaShe] M.A. Lavréntieff and V.M. Shepelev, *Sur la représentation conforme*, C. R. Acad. Sci. Paris **191** (1930), 1426–1427.
- [Law1] J. Lawrynowicz, *Parametrization and boundary correspondence for Teichmüller mappings in an annulus*, Proc. Romanian–Finnish Seminar on Complex Analysis, Bucharest 1976, Lecture Notes in Math., Vol. 743, Springer-Verlag, Berlin (1979), 165–183.
- [Law2] J. Lawrynowicz in cooperation with J. Krzyż, *Quasiconformal Mappings in the Plane: Parametrical Methods*, Lecture Notes in Math., Vol. 978, Springer-Verlag, Berlin (1983).
- [Le] C.Y. Lee, *Similarity principle with boundary conditions for pseudo-analytic functions*, Duke Math. J. **23** (1956), 157–163.
- [L1] O. Lehto, *On the differentiability of quasiconformal mappings with prescribed complex dilatation*, Ann. Acad. Sci. Fenn. A I **275** (1960), 1–28.

- [L2] O. Lehto, *Remarks on the integrability of the derivatives of quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I **371** (1965), 1–8.
- [L3] O. Lehto, *Homeomorphic solutions of a Beltrami differential equation*, Festband zum 70 Geburtstag von Rolf Nevanlinna, Springer-Verlag, Berlin (1966), 58–65.
- [L4] O. Lehto, *Quasiconformal mapping in the plane*, Proc. ICM Moscow 1966, Mir, Moskva (1968), 319–322.
- [L5] O. Lehto, *Schlicht functions with a quasiconformal extension*, Ann. Acad. Sci. Fenn. A I Math. **500** (1971), 1–10.
- [L6] O. Lehto, *Remarks on generalized Beltrami equations and conformal mappings*, Proc. Romanian–Finish Seminar on Teichmüller spaces and Quasiconformal Mappings, Braşov 1969, Ed. Acad. R. S. R. (1971), 203–214.
- [L7] O. Lehto, *Quasikonformale mappings and singular integrals*, Ist. Naz. Alta Mat., Symposia Math., Vol. 18, Academic Press, London–New York (1976), 429–453.
- [L8] O. Lehto, *On univalent functions with quasiconformal extensions over the boundary*, J. Anal. Math. **30** (1976), 349–354.
- [L9] O. Lehto, *Quasiconformal homeomorphisms and Beltrami equations*, Discrete Groups and Automorphic Functions, Academic Press, London (1977), 121–142.
- [L10] O. Lehto, *A historical survey of quasiconformal mappings*, Zum Werk Leonhard Eulers, Birkhäuser, Basel (1984), 205–217.
- [L11] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987).
- [LV1] O. Lehto and K.I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.
- [LV2] O. Lehto and K.I. Virtanen, *On the existence of quasiconformal mappings with prescribed complex dilatation*, Ann. Acad. Sci. Fenn. A I **274** (1960), 1–24.
- [LV3] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Grundlehren Math. Wiss., Vol. 126, Springer-Verlag, Berlin–Heidelberg–New York (1965), 2nd edn: *Quasiconformal Mappings in the Plane* (1973).
- [LVVä] O. Lehto, K.I. Virtanen and J. Väisälä, *Contributions to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I **273** (1959), 1–14.
- [L-F1] J. Lelong-Ferrand, *Représentation conforme et transformations à intégrale de Dirichlet bornée*, Gauthier-Villars, Paris (1955).
- [L-F2] J. Lelong-Ferrand, *Invariants conformes globaux sur les variétés riemanniennes*, J. Differential Geom. **8** (1973), 487–510.
- [L-F3] J. Lelong-Ferrand, *Étude d'une classe d'applications liées à des homomorphismes d'algèbres de fonctions, et généralisant les quasi-conformes*, Duke Math. J. **40** (1973), 163–186.
- [L-F4] J. Lelong-Ferrand, *Problèmes de géométrie conforme*, Proc. ICM Vancouver, BC 1974, Vol. 2, Canadian Math. Congress, Montreal, PQ (1975), 13–19.
- [L-F5] J. Lelong-Ferrand, *Regularity of conformal mappings of Riemannian manifolds*, Proc. Romanian–Finnish Seminar on Complex Analysis, Bucharest 1976, Springer-Verlag, Berlin (1979), 197–203.
- [Loe] C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414.
- [Lo] H. Looman, *Über eine Erweiterung des Cauchy–Goursatschen Integralsatzes*, Nieuw Archiv voor Wiskunde **14** (1924), 234–239.
- [Low] A.J. Lowther, *The boundary behaviour of a quasiconformal mapping*, J. Ration. Mech. Anal. **5** (1956), 335–342.
- [MM1] V. Marković and M. Mateljević, *New versions of Grötzsch principle and Reich–Strebel inequality*, Mat. Vesnik **49** (1997), 235–239.
- [MM2] V. Marković and M. Mateljević, *New versions of the main inequality and uniqueness of harmonic maps*, J. Anal. Math. **79** (1999), 315–334.
- [Mar] A.I. Markuševič, *On some classes of continuous mappings*, Dokl. Akad. Nauk SSSR **28** (1940), 301–304 (in Russian).
- [Ma1] O. Martio, *Boundary values and injectiveness of the solutions of Beltrami equations*, Ann. Acad. Sci. Fenn. A I **402** (1967), 1–27.

- [Ma2] O. Martio, *Partial differential equations and quasiregular mappings*, Quasiconformal Space Mappings. A Collection of Surveys 1960–1990, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 65–79.
- [MRV1] O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. A I Math. **448** (1969), 1–40.
- [MRV2] O. Martio, S. Rickman and J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. A I Math. **465** (1970), 1–13.
- [MRV3] O. Martio, S. Rickman and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. A I Math. **488** (1971), 1–31.
- [MSr] O. Martio and U. Srebro, *On the local behaviour of quasiregular maps and branched covering maps*, J. Anal. Math. **36** (1979), 198–212.
- [MŞ] M. Mitrea and F. Şabac, *Pompeiu's integral representation formula. History and Mathematics*, Rev. Roumaine Math. Pures Appl. **43** (1/2) (1998), 211–226.
- [Moc1] M. Mocanu, *Some sufficient conditions for the convergence of the derivatives of weakly quasiregular mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 151–165.
- [Moc2] M. Mocanu, *Harmonic morphisms with integrable dilatation*, Math. Reports **2** (52) (4) (2000), 533–545.
- [M1] Gr.C. Moisil, *Sur l'équation $\Delta u = 0$* , C. R. Acad. Sci. Paris **191** (1930), 984–986.
- [M2] Gr.C. Moisil, *Sur le système d'équations de M. Dirac du type elliptique*, C. R. Acad. Sci. Paris **191** (1930), 1292–1293.
- [M3] Gr.C. Moisil, *Les fonctions monogènes dans les espaces à plusieurs dimensions*, C. R. LXIV Congrès des Soc. Savantes Clermont-Ferrant (1931), 129.
- [MTh] Gr.C. Moisil et N. Théodoresco, *Fonctions holomorphes dans l'espace*, Mathematica (Cluj) **5** (1931), 142–152.
- [Mo1] A. Mori, *On an absolute constant in the theory of quasiconformal mappings*, J. Math. Soc. Japan **8** (1956), 156–166.
- [Mo2] A. Mori, *On quasi-conformality and pseudo-analyticity*, Trans. Amer. Math. Soc. **84** (1957), 56–77.
- [Mor] C.B. Morrey, *On the solution of quasilinear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
- [Nä] M. Näätänen, *Maps with continuous characteristic as a subclass of quasiconformal maps*, Ann. Acad. Sci. Fenn. A I **410** (1967), 1–28.
- [N] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).
- [Ne1] R. Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten*, Acta Math. **58** (1932), 295–373.
- [Ne2] R. Nevanlinna, *Eindeutige analytische Funktionen*, Grundlehren Math. Wiss., Bd. 46, Springer-Verlag, Berlin (1936); Zweite Auflage (1953); English edn: *Analytic Functions*, Grundlehren Math. Wiss., Bd. 162, Springer-Verlag, Berlin (1970).
- [Ne3] R. Nevanlinna, *Ein Satz über offene Riemannsche Flächen*, Ann. Acad. Sci. Fenn. A I **54** (3) (1940), 1–18.
- [Ne4] R. Nevanlinna, *Uniformisierung*, Grundlehren Math. Wiss., Bd. 64, Springer-Verlag, Berlin–Göttingen–Heidelberg (1953); Zweite Auflage (1967).
- [Ne5] R. Nevanlinna, *A remark on differentiable mappings*, Michigan Math. J. **3** (1955), 53–57.
- [Ne6] R. Nevanlinna, *Über fastkonforme Abbildungen*, Ann. Acad. Sci. Fenn. A I **251** (7) (1958), 1–10.
- [Ni1] M. Nicolescu, *Fonctions complexes dans le plan et dans l'espace*, Thèse Fac. Sci., Gauthier-Villars, Paris (1928).
- [Ni2] M. Nicolescu, *Sur les fonctions conjuguées sur une surface au sens de Beltrami*, Bull. Cl. Sci. Acad. Royale Belgique 5^e série **16** (1930), 1012–1016.
- [Nir1] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Commun. Pure Appl. Math. **6** (1953), 103–156, 395.
- [Nir2] L. Nirenberg, *On a generalization of quasiconformal mappings and its applications to elliptic partial differential equations*, Contributions to the Theory of Partial Differential Equations, Ann. of Math. Stud., Vol. 33, Princeton Univ. Press, Princeton (1954), 95–100.
- [No1] K. Noshiro, *A theorem on the cluster sets of pseudo-analytic functions*, Nagoya Math. J. **1** (1950), 83–89.

- [No2] K. Noshiro, *Cluster Sets*, Ergeb. Math., Neue Folge, Bd. 28, Springer-Verlag, Berlin (1960).
- [O1] M. Ohtsuka, *Sur un théorème étoilé de Gross*, Nagoya Math. J. **9** (1955), 191–207.
- [O2] M. Ohtsuka, *Dirichlet Problem, Extremal Length and Prime Ends*, Van Nostrand, New York, (1970).
- [On1] O. Onicescu, *Propriétés topologiques de la transformation définie par une fonction uniforme de la variable complexe z* , C. R. Acad. Sci. Paris **186** (1928), 563–565.
- [On2] O. Onicescu, *Sur les fonctions holotropes*, C. R. Acad. Sci. Paris **201** (1935), 122–123; Rev. Math. Union Interbalk. **1** (1936), 33–52.
- [On3] O. Onicescu, *Les fonctions holotropes*, Bull. Math. Soc. Sci. Math. Phys. **7** (55) (1963), 203–223.
- [OOn] S. Ozaki and I. Ono, *Second principal theorem of pseudo-meromorphic functions*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1952), 214–221.
- [OOnOz1] S. Ozaki, I. Ono and M. Ozawa, *A theorem of pseudo-meromorphic function*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1951), 203–205.
- [OOnOz2] S. Ozaki, I. Ono and M. Ozawa, *Second principal theorem of the pseudo-meromorphic mapping in the three dimensional spaces*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1951), 207–210.
- [OOnOz3] S. Ozaki, I. Ono and M. Ozawa, *On the pseudo-meromorphic mappings on Riemann surfaces*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1952), 211–213.
- [Pe1] I.N. Pesin, *On the theory of generalized Q -quasiconformal mappings*, Dokl. Akad. Nauk SSSR **102** (1955), 223–224 (in Russian).
- [Pe2] I.N. Pesin, *Metric properties of Q -quasiconformal mappings*, Mat. Sb. **40** (82) (1956), 281–294 (in Russian).
- [Pe3] I.N. Pesin, *Sets of removable singularities of analytic functions and quasiconformal mappings*, Researches on Contemporary Problems of Function Theory of a Complex Variable, Gos. Izd. Fiz.-Mat. Literaturny, Moscow (1960), 419–424 (in Russian).
- [Pf1] A. Pfluger, *Une propriété métrique de la représentation quasi-conforme*, C. R. Acad. Sci. Paris **226** (1948), 623–625.
- [Pf2] A. Pfluger, *Sur une propriété de l'application quasi-conforme d'une surface de Riemann ouverte*, C. R. Acad. Sci. Paris **227** (1948), 25–26.
- [Pf3] A. Pfluger, *Quelques théorèmes sur une classe de fonctions pseudo-analytiques*, C. R. Acad. Sci. Paris **231** (1950), 1022–1023.
- [Pf4] A. Pfluger, *Quasikonforme Abbildungen und logarithmische Kapazität*, Ann. Inst. Fourier Grenoble **2** (1951), 69–80.
- [Pf5] A. Pfluger, *Extremallängen und Kapazität*, Comment. Math. Helv. **29** (1955), 120–131.
- [Pf6] A. Pfluger, *Theorie der Riemannschen Flächen*, Grundlehren Math. Wiss., Bd. 89, Springer-Verlag, Berlin–Göttingen–Heidelberg (1957).
- [Pf7] A. Pfluger, *Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen*, Comment. Math. Helv. **33** (1959), 23–33.
- [Pi] U. Pirl, *Isotherme Kurvenscharen und zugehörige Extremalprobleme der konformen Abbildungen*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur Reihe **4** (1955), 1225–1252.
- [Pol] E.A. Poleckij, *Method modules for non homeomorphic quasiconformal mappings*, Mat. Sb. **83** (1970), 261–272.
- [P1] G.N. Položij, *On p -analytic functions of a complex variable*, Dokl. Acad. Nauk SSSR **58** (1947), 1275–1278 (in Russian).
- [P2] G.N. Položij, *On the application of generalized derivatives in the class of quasiconformal mappings*, Dokl. Akad. Nauk SSSR **63** (1948), 615–618 (in Russian).
- [P3] G.N. Položij, *A generalization of Cauchy integral formula*, Mat. Sb. (N.S.) **24** (1949), 375–384 (in Russian).
- [P4] G.N. Položij, *Theorem on domain invariance for certain elliptic systems of differential equations and its applications*, Mat. Sb. **32** (74) (1953), 485–492 (in Russian).
- [P5] G.N. Položij, *A theorem on boundary correspondence and variational theorems for certain elliptic systems of differential equations*, Dokl. Akad. Nauk SSSR **95** (1954), 927–930 (in Russian).
- [P6] G.N. Položij, *On the problem of (p, q) -analytic functions of a complex variable and their applications*, Rev. Roumaine Math. Pures Appl. **2** (1957), 331–363 (in Russian).

- [P7] G.N. Polozij, *Generalizations of the Theory of Analytic Functions of a Complex Variable. p -Analytic and (p, q) -Analytic Functions and Some of Their Applications*, Izd. Kievskogo Univ. (1965); 2nd edn (1973) (in Russian).
- [Pom1] D. Pompeiu, *Sur une classe de fonctions d'une variable complexe*, Rend. Circ. Mat. Palermo **33** (1912), 108–113.
- [Pom2] D. Pompeiu, *Sur une classe de fonctions d'une variable complexe et sur certaines équations intégrales*, Rend. Circ. Mat. Palermo **35** (1913), 277–281.
- [Pom3] D. Pompeiu, *Sur une classe d'intégrales doubles, I, II, III, IV*, Bull. Sect. Scient. Acad. Roumaine **1** (1912), 128–131, 265–271, 289–296; **3** (1914–1915), 272–277.
- [Pom4] D. Pompeiu, *Sur une définition des fonctions holomorphes*, C. R. Acad. Sci. Paris **166** (1918), 209–212.
- [Pomm] Chr. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen (1975).
- [Ra1] H.E. Rauch, *On the moduli of algebraic Riemann surfaces*, Proc. Nat. Acad. Sci. **41** (1955), 42–49; 236–238.
- [Ra2] H.E. Rauch, *Weierstrass points, branch points, and moduli of Riemann surfaces*, Commun. Pure Appl. Math. **12** (1959), 543–560.
- [Ra3] H.E. Rauch, *A transcendental view of the space of algebraic Riemann surfaces*, Bull. Amer. Math. Soc. **71** (1965), 1–39; errata **74** (1968), 767.
- [Re1] E. Reich, *On a characterization of quasiconformal mappings*, Comment. Math. Helv. **37** (1962), 44–48.
- [Re2] E. Reich, *On criteria of unique extremality for Teichmüller mappings*, Ann. Acad. Sci. Fenn. A I Math. **6** (1981), 289–301.
- [ReStr] E. Reich and K. Strebel, *On quasiconformal mappings which keep the boundary pointwise fixed*, Trans. Amer. Math. Soc. **138** (1969), 211–222.
- [ReW] E. Reich and H. Walczak, *On the behaviour of quasiconformal mappings at a point*, Trans. Amer. Math. Soc. **117** (1965), 338–351.
- [Rei] H.M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. **49** (1974), 260–276.
- [ReiRy] H.M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation*, Lecture Notes in Math., Vol. 487, Springer-Verlag, Berlin (1975).
- [Ren1] H. Renelt, *Über Extremalprobleme für schlichte Lösungen elliptischer Differentialgleichungssysteme*, Comment. Math. Helv. **54** (1979), 17–41.
- [Ren2] H. Renelt, *Quasikonforme Abbildungen und elliptische Systeme erster Ordnung in der Ebene*, Teubner-Texte Math., Bd. 46, Teubner, Leipzig (1982); English transl.: Chichester (1988).
- [RI] E. Rengel, *Über einige Schlitztheoreme der konformen Abbildung*, Schriftenreihe Math. Inst. Univ. Berlin **1** (4) (1933), 141–162.
- [Rgli1] H. Renggli, *Un théorème de représentation conforme*, C. R. Acad. Sci. Paris **235** (1952), 1593–1595.
- [Rgli2] H. Renggli, *Zur Definition der quasikonformen Abbildungen*, Comment. Math. Helv. **34** (3) (1960), 222–226.
- [Regli3] H. Renggli, *Quasiconformal mappings and extremal lengths*, Amer. J. Math. **86** (1964), 63–69.
- [R1] Yu.G. Reshetnyak, *Some geometric properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Zh. **7** (1966), 886–919 (in Russian).
- [R2] Yu.G. Reshetnyak, *Estimates of the modulus of continuity for certain mappings*, Sibirsk. Mat. Zh. **7** (1966), 1106–1114 (in Russian).
- [R3] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, Sibirsk. Mat. Zh. **8** (3) (1967), 629–658 (in Russian).
- [R4] Yu.G. Reshetnyak, *Liouville's conformal mapping theorem under minimal regularity assumptions*, Sibirsk. Mat. Zh. **8** (1967), 629–658 (in Russian).
- [R5] Yu.G. Reshetnyak, *General theorems on semicontinuity and convergence with functionals*, Sibirsk. Mat. Zh. **8** (1967), 1051–1069 (in Russian).
- [R6] Yu.G. Reshetnyak, *Generalized derivatives and differentiability almost everywhere*, Mat. Sb. **75** (117) (1968), 323–334 (in Russian).
- [R7] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Transl. Math. Monogr., Vol. 73, Amer. Math. Soc., Providence, RI (1989); Russian edn: Nauka, Moskva (1982).

- [Ri1] S. Rickman, *Characterization of quasiconformal arcs*, Ann. Acad. Sci. Fenn. A I **395** (1967), 1–30.
- [Ri2] S. Rickman, *Quasiconformal mappings*, Ann. Acad. Sci. Fen. A I Math. **13** (1988), 371–385.
- [Ri3] S. Rickman, *Picard's theorem and defect relation for quasiregular mappings*, Quasiconformal Space Mappings. A Collection of Surveys 1960–1990, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 93–103.
- [Ri4] S. Rickman, *Quasiregular Mappings*, Ergeb. Math., 3. Folge, Bd. 26, Springer-Verlag, Berlin–Heidelberg (1993).
- [Rie] B. Riemann, *Theorie der Abelschen Funktionen*, Gesammelte mathematische Werke, 2nd edn, Teubner, Leipzig (1892).
- [Ro] B. Rodin, *The method of extremal length*, Bull. Amer. Math. Soc. **80** (4) (1974), 587–606.
- [RoSa] B. Rodin and L. Sario (in collaboration with M. Nakai), *Principal Functions*, Van Nostrand, Princeton, NJ (1968).
- [RSY] V. Ryzanov, U. Srebro and E. Yakubov, *BMO-quasiconformal mappings*, J. Anal. Math. **83** (2001), 1–30.
- [Ša1] B.V. Šabat, *On generalized solutions of a system of partial elliptic differential equations*, Mat. Sb. **17** (59) (1945), 193–210 (in Russian).
- [Ša2] B.V. Šabat, *Cauchy's theorem and formula for linear classes of quasiconformal mappings*, Dokl. Akad. Nauk SSSR (N.S.) **69** (1949), 305–308 (in Russian).
- [Ša3] B.V. Šabat, *Generalizations and analogies of analytic function theory*, Matematika v SSSR za Sorok Let 1917–1957 I, Gos. Izd. Fiz.-Mat. Lit., Moskva (1959), 481–493 (in Russian).
- [Ša4] B.V. Šabat, *The modulus method in space*, Dokl. Akad. Nauk SSSR **130** (1960), 1210–1213 (in Russian); English transl.: Soviet Math. Dokl. **1** (1960), 165–168.
- [Sak] E. Sakai, *Note on pseudo-analytic functions*, Proc. Japan Acad. **25** (1949), 12–17.
- [Š] Z. Šapiro, *Sur l'existence des représentations quasi-conformes*, Dokl. Akad. Nauk SSSR (N.S.) **30** (1941), 685–687.
- [Sa1] L. Sario, *Über Riemannsche Flächen mit hebbarem Rand*, Ann. Acad. Sci. Fenn. A I **50** (1948), 1–79.
- [Sa2] L. Sario, *Sur le problème du type des surfaces de Riemann*, C. R. Acad. Sci. Paris **229** (1949), 1109–1111.
- [Sa3] L. Sario, *Capacity of the boundary and of a boundary component*, Ann. Math. (2) **59** (1) (1954), 135–144.
- [SaNa] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Grundlehren math. Wiss., Vol. 164, Springer-Verlag, Berlin (1970).
- [SaOi] L. Sario and K. Oikawa, *Capacity functions*, Grundlehren math. Wiss., Vol. 149, Springer-Verlag, New York (1969).
- [Sc] M. Schiffer, *On the modulus of doubly connected domains*, Quart. J. Math. Oxford **17** (1946), 197–213.
- [Sch] G. Schober, *Univalent Functions – Selected Topics*, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin (1975).
- [Shi1] K. Shibata, *Remarks on the sequence of quasiconformal mappings*, Proc. Japan. Acad. **32** (1956), 665–670.
- [Shi2] K. Shibata, *On boundary values of some pseudo-analytic functions*, Proc. Japan. Acad. **33** (1957), 628–632.
- [So] S.L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Amer. Math. Soc., Providence, RI (1963); Russian edn: Leningrad Univ., Leningrad (1950).
- [Sr1] U. Srebro, *Topological properties of quasiregular mappings*, Quasiconformal Space Mappings. A Collection of Surveys 1960–1990, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 104–118.
- [Sr2] U. Srebro, *Extremal lengths and quasiconformal maps*, Israel Math. Conf. Proc. **14** (2000), 135–158.
- [Sta1] V. Stanciu, *Quasiconformal BMO homeomorphisms between Riemann surfaces*, Progress in Analysis, Vol. I, Proc. 3rd Internat. ISAAC Congr., Berlin 2001, World Scientific, New Jersey–London–Singapore–Hong Kong (2003), 177–181.
- [Sta2] V. Stanciu, *Compact families of BMO-quasiconformal mappings, I, II*, Ann. Univ. Ferrara Sec. VII Sci. Mat. **40** (2003), 11–18, 37–42.

- [St1] S. Stoilow, *Les transformations continues et le théorème de M. Picard sur les fonctions entières*, C. R. Acad. Sci. Paris **185** (1927), 173–175.
- [St2] S. Stoilow, *Sur une classe de transformations continues à variation bornée*, C. R. Acad. Sci. Paris **186** (1928), 621–623.
- [St3] S. Stoilow, *Sur les transformations continues et la topologie des fonctions analytiques*, Ann. Sci. École Norm. Sup. Paris (3) **45** (1928), 347–382.
- [St4] S. Stoilow, *Remarques sur la définition des fonctions presque-analytiques de M. Lavrentieff*, C. R. Acad. Sci. Paris **200** (1935), 1520–1521.
- [St5] S. Stoilow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Collection Borel, Gauthier-Villars, Paris (1938), 2nd edn (1956); Russian edn: Nauka, Moscow (1964).
- [St6] S. Stoilow, *Sur la notion de pseudo-analyticité*, Lucrările celui de al IV-lea Congres al Matematicienilor Români 1956, Ed. Acad. R. P. R., București (1960), 69–70.
- [St7] S. Stoilow, *Représentation quasi-conforme et théorie géométrique des fonctions*, Lecture held in Paris, May 1960, [St8], 403–415.
- [St8] S. Stoilow, *Oeuvre Mathématique* (with a preface of Miron Nicolescu and an introductory study of Cabiria Andreian Cazacu), Ed. Acad. R. P. R., București (1964).
- [Str1] K. Strebel, *On the maximum dilatation of quasiconformal mappings*, Bull. Amer. Math. Soc. **61** (1955), 225; Proc. Amer. Math. Soc. **6** (1955), 903–909.
- [Str2] K. Strebel, *Eine Abschätzung der Länge gewisser Kurven bei quasikonformer Abbildung*, Ann. Acad. Sci. Fenn. A I **243** (1957), 1–10.
- [Str3] K. Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises, I, II*, Comment. Math. Helv. **36** (4) (1962), 306–323; **39** (1) (1964), 77–89.
- [Str4] K. Strebel, *On quadratic differentials and extremal quasiconformal mappings*, Proc. ICM Vancouver 1974, Vol. 2, Canadian Math. Congress Montreal, Que (1975), 223–227.
- [Str5] K. Strebel, *On the existence of extremal Teichmüller mappings*, J. Anal. Math. **30** (1976), 464–480.
- [Str6] K. Strebel, *Is there always a unique extremal Teichmüller mapping?* Proc. Amer. Math. Soc. **90** (2) (1984), 240–242.
- [Str7] K. Strebel, *Quadratic Differentials*, Ergeb. Math., 3. Folge, Bd. 5, Springer-Verlag, Berlin (1984).
- [Str8] K. Strebel, *Extremal quasiconformal mappings*, Results Math. **10** (1–2) (1986), 168–210.
- [Su1] G.D. Suvorov, *Families of Topological Mappings in the Plane*, Red.-Izd. Otdel Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk (1965) (in Russian).
- [Su2] G.D. Suvorov, *Generalized Length–Area Principle in the Theory of Mappings*, Naukova Dumka, Kiev (1985) (in Russian).
- [Sy] A.V. Sychev, *Moduli and Space Quasiconformal Mappings*, Izd. Nauka Sibirsk. Otdel., Novosibirsk (1983) (in Russian).
- [Sz] K. Szilárd, *Über die Grundlagen der Funktionentheorie*, Math. Z. **26** (5) (1927), 653–671.
- [Ta] O. Taari, *Charakterisierung der Quasikonformität mit Hilfe der Winkelverzerrung*, Ann. Acad. Sci. Fenn. A I **390** (1966), 1–42.
- [T1] O. Teichmüller, *Operatoren in Wachsschen, Raum*, J. Reine Angew. Math. **174** (1935), 73–124; *Gesammelte Abhandlungen*. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), pp. 1–52.
- [T2] O. Teichmüller, *Eine Anwendung quasikonformer Abbildungen auf das Typenproblem*, Deutsche Math. **2** (1937), 321–327; *Gesammelte Abhandlungen*. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 171–177.
- [T3] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. **3** (1938), 621–678; *Gesammelte Abhandlungen*. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 205–262.
- [T4] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Math.-Natur. Kl. **22** (1939), 1–197; *Gesammelte Abhandlungen*. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 335–531.
- [T5] O. Teichmüller, *Über Extremalprobleme der konformen Geometrie*, Deutsche Math. **6** (1941), 50–77; *Gesammelte Abhandlungen*. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 554–581.

- [T6] O. Teichmüller, *Vollständige Lösung einer Extremalaufgabe der quasikonformen Abbildung*, Abh. Preuss. Akad. Wiss. Math.-Natur. Kl. **5** (1941), 1–18; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 583–598.
- [T7] O. Teichmüller, *Skizze einer Begründung der algebraischen Funktionentheorie durch Uniformisierung*, Deutsche Math. **6** (1942), 257–265; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 599–607.
- [T8] O. Teichmüller, *Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen*, Abh. Preuss. Akad. Wiss. Math.-Natur. Kl. **4** (1943), 1–42; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 635–676.
- [T9] O. Teichmüller, *Beweis der analytischen Abhängigkeit des konformen Moduls einer Ringflächenschar von den Parametern*, Deutsche Math. **7** (1944), 309–336; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 677–704.
- [T10] O. Teichmüller, *Ein Verschiebungssatz der konformen Abbildung*, Deutsche Math. **7** (1944), 336–343; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 704–711.
- [T11] O. Teichmüller, *Veränderliche Riemannsche Flächen*, Deutsche Math. **7** (1944), 344–359; Gesammelte Abhandlungen. Collected Papers, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982), 712–727.
- [T12] O. Teichmüller, *Gesammelte Abhandlungen. Collected Papers*, L.V. Ahlfors and F.W. Gehring, eds, Springer-Verlag, Berlin (1982).
- [Th1] N. Théodoresco, *Quelques pas dans une théorie des fonctions de variable complexe au sens général, I, II*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (6) **11** (3, 4) (1930), 279–282, 382–384.
- [Th2] N. Théodoresco, *La dérivée aréolaire et ses applications physiques*, Thèse Fac. Sci. Paris, Gauthier-Villars, Paris (1931).
- [Th3] N. Théodoresco, *Le problème de Cauchy pour une classe de systèmes d'équations aux dérivées partielles. Application aux équations de Dirac*, Ann. Sc. Norm. Pisa Ser. II **4** (1935), 51–70.
- [Th4] N. Théodoresco, *La dérivée aréolaire*, Ann. Roumaines de Math. **3** (1936), 1–62.
- [Thi] L.V. Thiem, *Beitrag zum Typenproblem der Riemannschen Flächen*, Comment. Math. Helv. **20** (1947), 270–287.
- [Ti] M. Tienari, *Fortsetzung einer quasikonformen Abbildung über einen Jordanbogen*, Ann. Acad. Sci. Fenn. A I **321** (1962), 1–32.
- [TSh] Y. Tôki and K. Shibata, *On the pseudo-analytic functions*, Osaka Math. J. **6** (1954), 145–165.
- [Tu] P. Tukia, *A survey of Möbius groups*, Proc. ICM Zürich 1994, Vol. 2, Birkhäuser, Basel (1995), 907–917.
- [TuVä] P. Tukia and J. Väisälä, *Quasisymmetric embeddings in metric spaces*, Ann. Acad. Sci. Fenn. A I Math. **5** (1980), 97–114.
- [Vä1] J. Väisälä, *On normal quasiconformal functions*, Ann. Acad. Sci. Fenn. A I **266** (1959), 1–33.
- [Vä2] J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. A I **298** (1961), 1–36.
- [Vä3] J. Väisälä, *Remarks on a paper of Tienari concerning quasiconformal continuation*, Ann. Acad. Sci. Fenn. A I **324** (1962), 1–6.
- [Vä4] J. Väisälä, *Discrete open mappings on manifolds*, Ann. Acad. Sci. Fenn. A I Math. **392** (1966), 1–10.
- [Vä5] J. Väisälä, *Lectures on n -Dimensional Quasiconformal Mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag, Berlin–Heidelberg–New York (1971).
- [Vä6] J. Väisälä, *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. A I Math. **509** (1972), 1–14.
- [Vä7] J. Väisälä, *A survey of quasiregular maps in $\mathbb{R}^n$* , Proc. ICM Helsinki 1978, Vol. 2, Acad. Sci. Fenn., Helsinki (1980), 685–691.
- [Vä8] J. Väisälä, *Domains and maps, Quasiconformal Space Mappings. A Collection of Surveys 1960–1990*, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992), 119–131.
- [Ve] I.N. Vekua, *Generalized Analytic Functions*, Reading MA (1964); Russian edn: Gos. Izd. Fiz.-Mat. Lit., Moskva (1959); Pergamon Press, Oxford (1962); Akad. Verlag, Berlin (1963).

- [Vo1] L.I. Volkovyskij, *On the type problem of a simple connected Riemann surface*, Mat. Sb. **18** (60) (1946), 185–210 (in Russian).
- [Vo2] L.I. Volkovyskij, *Researches on the type problem of a simply connected Riemann surface*, Trudy Mat. Inst. Steklova **34** (1950), 1–171 (in Russian).
- [Vo3] L.I. Volkovyskij, *Quasiconformal Mappings*, Izd. L'vovskogo Univ., L'vov (1954) (in Russian).
- [Vo4] L.I. Volkovyskij, *On conformal modules and quasiconformal mappings*, Quelques problèmes des mathématiques et de la mécanique, Novosibirsk (1961), 65–68 (in Russian).
- [Vu1] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Anal. Math. **45** (1985), 69–115.
- [Vu2] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin (1988).
- [Vu3] M. Vuorinen (ed), *Quasiconformal Space Mappings. A Collection of Surveys 1960–1990*, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992).
- [We] A. Weil, *Modules des surfaces de Riemann*, Séminaire Bourbaki 10^e année, Exposé No 168 (1958), 1–7.
- [W] R.J. Wille, *An outer limit of non-conformalness, for which Picard theorem still hold*, Nederl. Akad. Wetensch. Proc. **50** (1947); Indag. Math. **9** (1947), 415–419.
- [Wi1] H. Wittich, *Ein Kriterium zur Typenbestimmung von Riemannschen Flächen*, Monats. Math. Phys. **44** (1936), 85–96.
- [Wi2] H. Wittich, *Zum Beweis eines Satzes über quasikonforme Abbildungen*, Math. Z. **51** (1948), 278–288.
- [Wi3] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Ergeb. Math., Vol. 8, Springer-Verlag, Berlin (1955).
- [Yo1] T. Yoshida, *On the behaviour of pseudo-regular functions in a neighbourhood of a closed set of capacity zero*, Proc. Japan. Acad. **26** (1950), 1–8.
- [Yo2] T. Yoshida, *Theorems on the cluster sets of pseudo-analytic functions*, Proc. Japan. Acad. **27** (1951), 268–274.
- [Yû1] Z. Yûjôbô, *On pseudo-regular functions*, Comment. Math. Univ. St. Paul **1** (1953), 67–80.
- [Yû3] Z. Yûjôbô, *On the quasiconformal mapping from a simply-connected domain on another one*, Comment. Math. Univ. St. Paul **2** (1953), 1–8.
- [Yû2] Z. Yûjôbô, *Supplements to my paper: On pseudo-regular functions*, Comment. Math. Univ. St. Paul **4** (1955), 11–13.
- [Yû4] Z. Yûjôbô, *On absolutely continuous functions of two or more variables in the Tonelli sense and quasiconformal mappings in the A. Mori sense*, Comment. Math. Univ. St. Paul **4** (1955), 67–92.
- [Za] J. Zajac, *Quasiconformal homeomorphisms in low dimensions*, Folia Sci. Univ. Techn. Resoviensis Mat. **23** (1999), 175–200.
- [Z] Yu.B. Zelinskij, *On a principle of boundary correspondence for quasiconformal mappings*, Some Problems on Contemporary Function Theory, Akad. Nauk SSSR, Novosibirsk (1976), 61–67 (in Russian).
- [Zi] W. Ziemer, *Extremal length and conformal capacity*, Trans. Amer. Math. Soc. **126** (1967), 460–473.
- [Zim] E. Zimmermann, *Quasikonforme schlichte Abbildungen im drei-dimensionalen Raum*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **5** (1955), 109–116.
- [Zo] V.A. Zorich, *A theorem of M.A. Lavrent'ev on quasiconformal mappings in space*, Mat. Sb. **74** (116) (3) (1967), 417–433 (in Russian).

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CHAPTER 18

Quasiconformal Mappings in Value-Distribution Theory

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1. Introduction

1.1. Introduction

The theory of quasiconformal (qc) mappings is considered to begin in the work of Grötzsch, with [Gr28a] the first of his series of works introducing the subject published in rapid succession in the same journal. This first paper as well as the standard texts in qc -theory [A66, LV73, Bel74] make it easy to overlook the deep and long-standing connections between this subject and the “modern” theory of meromorphic functions, whose contemporaneous development was led by R. Nevanlinna. To this day, qc mappings (and their extensions) continue to enrich the theory; there seems to be only one significant example of the reverse effect (see Section 5.3). Although connection between these two subjects was noted from the beginning, as each matured their more refined developments had less mutual impact.

This chapter centers on several themes, of which the following are highlights. Perhaps the oldest is (A): applications of qc maps to the type problem; it is here that some of the most significant modern developments are also occurring. We then have (B): applications to the inverse and related problems, and finally, (C): conformality at a point (this seems an example of meromorphic function theory influencing the direction of quasiconformal theory).

As the foundations of qc theory developed, there were many tentative definitions and modifications until an intensive study was made in the context of Teichmüller theory, and today [A66] and [LV73] present the current status. In the contexts considered here, almost any of these definitions could be used in showing that the mappings we construct are quasiconformal. However, one of the main goals is to produce a function meromorphic in the plane (or a finite disk), and this now follows most directly from the theory of the Beltrami equation (9). On the other hand, if $w = \psi(z)$ is a quasiconformal mapping of the plane, almost all information used in the questions considered here depends only on the behavior of $|\psi|$; rarely is $\arg w$ of significance.

It is impossible to discuss the topics of this chapter without considering the *type problem*, but the literature on the type problem is so extensive that we can only highlight what seem to us the most significant aspects in Sections 3 and 4. We normally limit our attention to simply-connected noncompact (“open”) surfaces $\mathcal{S}$. Thus $\mathcal{S}$ is the conformal image of the unit disk (hyperbolic or “Grenzkreistypus”) or the plane $\mathbb{C}$ (parabolic or “Grenzpunkttypus”). Many of the topics considered here were the focus of intensive research before 1960, and we have tried to present an overview of many of these. In particular, some arguments of Ahlfors, Blanc, etc. are now absorbed by more general theories, but we have tried to give a flavor of some of the especially ingenious methods these early authors introduced.

Given the scope of this chapter, we could have a bibliography containing hundreds of items. We have tried to include the most significant works, but in this era of modern search engines we have not tried to be encyclopedic.

For the most part we use standard notation. However, note that $S(r, a) = \{|z - a| = r\}$, $S(r) = S(r, 0)$, $S = S(1, 0)$, with similar understandings of $B(a, r) = \{|z - a| < r\}$, $B(r)$ and B . The plane is $\mathbb{C}$; the extended plane (Riemann sphere) $\hat{\mathbb{C}}$.

1.2. Nevanlinna theory

We recall some features of Nevanlinna's theory; the classical references are [N29,N70], with [Hay64,GO74] relatively contemporary. This theory originated as a penetrating refinement and interpretation of Picard's discovery that an entire function which omits two values is constant, but its spirit and calculus have had an impact far beyond this.

For the theory to have immediate content for $R < \infty$, it is necessary to consider functions f with some lower bounds on how slow $T(r, f)$ can grow as r tends to R . We do not discuss the situation $R < \infty$ here.

Now let f be nonconstant and meromorphic in the plane. Then Nevanlinna associates a nonnegative number $\delta(a)$ to each $a \in \widehat{\mathbb{C}}$ such that $\delta(a) = 1$ when a is omitted (Picard value), and $0 \leq \delta(a) \leq 1$ in general. Nevanlinna's second fundamental theorem asserts that

$$\sum_{\widehat{\mathbb{C}}} \delta(a) \leq 2, \quad (1)$$

which while yielding Picard's theorem at once, is itself rarely applied. In contrast, the calculus Nevanlinna developed to obtain (1) yields dividends to this day.

This theory depends on the asymptotic behavior of the nondecreasing functions

$$n(r, a), \quad r > 0, a \in \widehat{\mathbb{C}}, \quad (2)$$

which represent the number of solutions to each equation $f(z) = a$ for $z \in B(r)$. More refined data uses $\bar{n}(r, a)$, where multiple solutions are counted only once. If we set

$$N(r, a) = \int_0^r \left[\frac{n(t, a) - n(0, a)}{t} \right] dt + n(0, a) \log r, \quad (3)$$

$$\bar{N}(r, a) = \int_0^{2\pi} \left[\frac{\bar{n}(t, a) - \bar{n}(0, a)}{t} \right] dt + \bar{n}(0, a) \log r, \quad (4)$$

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{i\theta}) d\theta, \quad (5)$$

then (1) uses the first of $(\delta(a, f))$ is the *deficiency* of a)

$$\begin{aligned} \delta(a, f) &= 1 - \limsup_{r \rightarrow R} \frac{N(r, a)}{T(r)}, \\ \theta(a, f) &= \liminf_{r \rightarrow R} \frac{N(r, a) - \bar{N}(r, a)}{T(r)}. \end{aligned} \quad (6)$$

It is elementary that, for each a , we have

$$0 \leq \delta(a), \theta(a), \quad \delta(a) + \theta(a) \leq 1,$$

and (1) is an immediate consequence of the more precise

$$\sum_{\mathbb{C}} \delta(a) + \theta(a) \leq 2. \quad (7)$$

Inequality (1) is more often quoted than (7), but the proof of the stronger (7) follows just as naturally from Nevanlinna's original calculus. However, (1) and (7) are of a different character. The "stronger" (7) holds as an equality for most familiar functions in the theory, while (1) is rarely achieved (the quasiconformal methods discussed in this chapter have been essential to analyze functions extremal for (1); see Section 6.5). To see that equality in (7) is almost generic, in contrast to equality in (1), we need only consider a polynomial p of degree $N \geq 1$. Then $\delta(a) = 0$ for all a so that $\sum \delta(a) = \delta(\infty) = 1$, while $\sum \{\delta(a) + \theta(a)\} = 2 - 1/N$. Nevanlinna (cf. [N70, Chapter 11, Section 1]) sketches a formal argument which shows (7) as a limiting case of the Riemann–Hurwitz formula, and in many simple situations, this can be worked out explicitly (e.g., $e^z = \lim(1 + z/N)^N$). However, efforts to make this heuristic argument apply in a general way fails; cf. Sections 5.1 and 7.

Notice that so far there is no mention of quasiconformal mappings but some of the definitions above do not require the full strength of analyticity. In some generalizations (in particular to that of quasiregular mappings in space [R93]), there are satisfactory analogues of (1), but not of (7).

It is interesting that Nevanlinna considered the possibility of relation (1) holding even when the set of a is augmented to include all *functions* satisfying $T(r, a(z)) = o(T(r, f))$, but proved it only when $q = 3$. The general inequality (1), in the context of small functions, was established only relatively recently ([Ste86], see also [Osg85]); while the analogue of (7) is being settled only as this is being written (see [Y1] for the case of summing over just "small" rational functions and [Y2] for the general case). The methods especially of [Y2] make strong contact with notions of algebraic geometry and moduli space.

When $R = \infty$, there is one natural functional associated to (the increasing, real-valued function) $T(r)$: its growth (order)

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log T(r)}{\log r}; \quad (8)$$

ρ may assume any value $0 \leq \rho \leq \infty$.

2. Background

2.1. Early history

Grötzsch's original motivation (as presented at the session of 23 July, 1928 of the Saxon Academy [Gr28a]) was geometric; to find the most nearly-conformal mapping of quadrilaterals and annuli, and he derived what we today consider the standard modulus inequalities (cf. (12)). In the presentation [Gr28b] we can find the first tentative definition of a quasiregular mapping (called an *Abbildung von beschränkter infinitesimales Verzerrung* which

was phrased in the generality of mappings from a plane region to a Riemann surface $\mathcal{S}$). In addition, at the end of [Gr28b] Grötzsch presents a short proof of the Big Picard theorem for these mappings in the situation that the domain is the complex plane $\mathbb{C}$ (relying upon the Picard theorem in the analytic case and properties of the (conformal) modulus).

The procedure of transferring results between meromorphic functions and quasiregular mappings is now considered routine, due to the “Measurable Riemann Mapping theorem” [AB60] (Lavrentieff [Lav35] already understood this principle in less generality). Let μ be a fixed measurable function defined in the plane. Let $g: \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ with $g \in W_{\text{loc}}^{1,2}$ when $g(z) \neq \infty$ and $1/g \in W_{\text{loc}}^{1,2}$ when $g(z) = \infty$. If g satisfies the Beltrami equation

$$\frac{\partial g}{\partial \bar{z}} = \mu(z) \frac{\partial g}{\partial z}, \quad \|\mu\|_{\infty} = k < 1, \quad (9)$$

has the factorization

$$g(\zeta) = f \circ \psi(\zeta), \quad (10)$$

where ψ is a quasiconformal *homeomorphism* of the plane which also satisfies (9) and f is meromorphic. In addition, any normalization such as ψ fixing two points makes this representation unique. A consequence of (10) is that if $\|\mu\|_{\infty}$ satisfies the bound in (9), then the domain of f must be the full plane; later, for example, in Section 4.4, we find situations where μ fails to satisfy the second condition in (9), and the associated function ψ is no longer surjective onto $\mathbb{C}$.

The strategy is often to produce g first, usually a “formal” solution to the problem at hand. Lying in the background are one or several complex-analytic functions F , which are combined to produce g . Typical advantages are that for g “small” error terms might vanish identically. The factorization (10) then produces an analytic function f .

That $\|\mu\|_{\infty}$ satisfies (9) can be checked by local computations; usually it is only the existence of homeomorphic solution that is used. However, representation (10) may hold in important cases when the bound in (9) fails (cf. [Sto38, Da88, BJ98]). Then g is called of “finite distortion”.

Instances where $\|\mu\|_{\infty} = 1$ have appeared in several classical situations, especially when considering the type problem, in which relatively crude information is needed and length–area techniques apply. One such example is recalled in Section 4.2.

Producing analytic/meromorphic functions by this procedure has been a powerful method to create examples, as well as for proving theorems (cf. Section 6.5, for example). In general, it will be far easier to construct formal solutions g which satisfy (9) and then produce a meromorphic function “automatically” using (10). The form (10) suggests that additional information about ψ near ∞ might (and does) give insightful links between the asymptotic behavior of g and f (see Section 5.3). It turns out that the use of (9) eliminates any mention of Riemann surfaces, which is the language of most classical literature in our bibliography.

The study of quasiregular mappings in two and higher dimensions continues to develop, and has significant interactions with geometry, differential equations and dynamics.

It is appropriate to insert some comments about the *modulus*, which already played a role in [Gr28a], and is an important tool in geometric analysis [A73,J58] and in some generalizations [R93]. Since it is a *conformal invariant*, the notion of a mapping is implicit in the background, and so our first result below gives a flavor of many arguments which appear throughout this literature. Modulus is defined in terms of a fixed curve family Γ in a region Ω . Given any nonnegative function $\rho(z)$ such that $\int_{\gamma} \rho(z)|dz| \geq 1$ for each $\gamma \in \Gamma$, the *modulus* of Γ is defined as

$$M(\Gamma) = \inf_{\rho} \int_{\Omega} \rho^2(z) dx dy. \quad (11)$$

It can be shown that $M(\Gamma)$ does not depend on the open region Ω containing Γ , i.e., it only depends on the path family Γ .

Not only is (11) a conformal invariant, but it is quasiinvariant under quasiconformal mappings. That is, if $\psi: \Omega \rightarrow \Omega'$ is a (quasi)conformal (sense-preserving) homeomorphism, one can set Γ' and ρ' as the objects associated in Ω' corresponding under ψ to Γ and ρ . It follows that $M(\Gamma) = M(\Gamma')$ if ψ is conformal, while the ratio

$$\left| \log \frac{M(\Gamma')}{M(\Gamma)} \right|$$

is bounded in the *qc* situation (thus $M(\Gamma)$ is a *quasiinvariant*).

When Γ is the family of closed curves which separates the two boundary components of an annulus Ω , it is customary to write $\text{Mod}(\Omega) = M(\Gamma)$. Then two elementary but useful properties are:

1. *Monotonicity*: if Ω and Ω' are annuli with $\Omega' \subset \Omega$ separating the boundary components of Ω , then

$$\text{Mod}(\Omega') \leq \text{Mod}(\Omega);$$

2. *Superadditivity*: If Ω_1 and Ω_2 are disjoint annuli contained in the annulus Ω , each of which separates the boundaries of Ω , then

$$\text{Mod}(\Omega) \geq \text{Mod}(\Omega_1) + \text{Mod}(\Omega_2). \quad (12)$$

Another standard setting is that Γ join two sides of a quadrilateral Ω and an analog of (12) also holds. We then have an elementary but significant result (almost folklore)

THEOREM 1. *Let F be a noncompact simply-connected Riemann surface, and D a parameter neighborhood of some given point $z_0 \in F$. Let Γ consist of all locally rectifiable curves which “start” on ∂D and which leave any compact set of F . Then F is parabolic if and only if*

$$M(\Gamma) = 0.$$

Equivalently, let Γ' consist of all rectifiable closed curves which separate D from infinity. Then F is hyperbolic if and only if

$$M(\Gamma') < \infty.$$

The proof is immediate. Let $\psi : F \rightarrow F'$ conformally, where F' is $\mathbb{C}$ or $\mathbb{D}$ as appropriate, and suppose that z_0 is sent to 0 and D to $D' \ni 0$. Then for sufficiently small $\varepsilon > 0$, we may apply the monotonicity principle to $F'' = \{z \in F'; |z| > \varepsilon\}$. However, if we consider the family $\Gamma_{[a,b]}$ of curves joining the two boundary components of an annulus $\mathcal{A} = \{a \leq |z| \leq b\}$, it is standard that $M(\Gamma_{[a,b]}) = 2\pi / \log(b/a)$. Thus $M(\Gamma) = 0 \iff F' = \mathbb{C}$, which gives the first part of theorem. A similar argument gives the second statement.

There is no mention of quasiconformal mappings in this argument. However, if the uniformizing function ψ is only assumed to be quasiconformal, the proof goes through, since only the quasiinvariance of the modulus is necessary for the proof (this observation is originally due to Teichmüller [Te38]).

Teichmüller [Te38] discovered that this argument would give information in situations in which ψ might fail to be qc (cf. [L-V47, p. 276] or [V50, pp. 25–27]). This extension will be used in Section 4.2.

PROPOSITION 1. *Let $\psi : \mathbb{C} \rightarrow B$ be a homeomorphism and assume that $\psi \in W_{\text{loc}}^{1,2}$. Then*

$$\int_{\mathbb{C}} K(z) dx dy = \infty,$$

where

$$K(z) = \frac{1 + \mu(z)}{1 - \mu(z)}, \quad \mu(z) = \frac{\psi_{\bar{z}}(z)}{\psi_z(z)}.$$

2.2. Nevanlinna's class F_q

A new circle of ideas and techniques was introduced soon after in Nevanlinna's monumental 1932 paper [N32a] (a sketch appears in Chapter XI of [N70]): the class of Riemann surfaces of class F_q . These are simply-connected surfaces which encompass (the Riemannian image of) many familiar analytic functions of each conformal type, but still form a broad enough family to provide a rich laboratory for methods of testing type.

It is useful to view the surface $\mathcal{S}$ as “spread over the Riemann sphere”; that is, $\mathcal{S}$ is a two-dimensional surface such that there is a map $p : \mathcal{S} \rightarrow \widehat{\mathbb{C}}$ which is continuous, open and discrete, and $\mathcal{S}$ is endowed with the pull-back under p of the spherical metric. Then $\mathcal{S} \in F_q$ if there are $q < \infty$ points $\mathcal{A} = \{a_1, a_2, \dots, a_q\}$ such that

$$p : \mathcal{S} \setminus \{p^{-1}(a_i); a_i \in \mathcal{A}\} \rightarrow \widehat{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$$

is a covering map.

Choose disjoint disks V_i on $\widehat{\mathbb{C}}$ about each a_i . The various preimages U of $p^{-1}(V_i \setminus a_i)$ fall into two classes: either $p|_U$ is (a) conformally equivalent to the map $z \rightarrow z^k$ in $B \setminus \{0\}$ for some $k \in \mathbb{Z}^+$ (branch point, algebraic singularity) or (b) conformally equivalent to the map

$$z \rightarrow \exp(z) : \{\Re z < 0\} \rightarrow B \setminus \{0\} \quad (13)$$

(logarithmic singularity). For example, the (surface corresponding to the) entire function $w = \sin z$ is in F_3 ($\mathcal{A} = \{\pm 1, \infty\}$) as is the elliptic modular function. The universal cover over $\widehat{\mathbb{C}}$ with q points deleted, $S_q(\infty)$, is an extremal member of F_q , since then alternative (b) always holds. The universal cover is important in several contexts, since (dating back well into the 19th century) it was known how it is mapped into the disk (see the very first pages of [N70]). This yields many applications, especially in the works [A32,L-V47, N32a,Te38]. Of course, the original proof of Picard's theorem relied on the universal cover $S_q(\infty)$ when $q \geq 3$.

Nevanlinna [N32a] established a basic fact about an important subclass of F_q which, following the lead of [EM], we call $\mathcal{N}$ -surfaces (this in honor of both Rolf and Frithiof Nevanlinna).

THEOREM 2. *Let the (noncompact) surface S in F_q have finitely many, say $1 \leq q_1 < \infty$, logarithmic singularities and no algebraic singularities. Then S is parabolic. In addition, (1) holds in the sharp form*

$$\sum \delta(a, f) = 2, \quad (14)$$

and $\delta(a) > 0$ if and only if $a \in \mathcal{A}$. Each $\delta(a_j)$ is an integral multiple of $1/q_1$.

(That one can allow a finite number of algebraic branch points without changing the conclusion is due to [El34].) In contrast, $S_q(\infty)$ is hyperbolic, but has infinitely many logarithmic singularities. The class $\mathbb{S}$ is defined as

$$\mathbb{S} = \bigcup_q F_q; \quad (15)$$

this class also has significance in the study of dynamical systems (cf. [EL92]).

Only in relatively recent times [EF59a,EF59b,Dras87,Er93] has it been proved that (elementary modifications of) this class provide all functions for which there is equality in (1), among meromorphic functions of finite order. This will be discussed in more detail in Section 6.5.

The procedure introduced in [N32a] soon became standard. First, there is a “topological” portion, describing how the given surface is constructed, followed by an “analytic” part to determine conformal type. The topological arguments are based on the *Speiser graph* associated to any surface S in F_q (we discuss this in Section 3.1), but Nevanlinna's analytic argument to determine type depended upon the oscillation theory of second-order

differential equations. That is, if $\mathcal{S} \in \mathcal{N}$, then the uniformizing function f has the representation $f = g/h$, where g and h are linearly independent entire solutions to the equation

$$w''(z) + \frac{1}{2}P(z)w = 0,$$

where $P(z)$ (the Schwarzian derivative $\{f, z\} = (f''/f)' - (1/2)(f''/f')^2$) is a polynomial (once branch points are allowed, P will have poles). The asymptotic behavior at ∞ of solutions to this equation when P is a polynomial was well understood at the time, but its rigidity has limited its use in the questions considered here (in one of his final papers, Nevanlinna [N66] proposed this as a means to resolve his *inverse problem* (cf. Section 6); this approach has not yet been successful).

2.3. Enter Ahlfors

Ahlfors presented a radically different approach to Nevanlinna's Theorem 2 in his article [A32], which appeared immediately after [N32a] in the *Acta*. In hindsight, we can see that the connection with quasiconformal mappings, while not explicitly mentioned, had now been cemented. This is a far more flexible method for these problems: differential equations are out of the picture.

Ahlfors notes that the theorem is clear when $q_1 = 2$: $\mathcal{S}$ must be (conformally equivalent to) the surface of $w = e^z$. He then carefully considers the first nontrivial case $q = 3$. There is, up to Möbius transformation of the range, exactly one surface with one logarithmic singularity corresponding to each cubic root of unity a_ℓ , all other preimages of each a_ℓ being unramified. To view this surface, start out with the unit disk B with the $\{a_\ell\}$ marked on its boundary and give B the spherical metric viewing its boundary as the equator of the sphere. For instance, one can take the southern hemisphere H_0 of a sphere of radius one, with three equidistant points a_0, a_1, a_2 marked on the equator so as to form a spherical triangle. Now choose one side, say $s_1 = (a_1, a_2)$ (we define s_2 and s_3 similarly), along the equator and consider a sequence of hemispheres $H_1^+, H_1^-, H_2^+, H_2^-, \dots$, where H_i^+ are northern hemispheres and H_i^- are southern hemispheres, all of which have only the points a_1 and a_2 marked on the equator (we forget about a_3). Then glue H_1^+ to H_0 along the short side s_1 , glue H_1^- to H_1^+ along the long side $s_2 \cup s_3 \cup a_3$, and so on, alternating gluing along the short and long sides. Repeat this procedure with the other two sides of the spherical triangle H_0 (where now a_1 and a_2 are ignored in turn) and you obtain the surface $\mathcal{S}$ with three logarithmic singularities projecting to the cubic roots of unity.

In visualizing $\mathcal{N}$ -surfaces, an important orientation comes from the partitioning of the plane into 2π -horizontal strips as fundamental regions for $w = e^z$. The regions in which e^z is close to the deficient values $\{0, \infty\}$ are the right and left half-planes, and e^z tends to these values at a common specific rate in each of these half-planes. Between these sectors e^z assumes all values $a \neq 0, \infty$ with equal frequency, and each strip penetrates both half-planes. For functions in class $\mathcal{N}$ this description is modified: there are $q_1 \geq q$ congruent sectors inside each of which f tends to one of the $a \in \mathcal{A}$; in the region between sectors

associated to a' , a'' ($a' \neq a''$), the solutions to $f(z) = a$ ($a \neq a', a''$) occur with equal frequency.

In our example with $q_1 = 3$, let us first map $\mathcal{S}$ topologically onto $\mathbb{C}$, and see what happens to the tiling of $\mathcal{S}$ induced by the partitioning into southern and northern hemispheres. The spherical triangle H_0 is sent homeomorphically onto an unbounded “triangle” $\mathcal{T}_0$, whose sides are three curves which tend to infinity in both directions. The complement of $\mathcal{T}_0$ in $\mathbb{C}$ is the union of three unbounded sector-like regions, each of which is stratified by strips. In one such region, each strip will correspond to a hemisphere in the sequence $H_i^\pm$ described above when the side s_1 is chosen. The curves (also unbounded in both directions) that separate two such strips are mapped to the union of the other two equatorial sides s_2 and s_3 . Hence, on each such curve we can mark the point which is mapped to the vertex common to s_2 and s_3 : these are the unramified preimages of the cubic root of unity a_3 , call them $\{a_3^j\}_{j=1}^\infty$.

Of course this is only a homeomorphic change of variables (let’s call it the *planar model*), the trick is to do this with explicit conformal or quasiconformal maps. Ahlfors’ insight is to use the universal cover $\mathcal{S}(\infty) \equiv \mathcal{S}_3(\infty)$, having the same three a_ℓ as Picard values.

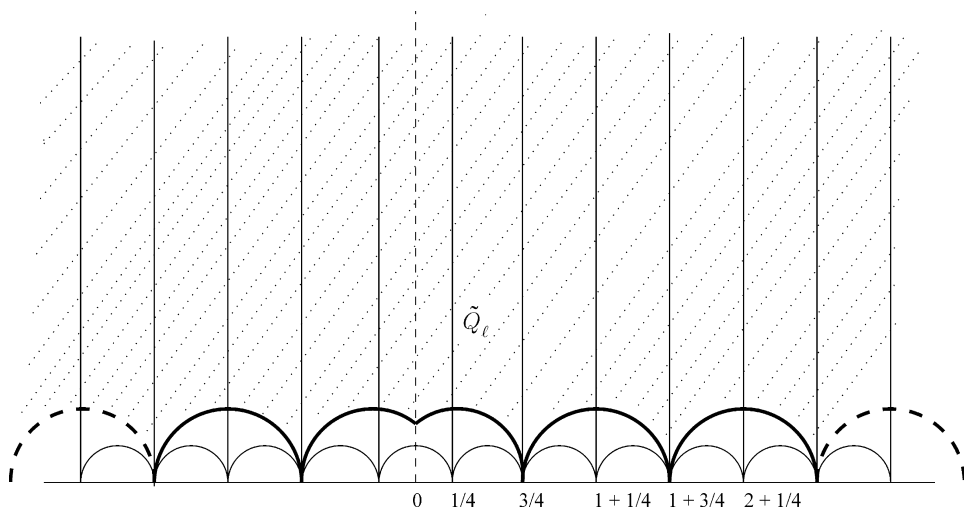
Recall $\zeta(w)$, the (inverse of) the modular function:

$$\zeta : \mathcal{S}(\infty) \rightarrow B.$$

The partitioning of the sphere into southern and northern hemispheres leads back in the unit ζ -disk B to the well-known tiling by Poincaré triangles $\{T_j\}$ all of whose vertices lie on ∂B . These vertices correspond to the (ideal) solutions of $f(z) = e^{2\pi ki/3}$, $k = 0, 1, 2$.

To exploit the uniformization of $\mathcal{S}_3(\infty)$, Ahlfors chooses three great circles Γ_ℓ on the Riemann sphere connecting the north and south pole and passing through each a_ℓ . Consider now Γ_3 , passing through a_3 . First, lift it to $\mathcal{S}$ as follows: start at the south pole, thought as a point of the spherical triangle H_0 , and run along Γ_3 not towards a_3 but in the opposite direction so as to exit H_0 through the mid-point of the side $s_1 = [a_1, a_2]$. Thus this lift enters H_1^+ , then exits through a_3 into H_2^- , etc.... The same lifting can be repeated for the other two curves Γ_1 and Γ_2 . Back in our planar model these curves correspond to three half-rays γ_ℓ . For instance γ_3 starts at the origin and passes through all the unramified preimages a_3^j of a_3 (these are the points on the boundary of the strips described above). The other two rays are obtained by rotating γ_3 by $2\pi/3$. With the exponential function one obtains only two such rays, i.e., the positive and negative imaginary axis.

Ahlfors now focuses his attention in turn to each of the three sectors that are thus created between two curves Γ , for instance the sector S_2 between Γ_3 and Γ_1 . The piece $S_2 \cap H_0$ is mapped conformally by the modular function to a portion of the fundamental triangle T_0 in B . Namely, in T_0 issue three straight segments from to origin perpendicular to the sides of T_0 . This splits T_0 into three pieces and the one with vertex A_2 corresponding to a_2 is the image of $S_2 \cap H_0$. The rest of the sector is mapped by Schwarz reflection to a subset Q_3 of the unit disk. One side of the sector, say Γ_3 , is mapped to a curve σ_3 , which is a broken curve made up consecutive hyperbolic geodesics which converge to A_3 , each geodesic corresponding to one full turn on Γ_3 starting and ending above a_3 (see Figure 1).

Fig. 1. The regions $\tilde{Q}_\ell$.

Finally Ahlfors considers the image of Q_3 under the map

$$z_3 = \frac{i\sqrt{3}}{4} \frac{A_3 + \zeta}{A_3 - \zeta},$$

and obtains a subregion of the upper half-plane $\tilde{Q}_3 \subset \{\Im z_3 > 0\}$, which lies above the union of all the semicircles of radius $1/2$ centered at the points $\pm(n + \frac{1}{4})$, $n \geq 0$ (notice that the first semicircle with center at $1/4$ is truncated when it reaches the imaginary axis). The last step is to apply the power $z^{2/3}$ so that $\tilde{Q}_3$ is now mapped to a sector-like region Σ_3 inside a sector of opening $2\pi/3$. Rotating Σ_3 by $2\pi/3$ one obtains Σ_1 and Σ_2 as well. In conclusion, after cutting $\mathcal{S}$ into three congruent pieces (sectors) using the curves Γ_ℓ , Ahlfors sends each conformally onto sector-like regions Σ_ℓ which fit together and cover the whole complex plane except for three sequences of ovals along three rays going to infinity separated by 120° . Assuming that one of these rays is the positive real axis the ovals on that ray can be enclosed by boxes of the form

$$b_n = \left\{ re^{i\theta} : \left(n - \frac{1}{4}\right)^{2/3} \leq r \leq \left(n + \frac{3}{4}\right)^{2/3}, |\theta| \leq \frac{C}{n} \right\},$$

where C is a large enough constant. In particular the angular opening $\theta(r)$ of each region Σ_ℓ approaches $2\pi/3$ as r tends to infinity.

At this point one can proceed in several different ways depending on taste and the degree of precision sought. Ahlfors intersects $\bigcup \Sigma_\ell$ with circles of radius ρ and pulls them back to closed Jordan curves in $\mathcal{S}$ so as to obtain an exhaustion of $\mathcal{S}$ by domains $\mathcal{S}(\rho)$. He then

explicitly computes the number of solutions to each equation $f(z) = a$ for $z \in S(\rho)$ and shows that

$$\lim_{r \rightarrow \infty} \frac{n(r, a)}{r^{3/2}}$$

exists for all a and is independent of a when $a \notin \bigcup a_\ell$. This makes it routine to check *formally* that f satisfies (1) with $\delta(a_\ell) = 2/3$, $1 \leq \ell \leq 3$, and $\delta(a) = 0$ otherwise; the caveat “formally” is necessary since we are computing in (1) using this *ad hoc* exhaustion $\{\Gamma(\rho)\}$, rather than the image of the circles $S(r)$ under an analytic mapping.

To see that these formal computations persist for the exhaustion of $\mathbb{C}$ using the circles $\{S(r)\}_{\{r>0\}}$, Ahlfors had a superb weapon at his call: his length–area method that was already the key ingredient in his thesis [A30]. He thus shows that if $r_1(\rho) = \max\{|z|, z \in \Gamma(\rho)\}$, $r_2(\rho) = \min\{|z|, z \in \Gamma(\rho)\}$, then

$$r_2(\rho) = (1 + o(1))r_1(\rho), \quad \rho \rightarrow \infty. \quad (16)$$

This implies that all limits

$$\frac{n(r, a)}{n(r, b)}, \quad r \rightarrow \infty,$$

may be computed by replacing r by ρ ; in particular, $\delta(a_i) = 2/3$, $1 \leq i \leq 3$, and $\delta(a) = 0$ otherwise. Thus (14) holds.

We note that Ahlfors has stopped one step short of constructing an explicit (asymptotically conformal) quasiconformal map of S onto the plane. All that one needs to do is to slightly stretch each region Σ_ℓ out to fill an entire sector of opening $2\pi/3$. This can be accomplished with “linear” stretchings depending only on $|z|$, and since the angular opening $\theta(r)$ of the regions tends to $2\pi/3$, it is immediate that this yields a global *qc* map of S onto $\mathbb{C}$. The parabolic type is therefore verified, although this weaker fact could have also been done by using the Euclidean metric on Σ_3 , pulling it back to S and then using it to estimate the modulus of all curves going to infinity as in Theorem 1.

Later work depending on a more careful estimate of the distortion of the *qc* map near infinity showed that (see Section 5.2)

$$r_2(\rho) = \{1 + o(1)\}r_1(\rho) = A\{1 + o(1)\}\rho, \quad \rho \rightarrow \infty,$$

for some nonzero constant A (cf. Teichmüller [Te38]), but this is not needed to compute the data for (1) or (7).

The distinction between the length–area method and quasiconformal mappings is somewhat arbitrary. In studying surfaces of class $\mathcal{N}$, Ahlfors and Nevanlinna were dealing with extremal configurations, and this extra rigidity was exploited by Ahlfors to show that this mapping was asymptotically conformal.

3. The type problem: Introduction

3.1. The type problem. Speiser graphs

The work discussed so far (up to 1932) can equally well be considered foundational for the *type problem* as well as for value-distribution theory.

Thus Theorem 2 is one of the significant early results in the classification of type. After that, the focus of “type determination” slowly bifurcates from what today is considered pure value-distribution theory, and so our presentation of research in this direction must be condensed. This is unfortunate since there is no modern account of the type problem; one is left with Chapter VII of [Wi55], the thesis [V50] (an exhaustive account of the situation up to 1950), and a large number of specialized articles. Chapter VII of [GO74] is focused on the inverse problem of Nevanlinna theory, but gives a beautiful description of many methods that have been important in resolving the type problem. This is to be expected, since the most precise tools used in studying type (especially those involving Riemann surfaces of class $\mathcal{S}$) play a role in Nevanlinna theory. Other methods have been used for the type problem through the years, but for the most part these have not had a significant impact in mainstream value-distribution theory; some will be considered in Section 7.

Many type criteria apply only to specific classes of surfaces, and involve convergence or divergence of certain series. In that sense, the likelihood of a nontautological solution to the problem appears as unlikely as to produce a universal test for convergence of series. However, the type problem presents a rich and interesting laboratory in which to test techniques and methods.

In 1929 Speiser [Sp29] proposed that the type of certain concretely presented surfaces be determined explicitly, and in [Sp30] introduced one of the key methods for describing these surfaces: the Speiser graph $\mathcal{G}$, as it is now called, and this perspective was the basis for the description of these surfaces in [N32a]. This initiated an extensive literature linking properties of the graph to conformal type. For twenty years, the centers of activity were in Germany, Switzerland, Japan and the former USSR (where, it appears, the first connections are due to Lavrentiev). There were a large number of papers carrying out these themes, and so it is often difficult to assign priority.

Thus let S be a surface of class F_q and $\mathcal{A} = \{a_1, \dots, a_q\}$ be the corresponding distinct points in the sphere (which we take to lie in $\mathbb{C}$). We construct a Jordan curve L passing through these points, choosing notation so that L is positively oriented and passes through them in cyclic order. (If desired, we may impose additional conditions, such as that in a neighborhood of each point L be a line-segment, etc.) Then L divides the plane into two components, $\mathcal{I}$ (inner) and $\mathcal{O}$ (outer), so that $p^{-1}(L)$ decomposes S into components which are homeomorphic to $\mathcal{I}$ and $\mathcal{O}$. Clearly S may be rebuilt by gluing these pieces along curves that project to the appropriate side (a_i, a_{i+1}) .

This can be modeled by a graph $\mathcal{G}$, also known as the *line complex*. We choose base points $\circ \in \mathcal{I}$ and $\times \in \mathcal{O}$, and whenever S contains a pair $\{\mathcal{I}, \mathcal{O}\}$ which may be connected by an arc which crosses L across the segment (a_i, a_{i+1}) , we construct a segment connecting $\circ$ with $\times$ (which we might label i). This complex determines S up to certain natural equivalences (cf. [GO74, p. 575] which cites the articles [Drap36, Hab52, Ta62]).

More formally, the definition of Speiser graph can be given as follows. For each i let γ_i be an open Jordan arc in the sphere connecting $\circ$ to $\times$ and crossing the curve L in exactly one point on the interval (a_i, a_{i+1}) . Assume also that $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$. Then define Γ to be $p^{-1}(\bigcup_i \gamma_i)$, where $p: \mathcal{S} \rightarrow \widehat{\mathbb{C}}$ is the projection defined in Section 2.2. Now apply a homeomorphism which sends $\mathcal{S}$ to the plane $\mathbb{C}$ to obtain an embedded planar graph $\mathcal{G}$ that is commonly known as the Speiser graph of $\mathcal{S}$. It is to be noted that two graphs that are graph-theoretically equivalent but which are embedded differently in the plane (meaning that one cannot be deformed into the other by isotopy) may correspond to different surfaces which can also have different type.

By abuse of notation we again write $\{\circ, \times\}$ for the *nodes* of $\mathcal{G}$ and call them *vertices*. The paths $\{\gamma\}$ connecting adjacent vertices form *bundles* of $1 \leq k \leq q - 1$ segments. An interior vertex of $\mathcal{G}$ is either ordinary (the endpoint of exactly two bundles), or of order $(m - 1)$, $3 \leq m \leq \infty$, at which m bundles are joined. The graph $\mathcal{G}$ may also contain (closed) polygons of $2p$ sides, $1 \leq p \leq \infty$. The case $p = 1$ corresponds to a closed path on $\mathcal{S}$ which loops once about some $a_i \in \mathcal{A}$; when $2 \leq p < \infty$ the corresponding point $a_i \in \mathcal{A}$ (the center of the polygon) is a branch point of multiplicity $p - 1$, and if $p = \infty$ then a_i corresponds to a logarithmic branch point. Only the polygons in the $p \geq 2$ case are counted as faces of the Speiser complex. And the sides of the faces are called edges of the Speiser graph even though sometimes they actually correspond to a bundle.

The graph of the exponential function is a line segment with vertices $\times$ and $\circ$ in cyclic order, while that of $\mathcal{S}_q(\infty)$, the universal cover of the q -punctured sphere, is a tree having q edges at each vertex (so that each vertex has valence q). In the rest of the chapter we refer to this tree simply as the *modular tree* or the *modular graph*. Figure 2 shows the graph of $\mathcal{S}_3(\infty)$.

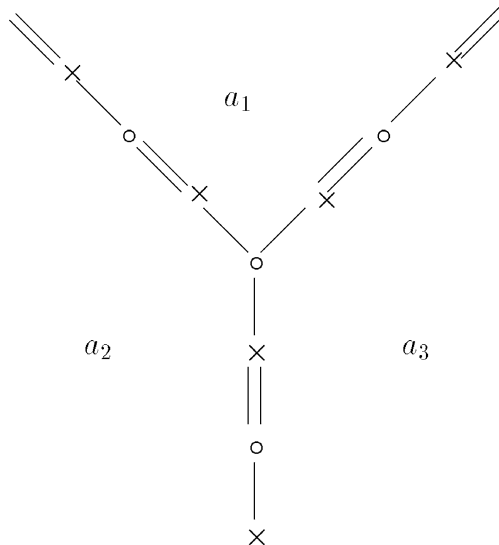


Fig. 2. Speiser graph of the surface $\mathcal{S}$ of Section 2.3.

3.2. The Nevanlinna–Wittich criterion

This gives a simple sufficient condition for parabolicity. Although this topic concerns the type problem as opposed to value-distribution theory, it was introduced by Nevanlinna [N32b] about the same time as his [N32a], and was the basis of an application by Teichmüller to a problem inspired by pure Nevanlinna theory (see Section 5.1).

Let $\mathcal{G}$ be the graph of the surface $\mathcal{S} \in F_q$. In a natural way we start with a given node, which we call of *generation zero*, and call nodes connected to this base node of generation 1, etc. If $\mathcal{G}_n$ is the portion of $\mathcal{G}$ with nodes of generation at most n , then $\partial\mathcal{G}_n$, viewed as a connected subset of the plane, has $s(n)$ vertices on its outer boundary. We insist that this is not the graph-theoretic boundary of $\mathcal{G}_n$: a vertex is in $\partial\mathcal{G}_n$ if it can be connected to infinity with a path in $\mathbb{C}$ which does not intersect $\mathcal{G}$.

We then have the following theorem.

THEOREM 3 (Nevanlinna–Wittich [Wi55, p. 110]). *If $\mathcal{S} \in F_q$ and*

$$\sum_n \frac{1}{s(n)} = \infty, \quad (17)$$

then $\mathcal{S}$ is of parabolic type.

Condition (17) is far from necessary, but it can be checked for many simple examples. In Section 5.1, following Teichmüller [Te38] we introduce a natural subclass of F_q and recall work of Le-Van [L-V47] which shows that (17) is necessary for this subclass.

Some examples may be appropriate to appreciate (17). The Weierstrass $\wp$ -function has $s(n) = 8n - 4$ for $n > 1$, while the surfaces of class $\mathcal{N}$ considered in [N32a] have $s(n)$ proportional to n for large n . On the other hand, the universal cover of the plane with q points deleted has $s(n) = q^n$.

To see that (17) is not necessary for parabolicity, we may consider the graph $\mathcal{G}$ of $w_0(z) = \exp(e^z)$, which corresponds to a parabolic surface in F_3 having infinitely many logarithmic branch points over 0 and ∞ and a single logarithmic branch point over $w = 1$. The graph corresponding to w_0 is constructed on two levels. First, we draw an infinite horizontal line L with alternating vertices $\times$ and $\circ$, say on the real axis. Connecting adjacent vertices relative to L are bundles each of one line. Then emanating from each such vertex is a vertical family of rays projecting upwards, with vertices connected by bundles consisting alternately of one or two segments; the segments which meet L have one. Since the only way to pass from different vertical portions is by passing through the base, we see that each $\mathcal{G}_n$ is part of a comb, and in this case $s(n)$ is the total number of vertices in $\mathcal{G}_n$. Thus $s(n) = (n+1)^2$, $n \geq 1$, so condition (17) fails in spite of $\mathcal{S}$ being parabolic.

The definition of $s(n)$ may appear contrived, but it links naturally to the well-known type criterion of Ahlfors (Section 4.1(A)). In fact, the sufficiency of the divergence (17) for parabolicity can be obtained from Ahlfors' test using the exhaustion provided by the Kobayashi net (cf. [V50, Theorems 9 and 10]). The reader who wants to get a feeling for how to construct and use the Kobayashi net is referred to Section 4.2, where we sketch the necessity of (17) for a suitable subclass.

The history of Theorem 3 is rather well documented. Nevanlinna [N32a] considers only the key test case $q = 3$ and obtained the first concrete result: the divergence of

$$\sum \frac{1}{ns(n)} = \infty$$

being a sufficient condition for parabolicity. This was another application of the length-area method of Ahlfors. In 1936–1937, Nevanlinna was in Göttingen as Visiting Professor (this is when he met Teichmüller) and at a meeting of the D.M.V. in fall 1936 proposed to Wittich that (17) should be enough. Wittich's proof (again only for $q = 3$) appears in [Wi39], and relies on a careful explicit exhaustion $\{\Gamma_\rho\}_{\rho \geq 0}$ of any Riemann surface $\mathcal{S}$ associated to F_3 ; in particular, this exhaustion respects the ranking $s(n)$ of the various vertices of the Speiser graph.

3.3. A necessary condition for parabolicity

As we have seen, proofs that conditions such as (17) imply parabolicity depend on finding an exhaustion of the associated surface $\mathcal{S}$ that reflects the combinatorial data of the surface. If that exhaustion $\{\Gamma(\rho)\}$ were the images of circle $S(\rho)$ under a quasiconformal homeomorphism onto $\mathcal{S}$, it would follow that this condition would be necessary (recall (9) and (10)).

This was already clear to Teichmüller [Te38, p. 255], who proposed studying a simple subclass of F_q , which we call $F_q(T)$. This class was characterized by:

- (1) $\mathcal{S}$ contains no algebraic branch points, and no unbounded lines in $\mathcal{G}$ have only ordinary vertices (i.e., in the language to be introduced in Section 4.3(A), $\mathcal{S}$ has no *logarithmic ends*);
- (2) from each vertex are exactly 2 or q bundles.

A model of such a surface is realized by starting with the modular graph $\mathcal{G}_q(\infty)$. Given any cyclic way of labeling the q segments linking $\times$ and $\circ$ using a base curve L as in Section 3.1, each edge in $\mathcal{G}_q(\infty)$ may be represented by a terminating q -decimal $\mathbf{q} = 0.q_1q_2 \cdots q_k$ representing the path originating from some fixed base vertex.

Teichmüller replaces each stage $k = k(\mathbf{q})$ connecting adjacent vertices v, v' with a series of $\ell_k(\mathbf{q})$ ($1 \leq \ell_k < \infty$) unramified vertices, arranging the bundles between successive vertices so that the resulting network remains a Speiser graph. Note that there is precisely one way to add edges in each bundle to obtain a Speiser graph in this process; the particular hypotheses (1) and (2) force each $\ell_k(\mathbf{q})$ to be odd. He proposed sufficient conditions for hyperbolicity and parabolicity for surfaces in $F_q(T)$, and used these to give a negative answer to a well-publicized problem of Nevanlinna (discussed in [N70, p. 312], as well as in the two earlier German editions); see Section 5.1, Theorem 5.

Later, in [L-V47], Le-Van used qc exhaustions to show (17) necessary for a simpler, symmetric subclass of $F_q(T)$. This study was intensified in [V50], which contains many other type criteria of this nature. In Section 4.2 we follow the ideas in [V50] to show Le-Van's proposition.

PROPOSITION 2. Let $\mathcal{S} \in F_q(T)$, and suppose that the $\{\ell_k(\mathbf{q})\}$ are independent of $\mathbf{q}$ and increase with k . Then $\mathcal{S}$ is parabolic if and only if

$$\sum_n \frac{1}{s(n)} = \infty. \quad (18)$$

4. The type problem: Basic methods

4.1. General methods in the type problem

A deep study of the type problem is given in the monograph [V50]. Unfortunately, this source was often overlooked by non-Soviet authors. It remains the fundamental reference for studying the type problem using the Grötzsch principle, Speiser graphs and the welding of elementary quasiconformal mappings. We outline these key tools, based largely on [V50]. While most of our discussion here is with surfaces of class $\mathbb{S} = \bigcup_q F_q$, these methods adapt to much greater generality, but we refer to [V50] and more recent accounts for details. After we introduce four techniques and three types of simple surfaces on which many constructions are based, we present some applications, including the proof of one direction in Proposition 2, namely that if the sum (18) converges then the surface is hyperbolic.

(A) *Conformal metric; Ahlfors test for parabolicity.* The first result is based on [A35a], as presented in [N70, Chapter XII]. It is in the spirit of Theorem 1, in that it is merely based on the existence of the change of variables implicit in a conformal map.

Many years later, Ahlfors [A82, p. 84], writes that what seems to have been important is “not . . . the result, but because it may have been the first use of arbitrary conformal metrics in relation to Riemann surfaces”. He credits antecedents due to Nevanlinna [N32b] and Kobayashi [Ko35] as motivation.

LEMMA 1. Let $\mathcal{S}$ be an open simply-connected Riemann surface on which is defined a function $U(w)$ such that:

- (1) $U(w) \rightarrow \infty$ as $w \rightarrow \infty$ (given $M < \infty$, the set $\{U < M\}$ is relatively compact);
- (2) U is continuous except at isolated points, at which $U = +\infty$;
- (3) ∇U is continuous except perhaps on a system of isolated piecewise C^1 curves on $\mathcal{S}$ with $|\nabla U| > 0$ except at isolated points.

For $\rho > \inf_{\mathcal{S}} U(w)$, let $\Gamma(\rho)$ be the level-set $\{U = \rho\}$, set

$$L(\rho) = \int_{\Gamma(\rho)} |\nabla U| |dw|,$$

and (cf. [N32a, p. 315]) assume that the curves $\Gamma(\rho)$ for sufficiently large ρ consist of a finite number of closed piecewise smooth curves which exhaust $\mathcal{S}$ (in [V50, p. 21], weaker conditions are required of the $\{\Gamma(\rho)\}$). Then, if

$$\int^\infty \frac{d\rho}{L(\rho)} = \infty,$$

the surface $\mathcal{S}$ is parabolic.

Given its heritage, it is not surprising that the proof uses length–area. If $A(\rho)$ is the area of $\{U < \rho\}$, then $(U_\eta = \partial U / \partial \eta)$

$$\int_{U=\rho} \frac{|dw|}{U_\eta} = \frac{dA}{d\rho}, \quad (19)$$

and this expression is invariant under conformal mappings.

Now suppose $\psi(w)$ were to map $\mathcal{S}$ to $B(R)$ for some $R < \infty$. Then the function $U_1(z) = U(w(z))$ satisfies the hypotheses already imposed on U , and $L(\rho)$ transfers to the z -plane since

$$\int_{U=\rho} |\nabla_w U(w)| |dw| = \int_{U=\rho} |\nabla_z U_1(z)| |z'(w)| |dw| = \int_{U_1=\rho} |\nabla_z U_1| |dz|.$$

For $\rho > \rho_0$ both $z = 0$ and some ball $B(r_0)$ are inside the level-set $\{U < \rho\}$. Hence for such ρ ,

$$\begin{aligned} 4\pi^2 r_0^2 &\leq \left(\int_{U=\rho} |dz| \right)^2 = \left(\int_{U=\rho} \sqrt{|U_\eta|} \frac{|dz|}{\sqrt{|U_\eta|}} \right)^2 \\ &\leq \int_{U=\rho} |U_\eta| |dz| \int_{U=\rho} \frac{|dz|}{|U_\eta|} \leq L(\rho) A'(\rho). \end{aligned}$$

We integrate with respect to ρ , and if $A(\rho) \leq \pi R^2$ for all ρ , it would follow that

$$\int^\rho \frac{d\rho}{L(\rho)} \leq \frac{1}{4\pi^2 r_0^2} [A(\rho) + O(1)] + O(R^2) = O(R^2) < \infty,$$

a contradiction. Hence R cannot be finite.

Volkovyskii introduced a strip condition for hyperbolicity in [V50, p. 22], using a family of closed curves “dual” to the family $\Gamma(\rho)$ of Lemma 1. He cites his candidate dissertation of 1937 as the original source.

(B) *Kobayashi network.* In order to obtain a useful density U to apply the test of (A), we use the *Kobayashi net* ([Ko35], cf. [N70, Chapter XII]) of the surface.

This net may be constructed in far more general situations, but we will apply it only to surfaces of class F_q .

In this situation, the net has a combinatorial structure which defines a natural density. For each branch point a of $\mathcal{S}$, let $Q = Q(a) \subset \mathcal{S}$ be those points closer (spherical metric) to a , than to any other branch point. This leads to a network bounded by arcs of great circles, which we call K (for Kobayashi).

The density introduced on $\mathcal{S}$, using K , will have the form

$$U(w) = \sigma(w) + \tau(w)$$

for appropriate functions σ, τ . The first term in this decomposition is easy to define:

$$\sigma(w) = \log \left| \frac{1 + \bar{a}w}{w - a} \right|, \quad w \in Q(a),$$

which is defined unambiguously even for points which belong to more than one $Q(a)$, as they will lie on the network K .

Note that any arc $\gamma \subset K$ bounds two polygons, Q', Q'' , centered at branch points a' and a'' , with γ equidistant between them. Thus as w varies on γ and $(a'w)$, $(a''w)$ are the great circles joining these points, then

$$d\tau(w) \equiv \left| d \arg \frac{1 + \bar{a}w}{w - a} \right| \quad (20)$$

is independent of the choice of $a = a'$ or $a = a''$.

This enables us to extend τ to $\mathcal{S}$, given a fixed exhaustion, much as when introducing generations in the context of Speiser graphs. Choose some node $w_0 \in K$ as base-point, and for $w \in K$ define

$$\tau(w) = \min \int_{[w_0, w]} d\tau,$$

where the integral is taken over all paths in K . Finally, τ may be extended to each $w \in Q(a) \setminus \{a\}$ by letting w' be the point of K at which the spherical geodesic (a, w) meets K , and taking $\tau(w) = \tau(w')$. It is straightforward to check that U satisfies the conditions of Lemma 1, but certain features should be noted. The first is that $U = \infty$ at points over $\mathcal{A}$ which lie in $\mathcal{S}$. The function ∇U is not continuous on K and on certain great-circles passing through various points $a \in \mathcal{A}$ at which τ has an extremum. In the next section, this will be illustrated as we establish Proposition 2.

Given the spirit of this chapter, it is appropriate to introduce the complex variable

$$t = \sigma + i\tau \quad (21)$$

notice that there are easy-to-identify cases in which the map $t(w)$ is conformal, and can be used to give *qc* exhaustions of $\mathcal{S}$. An important example of this is in Section 4.2, but it follows almost from the definition of algebraic branch point that $t(w)$ is conformal in a deleted neighborhood of a whenever a is an algebraic branch point.

(C) *Quasiconformal welding and pivoting.* This is a general method whose full scope is far beyond what can be introduced here. It is thoroughly discussed in Chapter 4 of [V50], and is systematically exploited there. It applies when $\mathcal{S}$ may be partitioned into a finite number of cells, each with a different quasiconformal mapping each to a standard plane region. These maps often may be welded to uniformize $\mathcal{S}$ to $B(R)$ for a suitable R , $0 < R \leq \infty$, and so identify type. Another application concerns determining the type of a strip Σ with a given welding function; see Section 4.4(A).

Pivoting is an effective means of shifting singular values. The principle is given in the elementary

LEMMA 2 [V50, Theorem 32]. (a) *Given $R > 0$, $\eta > 0$, there exists $\delta > 0$ such that, for any a , $|a| < \delta$, there is a qc homeomorphism ψ of $B(R)$ which is the identity on $B(R) \setminus B(R/2)$, $\psi(0) = a$ and*

$$\|\mu_\psi\| < \eta.$$

(b) *Similarly, given δ , η as above, there exists R_0 such that, for any $R > R_0$, there exists a qc map ψ as above.*

(c) *Finally, given $\delta < R/2$, there exists $\eta < 1$ such that given a , $|a| < \delta$, there is ψ whose dilatation satisfies the bound in (a).*

4.2. Interlude: A qc exhaustion and a weak converse of Theorem 3

We use the tools from the previous section to prove Proposition 2, following the approach of [V50, Section 68]. This presents an opportunity to display computations with the Kobayashi network K associated to $\mathcal{S}$.

We may suppose that the points of $\mathcal{A}$ are uniformly spaced on the equator S of $\mathcal{S}$ (post-composing if necessary with a qc homeomorphism of $\widehat{\mathbb{C}}$). We use as model the universal cover $\mathcal{S}(\infty)$ ($= \mathcal{S}_q(\infty)$) of the sphere minus q punctures (at the roots of unity a_ℓ , $1 \leq \ell \leq q$). Start say from the southern hemisphere H_0^- and glue q copies of the northern hemisphere, H_1^+ , to H_0^- , each one along one of the arcs s_ℓ determined by the a_ℓ . To each such northern hemisphere, glue $q - 1$ copies of the southern hemisphere, etc. . . .

The Speiser graph and the Kobayashi net in this case coincide with the modular graph in which each vertex has valence equal to q , although the Kobayashi net carries a metric as described in Section 4.1(B). Topologically, one can draw this graph in the plane, say with the origin corresponding to H_0^- , and think of $\mathcal{S}(\infty)$ as the plane itself. It is then clear how $\mathcal{S}(\infty)$ is divided by the graph into infinitely many simply-connected components which can be arranged into generations as follows: q components of 0th generation arise at the origin, then at each vertex of first generation the graph splits into $q - 1$ edges which form $q - 2$ new components. So in general the number of components of generation n equals the number of vertices of generation n times $q - 2$. Each component is bordered by a doubly-infinite polygonal line and they each correspond to a logarithmic singularity as defined in (13).

It is instructive to compute τ and σ on this representation of $\mathcal{S}(\infty)$. An edge connecting 0 to a vertex of first generation corresponds on the sphere to an arc of great circle connecting the south pole to the north pole, and so τ increases from 0 to π . More generally, along an edge connecting a vertex of generation n to a vertex of generation $n + 1$, τ will increase from $n\pi$ to $(n + 1)\pi$. On the other hand, since the curves forming the Kobayashi net correspond to a finite number of arcs of great circles on the sphere, σ is comparable to a constant on the whole graph and is Lipschitz as a function of τ . Therefore,

without loss of generality (postcomposing with a qc map of the plane), we can assume that σ is identically 0 on all of K .

Let now Ω_0 be a component of generation 0 of the complement of the graph. Draw lines in Ω_0 connecting each vertex of the graph on the boundary of Ω_0 to infinity. These are the level lines for $\tau = n\pi$, where n is the generation of the vertex. Thus Ω_0 is decomposed into a doubly-infinity collection of half-strips, and σ tends to $+\infty$ along each of these half-strips. In summary, the function $\tau + i\sigma$ is a two-to-one cover of Ω_0 onto the first quadrant with a fold over the positive real axis corresponding to each line connecting 0 to infinity. This is repeated for each Ω_n of generation n , except that now the fold is over the half-line $\{\tau = n\pi, \sigma \geq 0\}$.

For a surface S in $F_q(T)$ as in Proposition 2 the description of the modular graph lying in the plane, as given above, is modified by breaking up each edge of generation n into ℓ_n pieces, so that now τ will increase by $\ell_n\pi$ instead of just π , when going from a vertex of generation n to a vertex of generation $n+1$ of the modular graph. On the sphere this corresponds to replacing, say, a trip from the south pole to the north pole along the great circle which bisects the side $s = [a_j, a_{j+1}]$ with the same trip followed by $(\ell_n - 1)/2$ complete loops along that same great circle. Everything said above can now be repeated verbatim with the obvious modifications.

Now we are ready to introduce the exhaustion. For $\rho > 0$ consider the system of curves

$$\Gamma(\rho) = \{w \in S: \sigma + \tau = \rho\}. \quad (22)$$

For each $n\pi < \rho \leq (n+1)\pi$, $\Gamma(\rho)$ is a closed simple curve which contains in its interior (the bounded component of the complement) all the vertices of generation n of the Kobayashi net described above, because at those points $\sigma = 0$ and $\tau = n\pi$.

When S has a Kobayashi net, the curves (22) always exhaust S in the sense of Lemma 1 in Section 4.1(A). However, more restrictions are needed, e.g., no algebraic branch points, if one wants this exhaustion to correspond to a uniformizing map.

By abuse of notation, write $\Gamma(k)$ in place of $\Gamma(n_k\pi)$, where $n_k = \ell_1 + \cdots + \ell_k$. Consider the conformal ring A_k in S between $\Gamma(k-1)$ and $\Gamma(k)$. Its image under the map $\sigma + i\tau$ is a trapezoid-like region D_k in the first quadrant. A more careful analysis yields that A_k is decomposed into a string of alternating quadrilaterals and triangles, so that the quadrilaterals are mapped by $\sigma + i\tau$, in a two-to-one fashion, to one of the following trapezoids

$$D_{jk} = \{(\sigma, \tau): \sigma \geq 0, \tau \geq n_j\pi, n_{k-1}\pi \leq \sigma + \tau \leq n_k\pi\},$$

with $j = k-1, k-2, \dots, 1$ ($n_0 = 0$); while the triangles are mapped two-to-one onto D_{kk} , which is an isosceles right triangle. We stress that the map $\sigma + i\tau$ is conformal and hence carries no distortion, not even along the various folds or the Kobayashi net.

Le-Van's goal is to consider $U = S \setminus (E \cup \gamma)$, where E is a compact set and γ is a slit which starts on E and goes to infinity (leaves every compact set), and to construct a homeomorphism ψ of U onto a strip S of height 1 and finite side-length, so that the gluing induced by γ between the top and bottom horizontal boundaries of S is the ordinary identification of $(x, 0)$ with $(x, 1)$. The construction is ingenious, but more to the point

the resulting homeomorphism ψ is only locally quasiconformal. This is a very early, “historical”, appearance of *mappings of finite distortion* (as opposed to mappings of bounded distortion), as we observed in Section 2.1. The conclusion is reached by showing that the integral of the point-wise distortion of ψ^{-1} on S is finite and using Proposition 1 to deduce that ψ^{-1} could not possibly map S (with top and bottom identified) to $\mathbb{C} \setminus \mathbb{D}$. The proof of this last fact is again reminiscent of the length–area method.

To simplify the computations below we assume now that $q = 3$ so that the number of vertices of generation n ($n \geq 2$) is double the number of vertices of generation $n - 1$. This discussion will hold for general q as well, modulo replacing 2^k by $(q - 1)^k$ everywhere.

Since triangles alternate with quadrilaterals along the ring A_k , the trick is to pair every D_{jk} with one copy of D_{kk} and map this pair to a unit square Q_0 with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, so that the triangle D_{kk} goes to the triangle $(0, 0)$, $(1, 0)$, $(1, 1/2)$. This square will then be scaled by 2^{-k} so that it will fit in the following picture: start with Q_0 , then take two copies, Q_{11} and Q_{12} , half the size of Q_0 , and attach them to Q_0 along its vertical right side. Repeat with four squares of side $1/4$, etc. . . . One obtains the familiar Carleson grid, and the horizontal segments together with the slanted ones form the image of the modular graph. Here the squares Q_{kj} with k fixed are of generation k , and together they make a strip S_k of height 1 and side-length 2^{-k} , which is the image of the ring A_k .

This is the combinatorial picture. We now need to define the map explicitly on $D_{jk} \cup D_{kk}$, verify that it is continuous across the boundary, and also keep track of the distortion.

First, each D_{jk} may be mapped onto a quarter-annulus

$$\left\{ (n_{k-1} - n_j)\pi \leq |Z| \leq (n_k - n_j)\pi, 0 \leq \arg Z \leq \frac{\pi}{2} \right\},$$

with dilatation bounded by an absolute constant. Notice that D_{jk} is glued along the top slanted boundary segment to a $D_{j(k+1)}$, i.e., to a trapezoid in the same shifted quadrant $\{\sigma > 0, \tau > n_j\}$ (the image of a component Ω_j as defined above). So far the map is continuous on the complement of the Kobayashi net K .

Now one could apply a logarithm to get a rectangle L_{kj} of height $\pi/2$ and side-length $\log[(n_k - n_j)/(n_{k-1} - n_j)] = \log(1 + \ell_k/n_{kj})$ (here $n_{kj} = n_{k-1} - n_j$), but it would now be difficult to stack the L_{kj} into the strip S_k because of the varying side-lengths. Instead we use the change of variables into polar coordinates $(\sigma, \tau) \mapsto (r, \theta)$, where $r^2 = \sigma^2 + \tau^2$ and $\tan \theta = \tau/\sigma$, which takes the quarter-annulus above into a rectangle R_k of height $\pi/2$ and side-length ℓ_k . Notice that the shape of R_k is now independent of j . Finally, the map $(x, y) \mapsto (x/\ell_k, 2y/\pi)$ is used to send R_k into a unit square. Finally, this unit square is mapped to a square Q_k of side-length 2^{-k} .

Call f the inverse map from Q_k to L_{jk} . Then

$$f(x, y) = \left(\log(2^k \ell_k x), 2^k y \frac{\pi}{2} \right)$$

and f is defined for $2^{-k} n_{k-1}/\ell_k < x < 2^{-k} n_k/\ell_k = 2^{-k} (n_{k-1}/\ell_k + 1)$ and $0 < y < 2^{-k}$.

The pointwise distortion of f is computed by taking the square of the operator norm of the differential matrix divided by the determinant of the differential matrix, we find the distortion to be

$$K(x, y) \leq C \left(2^k x + \frac{1}{2^k x} \right) \leq C \left(\frac{n_k}{\ell_k} + \frac{1}{2^k x} \right),$$

and so the integral $\int_{Q_k} K(x, y) dx dy$ is dominated by

$$(2^{-k})^2 \left[\frac{n_k}{\ell_k} + \log \left(1 + \frac{\ell_k}{n_{k-1}} \right) \right] \leq (2^{-k})^2 \left[\frac{n_k}{\ell_k} + \log \left(1 + \frac{\ell_k}{\ell_{k-1}} \right) \right]. \quad (23)$$

We note that the map constructed so far has constant real part on each curve $\Gamma(\rho)$, which is important in proving continuity of the final map across the Kobayashi net. What is still missing is to specify the map on the triangles, i.e., on D_{kk} . The first step is the same as before, namely D_{kk} is mapped quasiconformally onto a quarter-circle

$$\left\{ Z = r e^{i\phi} : r \leq \ell_k \pi, 0 \leq \phi \leq \frac{\pi}{2} \right\},$$

then the map

$$r e^{i\phi} \mapsto \left(\frac{r}{\ell_k \pi}, \frac{r}{\ell_k \pi} \frac{2}{\pi} \phi \right)$$

sends this quarter-circle onto the triangle $(0, 0), (1, 0), (1, 1/2)$, and scaling by 2^{-k} we get a triangle T_k . To check that this map is of bounded distortion independent of k , note that the inverse map has the form $(x, y) \mapsto (x \cos(y/x), x \sin(y/x))$, and this distortion computes to $2 + (y/x)^2$, with $0 < y/x < 1/2$.

Now modify the square Q_k obtained above with a map that is constant in the first variable so as to get the trapezoid $\tilde{Q}_k = Q_k \setminus \bigcup T_k$. This can be done with a map of bounded distortion. Putting everything together we have constructed a map ψ as required which sends the ring A_k onto the strip S_k of height 1 and side-length 2^{-k} . Also, one can verify that the real part of this map only depends on ρ throughout A_k and therefore is continuous across the Kobayashi net; in particular the top and bottom of S_k are identified as usual with $(x, 0) \equiv (x, 1)$. On the other hand, continuity across the curves $\Gamma(k)$ has already been checked.

Finally the computation of the integral of the distortion of ψ^{-1} on the new square $\tilde{Q}_k$ is comparable to the bound (23) found previously on Q_k . Adding it all up over the 2^k squares that form S_k and letting k tend to infinity, we find that the integral of the distortion of ψ^{-1} is bounded above by the series

$$\sum_{k=1}^{\infty} 2^{-k} \left[\frac{n_k}{\ell_k} + \log \left(1 + \frac{\ell_k}{\ell_{k-1}} \right) \right].$$

We record this as an intermediate result.

LEMMA 3. Let $\mathcal{S} \in F_q(T)$, and suppose that the $\{\ell_k(\mathbf{q})\}$ are independent of $\mathbf{q}$. Then convergence of the series

$$\sum_{k=1}^{\infty} (q-1)^{-k} \left[\frac{n_k}{\ell_k} + \log \left(1 + \frac{\ell_k}{\ell_{k-1}} \right) \right] \quad (24)$$

implies that $\mathcal{S}$ is hyperbolic.

We conclude with a proof of the weak converse to the Nevanlinna–Wittich criterion.

PROOF OF PROPOSITION 2. It is enough to show that the convergence of the series $\sum_n 1/s(n)$ in Proposition 2 implies the convergence of (24), and hence implies hyperbolicity of $\mathcal{S}$.

We are given that the $\{\ell_k\}$ are monotonically increasing; hence $n_k \leq k\ell_k$, so the first series in (24) $\sum_k (q-1)^{-k} (n_k/\ell_k)$ is dominated by the convergent series $\sum_k k(q-1)^{-k}$.

Also, if $n_{k-1} < n \leq n_k$, then $s(n) = s(n_{k-1}) + (n - n_{k-1})(q-1)^k$, and so

$$\begin{aligned} \sum_{n=n_{k-1}+1}^{n_k} \frac{1}{s(n)} &= \sum_{j=1}^{\ell_k} \frac{1}{s(n_{k-1}) + j(q-1)^k} \geq \int_1^{\ell_k} \frac{1}{s(n_{k-1}) + x(q-1)^k} dx \\ &= (q-1)^{-k} \log \left(1 + (q-1)^k \frac{\ell_k - 1}{s(n_{k-1}) + (q-1)^k} \right). \end{aligned}$$

But since the ℓ_k increase,

$$s(n_{k-1}) = (q-1)^{k-1} \ell_{k-1} + (q-1)^{k-2} \ell_{k-2} + \cdots \leq (q-1)^k \ell_{k-1},$$

and so

$$\sum_{n=n_{k-1}+1}^{n_k} \frac{1}{s(n)} \geq (q-1)^{-k} \log \left(1 + \frac{\ell_k - 1}{\ell_{k-1} + 1} \right).$$

It follows that the second sum in (24) also converges, which completes the proof of Proposition 2. $\square$

One final remark. The pre-images $L(u)$ in $\mathcal{S}$ of the lines $\{w = u\}$ form a family of exhausting curves as described in Lemma 1. Since the surface $\mathcal{S}$ considered here is hyperbolic, the integral

$$\int \frac{d\rho}{L(\rho)}$$

converges, so the Ahlfors' test (Section 4.1) fails.

4.3. Some model classes of surfaces

We now outline general methods used in the type problem which depend on quasiconformal methods; they will also be relevant to Section 6. These methods are collected in [V50], see also [GO74, Chapter 7]. These techniques were developed at first for the type problem, but can also be used to construct surfaces to show that the Nevanlinna relations (1) or (7) are sharp. The principle is to use as building-blocks surfaces Ψ of relatively simple transcendental functions, and then let $\mathcal{S}$ consist of a finite number such surfaces Ψ welded together onto a simple *nucleus* (or kernel), which is a polygon having a finite number of sides.

The uniformization of these surfaces presents a good laboratory of quasiconformal techniques, and some will be outlined here. We also present a few applications.

(A) *Logarithmic ends (compare with (13)).* These may be identified in terms of the Speiser graph: each end corresponds to a ray $\gamma \subset \mathcal{G}$ with all vertices simple, in the sense that each vertex is connected to only two other vertices. Thus γ can be viewed as a path of the Riemann surface $\mathcal{S}$ which spirals without limit about two of the q base points, say a and b .

Consider the surface Ψ of the function

$$w = -\frac{be^z - a}{e^z - 1}, \quad (25)$$

that is, $z = \log\{(w - a)/(w - b)\}$. These equations show at once the Ψ is parabolic, with the uniformizing function of order one. Let σ be an open piecewise smooth arc on $\widehat{\mathbb{C}} \setminus \{a, b\}$ whose ends are at a and b , and Γ one preimage on the surface Ψ . Then either component of $\Psi \setminus \Gamma$ is a logarithmic end (based at a and b)

We will show that a surface $\mathcal{S}$ having only finitely many logarithmic ends is parabolic. It is interesting to compare this sketch with Ahlfors's argument in Section 2.3. Here we outline some details, summarizing [GO74, Chapter 7, in particular, Section 7.5].

A surface $\mathcal{S}$ with a finite number of logarithmic ends is based on p logarithmic ends Λ_k with logarithmic branch points over b_k, b_{k+1} for $1 \leq k \leq p$ (indexing mod p), and a nucleus K . In the framework of Section 2.3, we have $b_k \in \mathcal{A} = \{a_1, \dots, a_q\}$, but a given $a \in \mathcal{A}$ may occur several times as a b_k . If $\mathcal{S}$ does not have algebraic branch points, the argument principle shows that necessarily

$$b_k \neq b_{k+1},$$

an assumption that we make here. For example, the function $f(z) = \cos \sqrt{z}$ belongs to F_3 , with $\mathcal{A} = \{\pm 1, \infty\}$, but it has infinitely many algebraic branch points.

We assume that $\mathcal{A}$ is contained in the finite plane. Let L be a Jordan curve through the points of $\mathcal{A}$. We assume that L is a linear segment in some fixed 2ε -neighborhood of each a_j . Let w_0 be some point interior to L . Starting at b_1 , circuit L in the positive direction, and after $m \geq 1$ cycles, in which the points $b_2, \dots, b_p$ are encountered in that order, we return to b_1 . Let $\mathcal{S}_m$ be the Riemann surface of $w = w_0 + z^m$ and $\pi_m : \mathcal{S}_m \rightarrow \widehat{\mathbb{C}}$ – the

projection. Then $\mathcal{S}$ will be anchored on the bordered surface K , where K is the component of $\mathcal{S}_m \setminus \pi_m^{-1}(L)$ containing w_0 .

The definition of the logarithmic end Λ_k includes a boundary curve Γ_k connecting branch points b_k and b_{k+1} . Here we take Γ_k to be the appropriate arc(s) of L , and then $\mathcal{S}$ is completed by attaching Λ_k to K along Γ_k . For each k , let U_k be the component of $\pi^{-1}(B(b_k, 2\varepsilon))$ which intersects the logarithmic end Λ_k . Also, choose α_k so that the ray

$$S_k = \{|w - b_k| \leq 2\varepsilon, \arg w = \alpha_k\}$$

intersects the interior of K . We define the nucleus of $\mathcal{S}$ as

$$K_0 = K \setminus \bigcup_1^p U_k, \quad (26)$$

and write

$$U_k \setminus S_k = U_k^+ \cup U_k^-,$$

where U_k^+ has nonempty intersection with Λ_k , and U_k^- with Λ_{k-1} . Now define

$$\begin{aligned} \lambda_k &= \Lambda_k \setminus \pi^{-1}(B(b_k, \varepsilon) \cup B(b_{k+1}, \varepsilon)), \\ \Lambda'_k &= \lambda_k \cup (U_k^+ \cup U_{k+1}^-), \end{aligned}$$

and observe that

$$\mathcal{S} = K_0 \cup \left(\bigcup_k \Lambda'_k \right).$$

The uniformization of $\mathcal{S}$ is effected element by element. As first approximation, we map Λ'_k into the upper ζ_k -half-plane H_k . Indeed, U_{k+1}^- is mapped conformally to

$$H_k^- = \{\xi_k < -1, \eta_k > 0\}, \quad \zeta_k = \xi_k + i\eta_k,$$

using the function

$$\zeta_k = \log \frac{w - b_{k+1}}{2\varepsilon} - 1 - i\alpha_{k+1},$$

where the branch of logarithm is chosen so that S_{k+1} is mapped into the real negative axis of the ζ_k -plane. Similarly, U_k^+ may be transformed to

$$H_k^+ = \{\xi_k > 1, \eta_k > 0\}$$

using a suitable branch of

$$\zeta_k = \log \frac{2\varepsilon}{w - b_k} + 1 - i\alpha_k;$$

again chosen so that S_k is sent to the positive axis.

It is easy to overlook how the mapping is extended to the region between H_k^+ and H_k^- , but here is where the symmetry of the class (25) is significant. When it is absent, as below in Section 4.4, we find spiraling in the tracts corresponding to the various deficient values.

When $\mathcal{S}$ is the surface of e^z , the image of $H^+ \cup H^-$ covers all of the upper half-plane except a half-strip of planar density zero.

Here, to connect H_k^+ to H_k^- , we consider each $\lambda_k \setminus \Lambda_k$. If L_k is the portion of L joining b_k to b_{k+1} , then the inverse images $\pi^{-1}(L_k)$ partition λ_k into congruent quadrilaterals $\{Q_{k\ell}\}$, and in turn each may be mapped to

$$Q_k^0 = \{-1 < \xi_k < 1, 0 < \eta_k < 2\pi\},$$

by a qc homeomorphism ψ_k which is linear on each boundary segment. By letting ℓ range through the positive integers, the region λ_k is mapped in a qc rigid manner to the full strip $I = \{|\xi_k| \leq 1, \eta_k \geq 0\}$.

The map ζ_k as already defined on $U_k^+ \cup U_{k+1}^-$ maps the two components of $\partial\lambda_k$ onto the rays $\{\xi_k = -1, \eta_k \geq \Theta_{k+1}^-\}$ and $\{\xi_k = 1, \eta_k \geq \Theta_k^+\}$, where $\Theta_k^+, \Theta_{k+1}^- \geq 0$. Let T_k be the trapezoid with vertices $(\pm 1, 0)$, $(-1, \Theta_{k+1}^-)$, $(1, \Theta_k^+)$. We may then map I to $I \setminus T_k$ by a $2\pi i$ -periodic qc homeomorphism which agrees with ζ_k on $(U_k^+ \cup U_{k+1}^-) \cap \lambda_k$, and in this way extend ζ_k to Λ'_k , having range $\{\Im \zeta_k \geq 0\} \setminus T_k$.

We do this for each k , and observe that for suitable branches, the map

$$\psi : \mathcal{S} \setminus K_0 \rightarrow \mathbb{C} : \zeta = \zeta_k^{2/p} e^{2\pi i(k-1)/p},$$

may be extended to the compact kernel K_0 to give a quasiconformal uniformization of $\mathcal{S}$ onto the finite plane: $\mathcal{S}$ is parabolic.

In Section 6.4 we show how the Nevanlinna data $\{\delta(a), \theta(a)\}$ for the conformal uniformization of $\mathcal{S}$ is inherited from this computation.

(B) *Periodic and almost-periodic ends*. These were introduced respectively by Ullrich [U36a] and Gol'dberg (cf. [GO74, Chapter 7, Section 6]). In general, there is no simple analytic description of the basic building-blocks for surfaces with almost-periodic ends, so these classes will be introduced constructively. However, the precise definition of surfaces built with almost-periodic ends is too technical to be completely presented here; see [GO74, p. 503].

Consider a family $\{\mathcal{R}_k\}_{|k|=1,2,\dots}$ of Riemann surfaces of genus zero. More precisely, Riemann spheres endowed with the pull-back of the spherical metric under a rational map. For convenience suppose that $\mathcal{R}_k = \mathcal{R}_{-k}$ for each k . While the surfaces are allowed to vary, it is assumed that all branch points project to a fixed finite set $\mathcal{A} = \{a_j\}_1^q$ (of course

this is necessary for $\mathcal{S}$ to be in F_q). The ramification set of a given $\mathcal{R}_k$ may be a proper subset of $\mathcal{A}$.

Choose $b_1, b_2 \in \mathcal{A}$ (it is not excluded that these coincide). Let γ be a simple curve on $\widehat{\mathbb{C}}$ with endpoints at b_1 and b_2 , while otherwise disjoint from $\mathcal{A}$, and on each $\mathcal{R}_k$ choose a (nontrivial) curve Γ_k which projects on γ . We then consider a family of bordered surfaces $\tilde{\mathcal{R}}_k \equiv \mathcal{R}_k \setminus \Gamma_k$ (here we must assume that $\tilde{\mathcal{R}}_k$ is connected when $b_1 = b_2$), so that Γ_k has two representatives, Γ_k^+ and Γ_k^- which form the border of each $\tilde{\mathcal{R}}_k$. We glue each Γ_k^- to Γ_{k-1}^+ and Γ_k^+ to Γ_{k+1}^- ($k \neq -1$), with a natural gloss when $k = \pm 1$, obtaining a surface Ψ , with logarithmic branch points over b_1 and b_2 . Thus the curve $\Gamma_1 = \Gamma_{-1}$ divides Ψ into two components, either of which is called an *end*.

Volkovyskii [V50, p. 120] classifies these as first or second type depending on whether $b_1 \neq b_2$.

Surfaces consisting of a finite number of logarithmic ends are a special case of the construction. With just two logarithmic ends each $\{\mathcal{R}\}$ is a copy of the sphere, and Ψ is the surface of the logarithmic function.

If all $\{\mathcal{R}_k\}$ coincide, the end is called *periodic*. When $b_1 \neq b_2$, Ψ is the Riemann surface of

$$w = R(e^z), \quad (27)$$

where $R(\zeta)$ is a rational function mapping the ζ -plane to Ψ . When $R(0) = R(\infty)$, Ψ is the surface of $w = R(\sin z)$.

A surface $\mathcal{S}$ is said to consist of finitely many periodic ends if it may be decomposed as a nucleus having a finite number of (piecewise-smooth) edges, to which is attached a finite number of periodic ends.

Ullrich showed that surfaces with finitely many periodic ends are parabolic, and used these to give a partial solution to the inverse problem (cf. Section 6.2); this is not surprising since the form (27) allows the Riemann–Hurwitz formula to be applied to each function R . Indeed, a function of this form will exhibit equality in (7), with the terms $\theta(a)$ nonzero for multiple values a of $R(W)$, and the deficient values corresponding to the various possibilities of $R(0), R(\infty)$. The details are in [GO74, Section 7.6]. However, the form (27) indicates an important flexibility not present in (25). First, R is no longer a homeomorphism of $\widehat{\mathbb{C}}$, so the branching terms in (7), due to contributions from (4), will appear. In addition, we may choose R so that the rates at which R tends to $R(0)$ and $R(\infty)$ are different. This is quite different from what happens in (A), and some of its consequences will be seen in Sections 4.4 and 6.2.

Let us consider a simple example, say $R(W) = W^2/(W - 1)$. Then the function $w = f(z) = R(e^z)$ from (27) has order one, and has logarithmic branch points over ∞ and 0, while $W = \infty$ is an ordinary point of R . The regions which define the logarithmic branch points can be considered the images of the half-planes $\{\Re z > K\}$ and $\{\Re z < -K\}$ respectively for appropriate values of K . Notice, however, that the images of the boundary lines $\{x = \pm K\}$ wind about these branch points at different rates: on the right we have $d \arg w / dy = 1$, while on the left this ratio is 2. This will be reflected in the spiraling character of the two logarithmic ends in the uniformization of Ψ .

In our discussion of periodic ends, we take γ to be the real axis and let Ψ be the periodic end corresponding to the region above γ . The different rates of approach at $\pm\infty$ are captured by the notation $d^+ = 1$, $d^- = 2$.

In the case of *almost-periodic ends* the surface Ψ is constructed in the same spirit, but using more general classes of $\{\mathcal{R}_k\}$, chosen to satisfy certain stability criteria [GO74, p. 503]. Gol'dberg's original use of them is described in Section 6.2.

Analogous to the situation with logarithmic or periodic ends, an almost-periodic end is a component of $\Psi \setminus \Gamma_1$.

A surface Φ with finitely many almost-periodic ends is one which consists of finitely many almost-periodic ends fused to a nucleus.

A surprisingly recent application of these ends is in [CER93, Example 2.5], where the authors construct meromorphic functions f of any desired order $\rho \geq 1/2$ such that f'/f is never zero (they also show this cannot occur if f is transcendental and $\rho < 1/2$).

There is also the notion of doubly-periodic end and quarter-end, the former introduced by Ullrich [U36b] and Teichmüller [Te44], the latter by Künzi [Ku56]; these consist of half or a quarter of the Speiser graph of a doubly-periodic function. They can be used to create examples, but have had less impact in traditional Nevanlinna theory since surfaces which contain these as components in general will not have deficient values. A typical application (cf. [Ku55]) is to construct a surface whose uniformizing function has any prescribed order ρ , $0 \leq \rho \leq \infty$. However, while this provides an interesting use of these tools, there are far more elementary ways to construct functions of a given order, with the Lindelöf functions (see Section 6.3) being the most simple.

Huckemann [Hu56a, pp. 215–239], also saw the need to go beyond surfaces of class F_q . In the language of [N70, Section 1, Chapter XI], his surfaces $\mathcal{S}$ also allow *indirect singularities* over $w = a$. This happens when there is a decreasing chain of components $U(a, \varepsilon)$ of $f^{-1}(\{|w - a| < \varepsilon\})$ such that, for $0 < \varepsilon < \varepsilon_0$,

- (1) $U(a, \varepsilon) \neq \emptyset$;
- (2) $\bigcap_{\varepsilon > 0} U(a, \varepsilon) = \emptyset$;
- (3) the equation $f(z) = a$ always has a solution in each $U(a, \varepsilon)$.

On the other hand, there is a *direct singularity* over a if the last equation has no solution for ε sufficiently small. For example, $w = 0$ is an indirect critical value of $w = \sin z/z$, and a direct critical value of $w = e^z$. Huckemann develops a theory of line complexes for these surfaces (some anticipatory work appears in [Se54]) and in Section 6.2 we discuss one of his applications.

4.4. Spiraling

Conformal type can be controlled by other features on the Speiser graph, and one especially important method involves its asymmetry. This general principle seems to have first been observed by Blanc [Bl37], but exploited for Speiser graphs in [L-V49]; it is thoroughly considered in the thesis [V50].

We have seen in Sections 4.1 and 4.2 that the type problem may be converted to issues of mappings of strips, introducing the problem of *conformal welding* (conformal sewing).

(A) *Early work: Charles Blanc.* A remarkably early discussion of this appears in [B137, Section 18].

Suppose that $\varphi(u)$ is a homeomorphism of $[0, \infty)$, and identify $(u, 0)$ with $(\varphi(u), 1)$ on the boundary of the half-strip $\{0 \leq \Re z \leq 1, \Im z \geq 0\}$ to get a welded doubly-connected surface Σ . As suggested by Theorem 1, we may define Σ to be parabolic or hyperbolic depending on whether the modulus of all curves tending to infinity in Σ is zero or positive.

Blanc [B137, p. 363] showed that if

$$\phi(u) = ku, \quad k > 0, \quad (28)$$

then Σ is always hyperbolic except in the case $k = 1$ where it is parabolic. For $k \neq 1$, the image of the real axis is a spiraling curve in B .

We briefly outline his proof, much of which applies to more general welding functions $\phi(u)$; of course here we suppose $k \neq 1$. For $z \in \Sigma$ we want to find an analytic function $f(z) = re^{i\theta}$, so that r is constant on each line ℓ_u connecting $(u, 0)$ to $(\phi(u), 1)$. The line ℓ_u has the parametric form $x(t) = u[(k-1)t+1]$, $y(t) = t$, for $0 \leq t \leq 1$, hence along ℓ_u we have

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\partial\theta}{\partial x} \frac{dx}{dt} + \frac{\partial\theta}{\partial y} \frac{dy}{dt} \\ &= -\frac{r'}{r} \frac{\partial u}{\partial y} u(k-1) + \frac{r'}{r} \frac{\partial u}{\partial x} \\ &= \frac{r'}{r} (k-1) \frac{u^2}{x} u(k-1) + \frac{r'}{r} \frac{u}{x} \\ &= \frac{r'}{r} \frac{u}{x} [(k-1)^2 u^2 + 1], \end{aligned}$$

where in computing the partial derivatives of θ we used the Cauchy–Riemann equations for the analytic function $\log r + i\theta$ and the fact that r is a function of u , and in computing the partial derivatives of u we used the formula $u = x/[(k-1)y+1]$.

Now integrating $d\theta/dt$ along ℓ_u we find that

$$2\pi = \int_0^1 \frac{r'}{r} \frac{1}{(k-1)t+1} [(k-1)^2 u^2 + 1] dt = \frac{r'}{r} \frac{\log k}{(k-1)} [(k-1)^2 u^2 + 1].$$

So

$$(\log r)' = 2\pi \frac{(k-1)}{\log k} \frac{1}{1 + [(k-1)u]^2}, \quad (29)$$

and integrating with respect to du yields

$$\log r = 2\pi \frac{1}{\log k} \arctan[(k-1)u].$$

It is now clear that as $u \rightarrow +\infty$, $\log r$ tends to a finite constant, i.e., Σ is hyperbolic.

In addition, plugging the formula (29) for r'/r back into $\partial\theta/\partial x$ we find that on every horizontal line (x, y_0) ,

$$\frac{\partial\theta}{\partial x} = C(k) \frac{2x}{x^2 + B(k)}.$$

Hence,

$$\theta(x, y_0) \approx C(k) \log(x^2 + B(k))$$

so that the image of any ray $\{y = y_0\}$ spirals without bound as $x \rightarrow \infty$ in Σ .

Chapter 4 of [V50] carries this further, developing rather direct and explicit conditions in terms of φ at ∞ to determine type, cf. Theorems 22–24. For example, suppose the welding function $\phi(u)$ satisfies

$$\phi'(u) \downarrow 1, \quad u \rightarrow \infty,$$

and given $u_0 > 0$, let for $n \geq 1$,

$$u_n = \phi(u_{n-1}), \quad \Delta_n = u_n - u_{n-1}.$$

Then [V50, Theorem 22] Σ is hyperbolic or parabolic according to whether the series

$$\sum_1^\infty \frac{1}{\Delta_n}$$

converges or diverges.

Eremenko has pointed out that if

$$\phi(u) = u + u^a, \quad 0 < a < 1/2,$$

then ϕ induces a parabolic welding, and on the other hand, when $a > 1$ the welding is hyperbolic, so there remains a fairly big gap where the answer is not known to this day.

The example in [P53] uses the sufficient condition for hyperbolicity, given in [V50, Theorem 21]. Suppose that $\phi'(u) > 1$ ($u > 0$), and choose some fixed $u_0 > 0$. Then, setting $u_{n+1} = \phi(u_n)$, it follows that Σ will be hyperbolic if

$$\sum_{k=0}^\infty \frac{1}{\phi'(u_0)\phi'(u_1)\cdots\phi'(u_k)} < \infty. \quad (30)$$

Gol'dberg considers this problem in [Go64b] where $\phi' \neq \text{const}$, but, in a sense is “almost constant”. He presents necessary and sufficient conditions for type in terms of integrals associated with ϕ' and obtains asymptotic expansions for the mapping if ϕ'' has further restricted growth at ∞ , details which we omit.

In [Go64b] Gol'dberg applied these ideas to construct an example which denied a popular conjecture of Edrei and Fuchs: if f is an entire function of finite order ρ having $\delta(a) > 0$ for some complex number a , then

$$\log T(r, f) > (\rho/2 + o(1)) \log r, \quad r \rightarrow \infty.$$

Although this conjecture is false, we note that when equality holds in (1), then

$$\frac{\log T(r)}{\log r} \rightarrow \rho, \quad r \rightarrow \infty,$$

and it is likely a similar effect holds when f is meromorphic.

Nevanlinna sketches in [N52] an elementary approach to these ideas.

(B) *Periodic ends and asymmetry.* Surfaces having finitely many periodic ends are uniformized using arguments of the same spirit as introduced in Section 4.3(A), and new phenomena may be exhibited. Of course (27) implies that each multiple value $\{a\}$ of R propagates to an infinite sequence of points of ramification in the z -plane. Up to the appearance of the monograph [Wi55], a vast literature developed giving the uniformization of various classes of these surfaces, including precise data concerning the value-distribution of the uniformizing functions; in addition to several papers by Wittich and items in the bibliography of [Wi55], we mention interesting contributions by Künzi and Pöschl.

Quasiconformal exhaustions may be used to show that spiraling occurs in the uniformization of these surfaces $\mathcal{S}$ when *asymmetry* occurs in the line complex. We present a brief outline.

We first construct a qc homeomorphism

$$\psi: \mathcal{S} \rightarrow \mathbb{C}_\zeta$$

(the subscript indicates the coordinate system), which sends each end to a closed half-plane. However, when these half-planes are sent to sectors (so that the ψ -image of $\mathcal{S}$ lies in the plane) it will be necessary to introduce spiraling to make ψ continuous. We do not discuss the adjustments necessary to account for the nucleus since this is analogous to the situation in Section 4.3(A).

Consider a single periodic end Ψ . On the surface $\mathcal{R}$ of the rational function $w = R(W)$ (where $W = e^z$) make a cross-cut γ joining $R(0)$ to $R(\infty)$ (we assume here that they are distinct).

Let $\Gamma = \Gamma(t)$, $-\infty < t < \infty$ be the pullback of γ to the z -plane, and we may choose γ so that Γ is horizontal as $t \rightarrow \pm\infty$.

The image H_* of the upper component of $\mathbb{C}_z \setminus \Gamma$ is a periodic end Ψ . By a Möbius transformation of the dependent variable, we may assume that R fixes 0 and ∞ . Thus, using the notation from Section 4.3(B), there are positive integers d^+ , d^- so that

$$|R(W)| = C\{1 + o(1)\}|W|^{d^+}, \quad |z| \rightarrow \infty,$$

$$|R(W)| = C'\{1 + o(1)\}|W|^{d^-}, \quad |z| \rightarrow 0;$$

i.e., since $w = R(e^z)$,

$$\log |w(z)| = C_1 + o(1) + d^+(\log |z|), \quad \Re z \rightarrow +\infty, z \in \Gamma, \quad (31)$$

$$\log |w(z)| = C_2 + o(1) + d^-(\log |z|), \quad \Re z \rightarrow -\infty, z \in \Gamma. \quad (32)$$

These formula can be used to give qc homeomorphisms which will allow various ends to be fit together. First consider a single copy of H_* , and let H_+ be the upper half-plane. Each $S(r)$ for sufficiently large r meets Γ at a unique point $\Gamma(r)$ in the right half-plane. Thus for z near Γ , $\Re z$ large, we may introduce

$$\sigma^*(z) = \frac{1}{d^+} \log w(z) = s^*(z) + it^*(z),$$

and obtain, for an appropriate branch of the argument, that $\sigma^*(z) = (1 + O(1)) \log z$; in particular, $t^*(z) = 0$ for $z \in \Gamma$. A similar situation holds for $\Gamma \cap H_*$ near $-\infty$.

Thus we may define $\psi_0: H_* \rightarrow H_+$ which is quasiconformal, and in addition satisfies $\psi_0(z) \equiv \text{Sign}(\Re z) \cdot r$ when $z \in \Gamma$, $|z| = r$, for r sufficiently large. Moreover, the distortion of ψ_0^{-1} can be made to tend to zero at infinity. In fact, one can obtain

$$\mu_{\psi_0^{-1}}(z) = O\left(\frac{1}{|z|}\right).$$

Now suppose $\mathcal{S}$ consists of m periodic ends, Ψ_j ($1 \leq j \leq m$) and qc maps $\psi_j: \Psi_j \rightarrow H_j$. Here H_j is a half-plane $\{\zeta_j = s_j e^{it_j}\}$, where t_j varies over an interval of opening π , and to each j correspond d_j^+, d_j^- , as in (31) and (32). The various cuts Γ_j which form the border of each Ψ_j must be chosen so that for each j ,

$$\Gamma_j^- = \Gamma_{j+1}^+.$$

at least near $\pm\infty$.

To make a continuous global mapping we need rotation. This can be done using the various d_j^+, d_j^- . For $1 \leq j \leq m$, we first map each H_j onto a sector in the $\zeta = se^{it}$ -plane of angular opening $2\pi/m$ by

$$s = \left(\frac{d_1^+ \cdots d_j^+}{d_1^- \cdots d_{j-1}^-} \right) s_j^{2/m}, \quad (33)$$

$$t = \frac{2}{m} t_j + (j-1) \frac{2\pi}{m}. \quad (34)$$

As we pass from the ζ -sectors corresponding to H_j and H_{j+1} , the term (33) ensures that the mapping is continuous on the boundary. Finally, after applying this procedure to H_m , we find out that ζ is continuous on $\partial H_m \cap \partial H_1$; this is because our function $w = f(z)$, whose surface is $\mathcal{S}$, represents a single-valued function.

In summary, let the *characteristic* Δ of $\mathcal{S}$ be

$$\Delta = |\Delta_1|: \Delta_1 = \log \left(\frac{d_1^+ d_2^+ \cdots d_m^+}{d_1^- d_2^- \cdots d_m^-} \right), \quad (35)$$

and consider the map $(s, t) \rightarrow (\log r, \theta)$:

$$s = \left(1 + \frac{\Delta_1}{m} \right) \log r, \quad r > 1,$$

$$t = \theta + \frac{\Delta_1}{m\pi} \log r, \quad 0 \leq \theta \leq 2\pi.$$

This is clearly *qc* and induces a continuous map from $\{|z| > R\}$ to $\{|\zeta| > R\}$, at least for R large enough. The form of τ shows that, when $\Delta \neq 0$, the ζ -(radial)-sectors correspond to spiraling sectors in the z -plane. The same type of analysis shows that the order ρ of the uniformizing function f is

$$\rho = \frac{m}{2} \left(1 + \frac{\Delta^2}{m^2 \pi^2} \right). \quad (36)$$

That rearranging a Speiser graph allows ρ to increase is the key to the role these ends play in [CER93]. One source for an elementary exposition of these ideas is [KW59].

Here is a related example which sheds light on the Speiser graph of $\exp(e^z)$, which already was introduced in the discussion of Theorem 3. Take $q = 2$. The function

$$R_n(W) = \left(1 + \frac{W}{n} \right)^n, \quad W = e^z,$$

has order 1, independent of n , which is reflected in its having a symmetric graph, i.e., $d^- = d^+ = n$. By antisymmetrization (this term seems due to Wittich) we may rearrange the graph to obtain an antisymmetric surface S_n so that $d_1^- = d_2^+ = 1$, $d_1^+ = d_2^- = n$, and in this way obtain a surface with two periodic ends in F_3 with order $\rho = 1 + (\log n/\pi)^2$.

The formal limit of this process is the (parabolic) surface of

$$w = \exp(e^z),$$

of infinite order. However, the “limit” surface of the $\{S_n\}$ grows even more rapidly: it corresponds to a surface of hyperbolic type [B137], [L-V49, p. 38]. (Other early examples which show such subtleties concerning the conformal type were developed by Myrberg [My35].)

4.5. Parking–Garage surfaces

These surfaces are discussed at several junctures in [V50], as well as by many other authors. They represent various possible extensions of the class F_q . Since this is ancillary to our main themes, our approach is more anecdotal, but acknowledges many of the original authors.

We take an infinite number of copies $\{H_n\}$ of the plane, and connect them by various strategies on branching. The model is the surface $\mathcal{S}$ of

$$w = \arcsin z$$

in which we make slits on the real segments $I_- = [\infty, -1]$, $I_+ = [1, \infty]$ and attach H_{2n} to H_{2n+1} along I_+ and H_{2n} to H_{2n-1} along I_- , so that $\mathcal{S}$ has infinitely many algebraic branch points (of first order) over ± 1 , so that ± 1 are *totally ramified*. In [V50, p. 123] *et seq* are generalizations of this construction, in which ± 1 are replaced by sequences $\{a_n\}$, at times using only planes $\{H_n\}_{n \geq 0}$ where now the gamma function is a well-known model.

For a typical case (this is called class A_1 , [V50, p. 125] *et seq*), for each $n \in \mathbb{Z}$ choose real a_n subject to $a_{2n} < a_{2n \pm 1}$. On the copy of the plane H_n , construct radial cuts receding from a_n and a_{n+1} to ∞ , and connect H_n to H_{n+1} along the common slit $[a_{n+1}, \infty]$. We may also perform an analogous construction for $n \in \mathbb{N}$, the natural numbers (this is the generalization of the gamma function); in Theorem 39 [V50] shows that all such surfaces are parabolic.

Matters become more delicate if the $\{a_n\}$ are not forced to be real. For example, suppose that no a_n is zero and $\arg a_n \neq -\arg a_{n+1}$. Then we have classes $\tilde{A}_1, A_2, A'_2$; for example, A_2 corresponds to the case that $n \in \mathbb{Z}$, A'_2 when $n \in \mathbb{N}$. In this setting, we may introduce α_n , the angle from the directed ray $[a_{n-1}, a_n]$ to $[a_n, a_{n+1}]$ with $0 \leq \alpha_n < 2\pi$, and $\beta_n = 2\pi - \alpha_n$. A typical result [V50, Theorem 45, p. 138] may be quoted:

THEOREM 4. *Let $\mathcal{S} \in A'_2$, and assume that $|a_n|$ is nonincreasing. Suppose that*

$$\alpha_n \leq \beta_n, \quad \sum \alpha_n = \infty$$

and, if we define

$$d_n = \log \frac{|a_n|}{|a_{n+1}|} + \alpha_n,$$

assume that the sequence d_n decreases monotonically to zero. Then $\mathcal{S}$ will be of hyperbolic type if there is $q > 1$ so that for all n ,

$$\frac{d_0 + d_1 + \cdots + d_n}{nd_n} \geq q.$$

Necessary and sufficient type conditions are difficult in general, unless the arguments are restricted to two or three rays (in the latter case, this is the focus of his class A_3). Volkovyskii views the class A_3 as another generalization of the modular function (with $q = 3$), in which the three branch points may vary from sheet to sheet. Theorems 50, 52 and 53 give necessary type criteria for special subclasses of A_3 .

In [P53], a related construction is given. We place all branch points $\{a_n\}$ on the base surface H_0 , say all on the imaginary axis. Then for $n \neq 0$ we place a copy of a_n on H_n , and joining H_n to H_0 along a slit from a_n to $-\infty$ parallel to the negative real axis. This construction is used in a crucial way by Osserman, see Section 5.2. In several articles Potyagailo produced other type criteria.

5. Nevanlinna theory: Classical methods

5.1. On a problem of Nevanlinna

Almost from the beginning, Nevanlinna realized that his theory could formally be viewed as a transcendental substitute for the Riemann–Hurwitz formula (this is also outlined in the beginning of Chapter 12 of [N70]). This philosophy in part lay in the background to the method of [N32a], which we discuss in Section 6.2. In 1932 Nevanlinna proposed, [N32c], a problem which might reveal this connection concretely, at least for surfaces of class F_q or $\mathcal{N}$. Let $\mathcal{G}$ be a Speiser graph and V its vertices. For each $v \in V$, define the *excess* as

$$E(v) = 2 - \sum_{F: v \in \partial F} \left(1 - \frac{1}{k}\right), \quad (37)$$

where the sum is over all faces F having v on the boundary and $2k$ is the number of edges of F , so that $1 \leq k \leq \infty$. For the modular graph of valence at least 3, one sees at once that $E < 0$ at every vertex, while $E = 0$ at vertices of the Weierstrass $\wp$ -function; if $\mathcal{G}$ represents a surface of $\mathcal{N}$, then $E = 0$ except for an at most finite number of vertices v .

Now choose some base vertex $v_0 \in \mathcal{G}$ and exhaust $\mathcal{G}$ by subgraphs $\mathcal{G}_n$ of generation n , starting from v_0 . We then may define $E = E(\mathcal{G})$ as, appropriately chosen, the limit, limit superior or limit inferior, of the sequence of excesses E_n corresponding to $\mathcal{G}_n$. We refer the reader to the interesting discussion in [N70, Chapter XII, Section 1], which culminates in his “question”:

PROBLEM 1 (Nevanlinna). Is $\mathcal{G}$ being parabolic or hyperbolic equivalent to E being zero or negative?

However, the answer to this question is a rather emphatic no. While in one direction this has been known for many years [Te38, p. 257] (this is the origin of the developments culminating in what we present as Proposition 2), the reverse counterexample is very recent [Ben] and is deferred to Section 7.

THEOREM 5 (Teichmüller). *There is a hyperbolic surface whose excess is zero.*

PROOF. We record this as an immediate consequence of Lemma 3; the main ideas go back to Teichmüller’s thesis [Te38]. Again we will take a surface in the class $F_q(T)$ as described in Section 3.3, i.e., one whose Speiser graph looks like the modular graph, but where the edges of generation k are replaced by ℓ_k edges. Notice first that if the ℓ_k are increasing, and hence tend to infinity, the total excess will necessarily be 0. This is because, at each vertex v of the modular graph of generation k the excess is equal to a negative constant independent of the generation, but this constant must be averaged with the null excess at the $\ell_{k-1} - 1$ ordinary vertices which precede v . We now only need only have to choose an increasing sequence ℓ_k for which the series (24) converges, so that the corresponding surface will be hyperbolic. Thus there is a large collection of choices $\{\ell_k\}$ to give counterexamples. $\square$

5.2. An application to surfaces in $\mathbb{R}^3$

Consider the surface $\mathcal{S}$,

$$z = f(x, y), \quad (x, y) \in \mathbb{R}^2. \quad (38)$$

If f is C^2 , for example (cf. [Oss53, p. 232]), each point has a neighborhood conformally equivalent to a disk, and these neighborhoods induce a conformal structure on $\mathcal{S}$.

It was asked by Loewner in 1954 whether every such surface is parabolic. It appears (communication from Osserman) that Loewner formulated this question noting from a theorem of Bernstein that when $\mathcal{S}$ is a minimal surface over the plane, it is conformally equivalent to $\mathbb{C}$.

In his thesis, Osserman [Oss53] showed that the answer is negative. The method depends on showing that the class of functions described in (38) is essentially [Oss53, p. 221] equivalent to the class of Riemann surfaces of entire functions having no finite asymptotic values (call this class A), which is shown by a careful scissors-and-paste construction. That is, each surface of class A can be embedded in $\mathbb{R}^3$ in the form (38).

With this as background, he has a large *repertoire* of candidate surfaces $\mathcal{S}$, and he selects one due to Potyagailo [P53]. As was mentioned at the end of Section 4.5 this is a hyperbolic surface which consists of infinitely many copies of the plane with a nonsymmetric arrangement of first-order branch points on the imaginary axis.

Osserman applies his method to convert this surface to one of the form (38). This embedding is smooth at nonramified points, but by making smooth coordinate changes suitably near each branch point, so that f is smooth, he shows that the conformal type remains hyperbolic.

NOTE. After this chapter was prepared our editor R. Kühnau noted that Lavrentiev in [Lav35] constructs a hyperbolic graph. His construction is quite different than Osserman's; rather than relying on asymmetry his surface satisfies a linear isoperimetric inequality.

5.3. The Teichmüller–Wittich–Belinskii theorem

It is classical that a quasiconformal homeomorphism $w = \psi(z)$ of the plane which fixes 0 and 1 satisfies a Hölder condition:

$$a|z|^{1/K} < |\psi(z)| < A|z|^K,$$

where K depends only on the maximal dilatation of ψ . This estimate shows at once that conformal type is invariant under quasiconformal mappings.

For applications to Nevanlinna theory, a more refined estimate is needed. This was recognized early by Teichmüller [Te38] who showed that if

$$\mu(z) = \frac{\psi_{\bar{z}}}{\psi_z},$$

and $m(r) = \sup_{S(r)} |\mu(z)|$, then

$$|\psi(z)| = \{A + o(1)\}|z|, \quad z \rightarrow \infty, A \neq 0, \quad (39)$$

when $\int^\infty m(r)r^{-1} dr < \infty$.

However, this integral will diverge except in very elementary cases. In the late 1930s, Teichmüller with Wittich had shown [Wi48] that (39) holds under the weaker hypothesis

$$\int_{\{|z|>1\}} |\mu(z)| \frac{dx dy}{|z|^2} < \infty, \quad (40)$$

to which Belinskii (cf. [Bel74]) added that (40) also implied that

$$\arg \psi(z) - \arg z \rightarrow \alpha$$

for some fixed α .

One further advance was needed. Since Nevanlinna theory is based on the exhaustion of the plane by circles, condition (39) may give more information than needed, and later constructions make the condition (40) unwieldy and awkward to verify. In particular, any construction which involves an infinite number of stages will challenge (40), see Section 6.4. However we have [DW75].

LEMMA 4. *Let $w = \psi(z)$ be a quasiconformal homeomorphism of the finite plane. Suppose that*

$$\int_0^{2\pi} |\mu(re^{i\theta})| d\theta \rightarrow 0, \quad r \rightarrow \infty. \quad (41)$$

Then, for $1/2 \leq |\tau| \leq 2$, we have uniformly as $z \rightarrow \infty$ that

$$\frac{\psi(\tau z)}{\psi(z)} = \{1 + o(1)\}\tau.$$

The proof of this lemma is now extremely elementary, as it follows from normal family considerations. Its value is that circles still correspond to “asymptotic” circles, and for functions of finite order with rather regular growth, the Nevanlinna data from the quasiregular model g will transfer to the meromorphic function f which is obtained by solving the equation (9). The more refined conclusion which follows from (40) is rarely needed, although it is natural when studying functions in a sector (cf. [DH84]).

NOTE. After this chapter was prepared, the authors realized that Lemma 4 was understood by Teichmüller [Te38, p. 662], and applied by Huckemann [Hu56b, p. 255]. However, these authors only considered functions of finite order.

6. Nevanlinna theory: Modern developments. Miscellany

6.1. Nevanlinna's inverse problem

We have already referred to the issue of the precision of (1) in Section 1.2. We state it as the following problem

PROBLEM 2. Given data $\{\delta_n, \theta_n, a_n\}$ with $\delta_n \geq 0$, $\theta_n \geq 0$, $0 < \delta_n + \theta_n \leq 1$, consistent with (7), find a meromorphic function f with $\delta(a_n) = \delta_n$, $\theta(a_n) = \theta_n$ and $\delta(a) = \theta(a) = 0$ for $a \notin \bigcup_n a_n$.

This issue was raised early by Nevanlinna, but it is treated as the main theme in [N32a] and [A32], where an important partial solution was obtained (cf. Theorem 2): equality can hold in (1) for any desired finite set of data $\{\delta(a)\}$ so long as each $\delta(a)$ is rational.

This led to a long literature having this and the type problem as subject, usually using Speiser graphs. In retrospect, this had the unfortunate effect concentrating attention to the class $\mathbb{S} = \bigcup_q F_q$ as basis for examples.

The full problem was solved much later [Dras76]. A far shorter proof of a special case where equality holds in (1) is in [Dras98]. Chapter 7 of [GO74] presents a thorough account of the situation up to [Dras76], and in particular gives a patient and completely self-contained account of the relevant material on quasiconformal mappings and versions of Teichmüller's theorem which had been used.

The situation for entire functions is often simpler. The restricted problem of finding an entire function f with given deficiencies subject to the inequality

$$\sum_{a \in \mathbb{C}} \delta(a) \leq 1 \tag{42}$$

was settled around 1960 by Fuchs and Hayman (cf. Chapter 4 of [Hay64]); their construction always gives functions f of infinite order. Interestingly, to prove that f solves this restricted problem, the authors showed in fact that their solution f is extremal for a variant of (1), in the sense that

$$\sum \delta(a_n, f) + \delta(a(z), f) = 2,$$

where $a(z) \equiv z$. This solution f is given by an explicit formula, but from the beginning it seems to have been understood that more powerful methods are required for the general meromorphic case.

Once we allow infinitely many nonzero terms $\{\delta(a)\}$, the problem inverse to (1), (7) cannot, in general, be solved using functions of finite order ρ . This was recognized already by Teichmüller in his pioneering article [Te39] (although he offered no proof); Weitsman [We72] showed that if $\rho < \infty$ then the deficiencies satisfy the sharp condition

$$\sum_a \delta(a)^{1/3} < \infty, \tag{43}$$

a conclusion stronger than conjectured by Teichmüller. To see that Weitsman's condition (43) is sharp we note that Eremenko [Er86] showed that the inverse problem for meromorphic functions of finite order may be solved for any data $\delta_n = \delta(a_n, f)$ such that $0 \leq \delta_n \leq 1$, $a_n \in \overline{\mathbb{C}}$, which satisfy (1) and (43). His proof makes efficient use of quasiconformal modifications and of the Lindelöf ends discussed below.

All work subsequent to [N32a] used quasiconformal mappings. In view of the recent work of Yamanoi [Y1, Y2] it is natural to ask for solutions with the $a(z)$ preassigned small functions. Once nonconstant functions $a(z)$ are allowed as targets, quasiconformal changes of variable seem not useful: $a((1 + o(1))z) \neq a(z)$!

6.2. Inverse problem: Earlier work

We recall the history given in [GO74]. The first general solutions are due to Nevanlinna [N32a], which settled the restricted problem of finding entire function with a given finite number of rational deficiencies satisfying (14). Nevanlinna's solution used his surfaces of class $\mathcal{N}$ having only logarithmic branch points, while Elfving [El34] showed how the theory is preserved if a finite number of algebraic branch points is also allowed.

Ullrich in [U36a] generalized the Nevanlinna–Elfving class to surfaces $\mathcal{S}$ whose Speiser graph, outside a finite portion (nucleus), consists of a finite number of logarithmic and periodic ends. The most familiar example of this class is $w = \sin z$, which exhibits equality in (7) but not (1).

By extending the approach of [L-V49] to include his almost periodic ends, Gol'dberg was able to settle the general inverse problem with data satisfying $\sum^N \delta_n(a_n) + \theta_n(a_n) < 2$, $N \leq N_0 < \infty$, see [GO74].

Independent of Gol'dberg [Go54], Huckemann [Hu56b] used his more general classes of Riemann surfaces, which were mentioned in Section 4.3, to shed light on the inverse problem. His building blocks are, in addition to logarithmic ends, what are called “simple” or “multiple” sine ends, in which the algebraic branch points are allowed to approach a given transcendental singularity a . The rate of convergence is measured by what Huckemann calls the *Wachstumsstärke*, and in turn this will control $\delta(a)$. Using this class, Huckemann solves the restricted problem

$$\sum \delta(a_j) < 2$$

for a finite number of data $\{a_j, \delta_j\}$ with the $\{a_j\}$ real.

Before discussing [Dras76] in Section 6.5 we need one more background consideration.

6.3. The deficiency problem

Nevanlinna theory asks for far more precise information than required to determine the type. And the full solution to Problem 2 needed notions which are irrelevant for the type problem, but are natural when considering (1) and (7), especially under restrictions on the order (8). One example was introduced in Section 5.3. The other we state as (the largely open):

PROBLEM 3 (Deficiency problem [Ed]). Given $0 \leq \rho < \infty$ as in (8), determine the precise bounds $\Lambda_1(\rho)$ (respectively $\Lambda_2(\rho)$) so that if f is meromorphic in the plane (resp. entire), then

$$\sum \delta(a) \leq \Lambda_1(\rho)(\Lambda_2(\rho)).$$

Characterize cases of equality.

It is in principle possible to raise the same problem in terms of the possible sum $\sum \delta(a) + \theta(a)$, the sum over functions of a given order, but, as we indicated in Section 1.2 the sharp bound is always two. Of course when ρ is a positive integer, $\Lambda_1(\rho) = \Lambda_2(\rho) = 2$, and Nevanlinna in [N32a], following the path initiated by Hille and Nevanlinna, showed that $\Lambda_1(\rho) = 2$ when 2ρ is an integer.

Although the classes of surfaces used to study the type problem contained those of functions of any given order $\rho \in (0, \infty)$, one of the earliest examples studied by Nevanlinna in [N29], and considered “classical” already at that time, was not readily included: the surfaces corresponding to the entire function $L(\rho)$, the Lindelöf functions. They are not of class $\mathbb{S}$, cf. (15), and in fact the Riemann surfaces can be complicated to describe. However, they are defined as simple canonical products

$$L(\rho) := \prod \left(1 - \frac{z}{a_n}\right) e^{(z/a_n) + \dots + (z/a_n)^p/p}, \quad p = [\rho] < \rho,$$

with zeros on a ray $\arg z = \alpha$, whose counting function $n(r, 0)$ satisfies

$$n(r, 0) \sim cr^\rho, \quad c > 0,$$

for a given $0 < \rho < \infty$ (it is possible to extend this to $\rho = 0, \infty$ in several ways). Note that when $\rho < 1$ there are no exponential convergence factors needed in this canonical product. When $\rho = 1/2$, this gives the surface of $\cos \sqrt{z}$, which is in F_3 , but, for example, when $0 < \rho < 1/2$, the critical points form sequences on the positive and negative real axis tending to ∞ , while for $1/2 < \rho < 1$ they approach the origin alternating from the right and left.

While the combinatorial description of these surfaces is a bit cumbersome, there is a simple asymptotic expansion for their uniformizing functions (cf. Chapters 1 and 2 of [Le80]):

$$\log |f_\rho(re^{i\theta})| = (1 + o(1))r^\rho h_\rho(\theta), \quad (44)$$

where $h_\rho(\theta)$ is (up to a positive multiple) the unique 2π -periodic solution of the equation

$$h''(\theta) + \rho^2 h(\theta) = 0, \quad \theta \neq 0,$$

whose maximum is 1, is continuous on $[-\pi, \pi]$ and whose derivative has a positive jump at $\theta = \alpha$ (of course it is necessary to exclude small circles about the zeros $\{a_n\}$ in this formula).

Thus we may choose $\alpha = \alpha(\rho)$ so that the trigonometric functions

$$(-1)^{2n-1} \cos n\theta$$

are limiting cases of the $\{h_\rho\}$ as $\rho \rightarrow n$. It is routine to check that $\alpha = \pi$ for $2n < \rho < 2n + 1$ and $\alpha = 0$ for $2n + 1 < \rho < 2n + 2$ are solutions.

The magnitude of the jumps of h'_ρ at $\theta = 0$ show an explicit relation between the density of zeros of the corresponding f_ρ : if $\max_\theta h_\rho(\theta) = 1$, then

$$n(r, 0) = (1 + o(1)) \frac{|\sin \pi \rho|}{\pi} r^\rho.$$

Notice that in terms of these graphs, the family $\{f_\rho\}$ as ρ varies “interpolates” the exponential functions of order $\rho = n \in \mathbb{Z}$; see Figure 3 for the range $1 \leq \rho \leq 2$.

The significance of these functions to the deficiency problem is that they provide the extremals for $1/2 < \rho < 1$ (when $\rho < 1/2$, $\Lambda_1(\rho) = \Lambda_2(\rho) = 1$, with, for example, any entire function of order ρ an extremal). For entire functions, the solution for $\rho \in (1/2, 1)$ is due to Edrei and Weitsman (cf. [EW68]), while for meromorphic functions (cf. [Ed73]) it is due to Edrei and Baernstein: it was for this purpose that Baernstein developed his star function, which has had deep consequences in many areas of analysis, and which is discussed elsewhere in these volumes [Ba02].

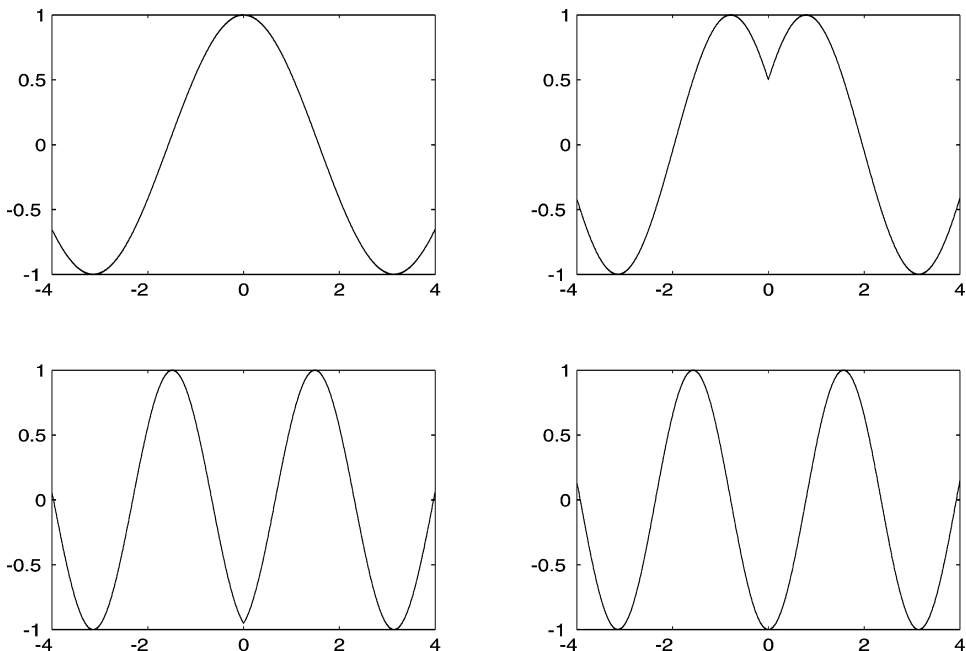


Fig. 3. Four graphs of $h_\rho(\theta)$: $\rho = 1, 1.3, 1.8, 2$.

6.4. Lindelöf ends and the solution

Drasin and Weitsman [DW75] noticed that Lindelöf functions were not part of the normal repertoire of Riemann surfaces used for constructions as in Section 4.3, hence developed a theory of “Lindelöf ends”. In general these ends have either 0 or ∞ as an indirect critical singularity.

This theory was compatible with the existing procedures as, for example, outlined in Section 4.3(A). In that section, we used as building block the functions e^z in the half-planes in which $\log |e^z| = x = r \cos \theta \neq 0$. [DW75] uses the sectors in which $\log |f_\rho(z)|$ is of one sign, and the “ $o(1)$ ” “error term” in (44) can be removed using qc compositions with fixed small dilatation so that these ends fit together.

For entire functions this generalization offers nothing new, since the product representation of f_ρ is already so explicit, but for meromorphic functions this is a class of any given order ρ which, as 2ρ approaches an integer greater or equal to 2, will in a natural way approach a meromorphic function for which (14) holds in (1). Drasin and Weitsman conjectured that this class gives the solution $\Lambda_1(\rho)$ to the deficiency problem for $\rho > 1$, $\rho \notin \mathbb{Z}$, but there is no confirmation of this for any such ρ (indeed, the presumably “simpler” problem to identify $\Lambda_2(\rho)$ is completely open for $\rho > 1$, $\rho \notin \mathbb{Z}$). One consequence of the approach in [DW75] is that it is likely that $\Lambda_2(\rho) = \Lambda_1(\rho)$ for certain ρ -intervals about each integer n .

Drasin used this type of ends in [Dras76]; they were the missing ingredient for the complete solution to the inverse problem. To simplify the discussion, we first consider only entire functions, and return to the general inverse problem in Section 6.5. The construction will introduce an increasing unbounded number of tracts in which f is allowed to tend to the various deficient values. In general, that will force the solution f to have infinite order (to be expected in view of [We72]). However, in certain situations, the order can be reduced. For example, the solution to the restricted problem $\sum \theta(a) \leq 2$ may be constructed to have order zero (for earlier results see [L-V49]).

Here is the principle of the construction. Suppose we wish to solve the restricted inverse problem (1) for entire functions. If we consider

$$f_n(z) = \exp(z^n),$$

it is clear that the plane divides into $2n$ sectors of equal angular measure, in which f_n tends alternately to 0 and ∞ . By the pivoting argument given above in Lemma 2 (cf. [Dras86]) one can make a quasiconformal modification of f_n and so obtain first a quasiregular map g_n and then using (9) and (10) obtain an entire function F_n for which the n tracts corresponding to 0 are sent to preassigned targets $\{a_j\}$, $1 \leq j \leq n$. Therefore we may solve the problem (14) with data $\delta(\infty) = 1$, $\delta(a) = p(a)/n$, where the $p(a)$ are nonnegative integers. The full solution F is a limiting case of this sort of construction, in which the plane is exhausted by concentric annuli $\mathcal{A}_n$, with modulus $M(\mathcal{A}_n) \rightarrow \infty$, where g is well approximated by an appropriate g_n . However, these annuli must be separated by annuli $\mathcal{B}_n$, since the asymptotic expansion of F_n is not continuous in n , and we will see that in general $M(\mathcal{B}_n)$ is forced to tend to infinity with n . In particular we have $F = F_n$ only on $\mathcal{A}_n$.

Quasiconformal modifications of the Lindelöf ends will enable us to “interpolate” these $\{F_n\}$ in the annuli $\mathcal{B}_n$ (in [GO74, Chapter 1, Section 5], this is almost achieved, except that each order $\rho = n$ forms a barrier since what is constructed is a single canonical product determined by the zero-counting function $n(r, 0)$ rather than the graphs $\{h_\rho(\theta)\}$ introduced in Section 6.3).

The conformal moduli of the $\{\mathcal{A}_n\}, \{\mathcal{B}_n\}$ are forced to be large (and unbounded as $n \rightarrow \infty$) from the need to control (41). The contributions to the dilatation of the quasiregular function g constructed by this procedure arise from allowing the local order ρ to change, as well as from moving the targets from $w = 0$ to the various a_j , each time Lemma 2 is applied. However, the value of the integral in (41) may be diminished for $z \in \mathcal{A}_n$ using several tools, all of which force the various moduli to grow rapidly: on the one hand, the local order then varies slowly (diminishing the dilatation at each point) and, since the number of sectors increases without bound, the contribution to the integral (41) from any fixed sector will degenerate as $r \rightarrow \infty$. This is why the condition (40) is unlikely to hold as we increase each $\mathcal{M}(\mathcal{A}_n)$, although (41) still holds.

6.5. *F. Nevanlinna conjecture. Extremal functions*

Quasiconformal methods have been a key ingredient in settling the cases of equality in (1), at least for function of finite order (for infinite order, there should be many extremals, and there seems no simple way of describing them).

In order to describe the solutions, we first consider the situation for entire functions, where $f_n(z) = \exp(z^n)$ shows equality possible whenever $n \in \mathbb{Z}$. The are two modifications of the $\{f_n\}$ which do not affect this property: we may multiply f_n by a function of smaller growth (even a canonical product of order n , minimal type), or make quasiconformal modifications of the target value $w = 0$ to obtain up to n deficient values.

For meromorphic functions, there are extremals f_ρ of each order ρ with 2ρ an integer at least two; these were discovered by Nevanlinna, following work of Hille, and are discussed in [N70]. These f_ρ may be modified in similar ways to achieve additional solutions to (14) in (1) of order $n/2$.

Drasin [Dras87] showed that (14) can hold with $\rho < \infty$ only when 2ρ is an integer at least two, and that the solution must be (modulo the type of modification just discussed) based on the $\{f_\rho\}$ discovered by Nevanlinna. Soon after, Eremenko [Er93] established this deep theorem:

THEOREM 6. *Let f be meromorphic in the plane of order $\rho < \infty$, and suppose the ramification term*

$$R(r, f) = N(r, 0, f') + N(r, f) - \overline{N}(r, f)$$

satisfies

$$R(r, f) = o(T(r, f)).$$

Then 2ρ is an integer at least two and (14) holds.

It is remarkable that this theorem was conjectured by Frithiof Nevanlinna in 1929 [fN29]. For entire functions Theorem 6 subject to ρ being an integer can be found in [EF59a] and [EF59b].

Nevanlinna's original calculus had shown that the hypotheses of Theorem 6 hold whenever f has finite order and $\sum \delta(a) = 2$. Thus [Er93] strengthens [Dras87], and his proof is independent of the results in [Dras87]. However, in both papers the spirit is in a sense inverse to that of Section 6.4. In that section we used quasiconformal techniques to spray the deficient and ramified values among the chosen targets $\{a_n\}$. In the case at hand, there is the opposite goal: to collapse the deficient values to as small a set as possible.

For entire functions, this is by now elementary, although much of the now standard techniques were introduced to realize this program. Most simply, if f is entire of finite order, then $f'(z)$ will have the same order as f , and will have $\delta(0) = \delta(\infty) = 1$. Since this suggests that f' has few zeros (and no poles when f is entire), it is possible to construct a product $p(z)$ with the same zeros as f' and then analyze

$$g(z) = \frac{f'(z)}{p(z)},$$

in an annulus $\mathcal{A}$ (with $M(\mathcal{A})$ large), where g is free of zeros and poles. Thus one can expand g in a Laurent series (with perhaps an inessential $\log z$ term) and read all asymptotic information from the Laurent coefficients: in fact, near $S(r)$

$$\log |g(z)| = \{1 + o(1)\} r^N \cos(\theta - \theta_N),$$

for an appropriate θ_N , from which we can deduce at once that the order of g is N .

In the meromorphic case it is necessary as in Section 6.4 to “double” the surface (that is, again consider $f(z^2)$). This ensures that the number of tracts for which the sets $\{|f(z) - a| < \varepsilon\}$ meet any circle $S(r)$ will always be even, and by appropriately transforming the values a associated to these tracts alternately to $w = 0$ and $w = \infty$, one arrives at a function g which in an annulus $\mathcal{A}$ (with $M(\mathcal{A})$ large) is single-valued and free of zeros and poles. Thus one can introduce $\log g$ as in the entire case and find that g has order N (and so f has order $N/2$).

7. Some recent advances

There have been other methods introduced to treat the topics considered in this chapter, and we mention some of these in this section. We conclude with a few open questions.

(A) *Graph theory.* We first return to the Nevanlinna problem introduced in Section 5.1, Problem 1.

Thus far Speiser graphs have been used to visualize a given surface $S \in F_q$. Given that graphs carry their own potential theory, it is natural to consider relations between parabolicity and hyperbolicity in these contexts, see for instance the book [DS84].

We recall the definition of a superharmonic function on a graph $\mathcal{G}$. Let u be a function defined on the vertices $V(\mathcal{G})$ of $\mathcal{G}$, and for each vertex v let $B(v)$ be the $\deg_v \mathcal{G}$ vertices w

which are linked to v by one edge of $\mathcal{G}$ (thus $B(v)$ is the $\mathcal{G}$ -ball of radius 1). We define the (graph-theoretic) Laplace operator as

$$\Delta u(v) = \frac{1}{\deg_v \mathcal{G}} \sum_{B(v)} (u(w) - u(v)), \quad v \in V(\mathcal{G}), \quad (45)$$

and u is (super)harmonic if $\Delta(u)(\leq) = 0$ on $\mathcal{G}$.

The hyperbolic–parabolic dichotomy in the graph-theoretic context asks whether $\mathcal{G}$ supports a positive nonconstant superharmonic function. This is formally precisely the classical test in Riemann surface theory, and is equivalent to the transience–recurrence of simple random walk on $\mathcal{G}$ [Ka49].

However, hyperbolic surfaces $\mathcal{S}$ may have parabolic Speiser graphs $\mathcal{G}$. To remedy this, Doyle [Do84] introduced the *McKean–Sullivan* random walk on $\mathcal{G}$. This is equivalent to simple random walk on $\mathcal{G}_1$, a certain extension of $\mathcal{G}$. Doyle shows that the type criteria for a surface $\mathcal{S} \in F_q$ coincides with the type as determined using recurrence on the extended graph $\mathcal{G}_1$. In his thesis [Me03], Merenkov presents a more function-theoretic interpretation of this modification and uses extremal length to show its link to the classical-type problem.

This concept leads to a striking example [Me03] related to Nevanlinna’s Problem 1, see also [Ben] for a different example.

THEOREM 7. *There is a parabolic surface $\mathcal{S}$ with negative excess.*

Thus there is no equivalent relation between excess and conformal type, however, Oh [Oh04] has shown that these notions coincide with certain additional conditions on the geometry of the building-blocks of $\mathcal{S}$, which are nearly best possible.

The surface $\mathcal{S}$ is based on an appropriate Speiser graph $\mathcal{G}$. First consider $\mathcal{G}_3(\infty)$ corresponding to the 3-regular modular graph. Let $\gamma = \langle v_0, v_1, \dots \rangle$ be an infinite simple path in $\mathcal{G}_3(\infty)$ and $\Gamma \supset \gamma$ consist of vertices $v \in \mathcal{G}_3(\infty)$ with (combinatorial distance) $d(v, v_n) \leq n$ for all sufficiently large n , together with the edges linking these vertices. Vertices of Γ that lie on only one edge are called *leaves*. Then make a Speiser graph $\mathcal{G}^0$ by replacing each vertex by a hexagon, and adding two additional edges interior to each hexagon associated to a leaf so that $\mathcal{G}^0$ has degree three at each vertex.

In order to exploit the equivalence of type criteria for $\mathcal{S}$ and an appropriate graph, Merenkov introduces generalizations $\mathcal{G}_k$ ($k = 1, 2, 3, \dots$) of the extended graph $\mathcal{G}_1$ and uses the case $k = 4$. More precisely, he deals with the dual graph $\mathcal{G}_4^*$ (the dual graph arises on interchanging vertices and faces while preserving adjacency relations). This graph is parabolic, but $E(v) < 0$ at all vertices v other than those corresponding to hexagons paired with leaves, where two have excess zero and two have excess $1/2$. A direct computation establishes that the mean excess is negative.

(B) *Ahlfors’s five-islands theorem.* Picard’s theorem that a nonconstant meromorphic function can omit at most two values $a \in \overline{\mathbb{C}}$ can also be viewed as a type criterion for surfaces spread over the sphere. A more refined result is Nevanlinna’s 5-values theorem:

A nonconstant meromorphic function can have at most four totally ramified values.

(The value $a \in \overline{\mathbb{C}}$ is *totally ramified* if f' vanishes at every preimage of a under f .) There is a doubly periodic function with exactly 4 totally ramified values.

Ahlfors obtained a great generalization of Nevanlinna's result as a consequence of his theory of covering surfaces, see [A35b] or [A82]:

A nonconstant meromorphic function can have at most four disjoint totally ramified image Jordan domains.

(An *image Jordan domain* D is a Jordan domain in $\overline{\mathbb{C}}$, and it is *totally ramified* if there are no components U of $f^{-1}(D)$ which are mapped one-to-one and analytically onto D , such components are also known as *simple islands*.) It is not known if the hypothesis "Jordan" can be dropped. For instance, consider five concentric slit annuli $D_n := \{z: 2n < |z| < 2n + 1, |\arg(z)| < \pi\}$ for $n = 1, 2, 3, 4, 5$, is it true that a nonconstant meromorphic function must have a simple island over one of the D_n 's?

In [Ber98] Bergweiler offers a new proof of Ahlfors's theorem which is based on (9) and (10). He first reproves Nevanlinna's 5-values theorem using the renormalizations techniques developed by Zalcman in the last few decades [Z98]. Thus he establishes an intermediate result where the image domains D are small disks D_ϵ . The step from small disks to arbitrary image domains is where the quasiconformal technique comes into play.

(C) *A Bloch-type criterion for hyperbolicity.* The recent work of Bonk and Eremenko [BE00] gives a different kind of condition which influences type:

THEOREM 8. *A nonconstant meromorphic function in the plane covers in a one-to-one fashion (schlichtly) spherical disks of radii arbitrarily close to $\beta = \arctan \sqrt{8} \approx 70^\circ 32'$.*

They also obtain Ahlfors' five-islands theorem as a corollary.

The proof displays a rich collection of techniques. We note especially Mori's theorem (see [A66, p. 35], this is used here probably for the first time in the context of value-distribution), and Ahlfors's generalization of Schwarz's lemma (see [A82]).

We briefly describe the main ideas of the proof. If D is a plane domain and $f: D \rightarrow \widehat{\mathbb{C}}$ is nonconstant meromorphic in D , Bonk and Eremenko define $b_f(z_0)$ at each $z_0 \in D$ as the radius of the largest open spherical disk for which there is a branch ϕ_{z_0} of the inverse f^{-1} having $\phi_{z_0}(f(z_0)) = z_0$ (at a critical point z_0 of course $b_f(z_0) = 0$). The "spherical Bloch radius" of f is

$$\beta(f, D) = \sup\{b_f(z_0); z_0 \in D\},$$

so that Theorem 8 asserts that $\beta(f, \mathbb{C}) \geq \beta$.

The authors prove this by contradiction: assume that $\beta(f, \mathbb{C}) < \beta - \epsilon$ for some nonconstant meromorphic function f . Then $\beta(f, B(R)) < \beta - \epsilon$ for every $R < \infty$ as well. However, they show that there is a function $C_0: (0, \beta) \rightarrow (0, \infty)$ such that if f is meromorphic in $B(R)$ and $\beta(f, B(R)) < \beta - \epsilon$ for some $\epsilon > 0$, then

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq C_0(\beta - \epsilon) \frac{R}{R^2 - |z|^2}, \quad (46)$$

and this yields the contradiction upon letting R tend to infinity.

Inequality (46) suggests the use of the Ahlfors's version of the Schwarz lemma, yet its proof requires considerable effort. The authors view the Riemann surface $\mathcal{S}$ of f as a metric space with the pull-back of the spherical metric under f , and study the curvature of this metric: at ordinary points z the length element $2|f'(z) dz|/(1 + |f(z)|^2)$ yields curvature $+1$; on the other hand, a branch point z_0 of order $m - 1$ (so that the local map is an m -to-one cover near z_0 , $m \geq 2$) contributes $-2\pi(m - 1)$ to the integral curvature. The interplay between positive and negative curvature is what is at stake. The ad absurdum hypothesis $\beta(f, \mathbb{C}) < \beta - \epsilon$ implies that the negative curvature is quantitatively predominant (a preliminary argument shows that it is of no loss of generality to assume that f has no asymptotic values), and hence an appeal to Ahlfors' Schwarz lemma will yield (46).

The authors develop an ingenious factorization of such an f much as in (10), but rely on a careful decomposition of $\mathcal{S}$ into spherical triangles Δ such that the singular points of $\mathcal{S}$ appear only as vertices of the triangles. This and the hypothesis on f allows them to construct a map

$$\psi : \mathcal{S} \rightarrow B$$

so that ψ is K -qc and ψ^{-1} is L -Lipschitz ($\text{Lip}(L)$), with K and L depending only on $\epsilon > 0$. To obtain the map ψ they first send $\mathcal{S}$ to a new singular surface $\mathcal{S}'$ with a bi-Lipschitz homeomorphism. This is done by replacing each spherical triangle Δ (having spherical sidelengths α_i , $1 \leq i \leq 3$) by a Euclidean triangle $\tilde{\Delta}$ having sidelengths $F(\alpha_i)$. Here F is subadditive with $F'(0) < \infty$, and a very hands-on technical argument is needed to produce such an F having the crucial property that if γ and $\tilde{\gamma}$ are corresponding angles of Δ and $\tilde{\Delta}$, then

$$\tilde{\gamma} \geq \left(\frac{1}{2} + \delta\right)\gamma,$$

with $\delta = \delta(\epsilon) > 0$. These $\tilde{\Delta}$ then fit together to produce a new surface $\mathcal{S}'$, while Ahlfors's Schwarz lemma shows that $\mathcal{S}'$ is conformally equivalent to B : moreover, the conformal map $\psi' : \mathcal{S}' \rightarrow B$ has an inverse in $\text{Lip}(L)$, $L = L(\delta)$.

Take $R = 1$, for simplicity, and observe that the map

$$\varphi : B \rightarrow \mathcal{S} \rightarrow B,$$

where the left arrow is given by the identity map f_1 from B as metric space with hyperbolic distance dist_h to $\mathcal{S}$, and the right arrow by ψ , is a selfmap of B (one may assume that $\varphi(0) = 0$). Then Mori's distortion theorem [A66] shows that,

$$\text{dist}_h(0, \varphi(z)) \leq 43r^{1/K}, \quad z \in B(r), 0 \leq r \leq 32^{-K}.$$

In turn ψ^{-1} being L -Lipschitz shows that $f_1 = \psi^{-1} \circ \phi$ maps such $B(r)$ onto a spherical disk $B(f_1(0), 43Lr^{1/K})$, so that at the end we find that the image of $B((32L)^{-K})$ under f is contained in a hemisphere with center $f(0)$. Now Ahlfors' Schwarz lemma yields

a bound on the “derivative” at zero for f when considering the hyperbolic metric in the domain and the spherical metric in the codomain and this gives (46).

(D) *Other examples.* Langley has produced interesting examples of meromorphic functions using quasiconformal modifications. In particular, [Lan92] produces composite entire functions of finite order having a preassigned infinite sequence $\{a_n\}$ among its deficient values. The paper [Lan98] finds nontrivial examples of Bank–Laine functions of finite order. We recall that an entire function f is Bank–Laine if $f'(z) = \pm 1$ whenever $f(z) = 0$. These functions have a significant relationship with solutions to the differential equations introduced at the end of Section 2.2. The variant here is that in the equation

$$w''(z) + A(z)w = 0,$$

we assume that A is transcendental. There are linearly independent solutions f_1, f_2 such that $E(z) = f_1 f_2$ is Bank–Laine, and the conjecture is that if the Nevanlinna counting function $N(r, E)$ is of finite order, then A has order $n \in \mathbb{Z}$. The rigidity condition defining Bank–Laine functions makes them hard to construct once some elementary cases are enumerated (e^z and $\sin z$ being the most familiar).

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NOTE ADDED IN PROOFS (October 2004). We have learned from Mario Bonk that Charles Chrissman has proved that the welding $\phi(u) = u + u^a$ gives a hyperbolic welding of the strip whenever $a < 1/2$ (see Section 4.4(A)). Thus only the case $a = 1/2$ remains open.

References

- [A30] L. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildungen und der ganzen Funktionen*, Acta Soc. Sci. Fenn. (N.S.) **1** (9) (1930) 1–40.
- [A32] L. Ahlfors, *Über eine in der neueren Wertverteilungstheorie betrachtete Klasse transzendenter Funktionen*, Acta Math. **58** (1932), 375–406.
- [A35a] L. Ahlfors, *Sur le type d'une surface de Riemann*, C. R. Acad. Sci. Paris **201** (1935), 30–32.
- [A35b] L. Ahlfors, *Zur Theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.
- [A66] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton (1966).
- [A73] L. Ahlfors, *Conformal Invariants*, McGraw-Hill, New York (1973).
- [A82] L. Ahlfors, *Collected Papers*, Vol. 1, Birkhäuser, Boston (1982).

- [AB60] L. Ahlfors and L. Bers, *Riemann mapping theorem for variable metrics*, Ann. of Math. **72** (2) (1960), 385–404.
- [Ba73] A. Baernstein, *Proof of Edrei's spread conjecture*, Proc. London Math. Soc. (3) **26** (1973), 418–434.
- [Ba02] A. Baernstein, *The $\ast$ -function in complex analysis*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, North-Holland, Amsterdam (2002), 229–271.
- [Bel74] P.P. Belinskii, *General Properties of Quasiconformal Mappings*, Nauka, Siberian Section, Novosibirsk (1974) (in Russian).
- [Ben] I. Benjamini, S. Merenkov and O. Schramm, *A negative answer to Nevanlinna's type question and a parabolic surface with a lot of negative curvature*, Proc. Amer. Math. Soc. **132** (2004), 641–647.
- [Ber98] W. Bergweiler, *A new proof of the Ahlfors five islands theorem*, J. Anal. Math. **76** (1998), 337–347.
- [BI37] Ch. Blanc, *Les surfaces de Riemann des fonctions méromorphes*, Comment. Math. Helv. **9** (1937), 193–216, 335–363.
- [BE00] M. Bonk and A. Eremenko, *Covering properties of meromorphic functions, negative curvature and spherical geometry*, Ann. of Math. **152** (2) (2000), 551–592.
- [BJ98] M. Brakalova and J. Jenkins *On solutions of the Beltrami equation*, J. Anal. Math. **76** (1998), 67–92.
- [CER93] J. Clunie, A. Eremenko and J. Rossi, *On equilibrium points of logarithmic and Newtonian potentials*, J. London Math. Soc. **47** (2) (1993), 309–320.
- [Da88] G. David, *Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 25–70.
- [DS84] P. Doyle and J.L. Snell, *Random walks and electric networks*, Carus Math. Monogr., Vol. 22, Math. Assoc. Amer., Washington, DC (1984).
- [Do84] P. Doyle, *Random walk on the Speiser graph*, Bull. Amer. Math. Soc. **11** (1984), 371–377.
- [Drap36] E. Drape, *Über die Darstellung Riemannscher Flächen durch Streckenkomplexe*, Deutsche Math. **1** (1936), 805–824.
- [Dras76] D. Drasin, *The inverse problem of the Nevanlinna theory*, Acta Math. **138** (1976), 83–151.
- [Dras86] D. Drasin, *On the Teichmüller–Wittich–Belinskii theorem*, Result. Math. **10** (1986), 54–65.
- [Dras87] D. Drasin, *Proof of a conjecture of F. Nevanlinna concerning functions which have deficiency sum two*, Acta Math. **158** (1987), 1–94.
- [Dras98] D. Drasin, *On Nevanlinna's inverse problem*, Complex Var. (Gol'dberg issue) **37** (1998), 123–143.
- [DH84] D. Drasin and W.K. Hayman, *Value-distribution of functions meromorphic in an angle*, Proc. London Math. Soc. (3) **48** (1984), 319–340.
- [DW75] D. Drasin and A. Weitsman, *Meromorphic functions with large sum of deficiencies*, Adv. Math. **15** (1975), 93–126.
- [Ed] A. Edrei, *Sums of deficiencies of meromorphic functions*, J. Anal. Math. **19** (1967), 53–74.
- [Ed73] A. Edrei, *Solution of the deficiency problem for functions of small lower order*, Proc. London Math. Soc. (3) **26** (1973), 435–445.
- [EF59a] A. Edrei and W.H.J. Fuchs, *On the growth of meromorphic functions with several deficient values*, Trans. Amer. Math. Soc. **93** (1959), 292–328.
- [EF59b] A. Edrei and W.H.J. Fuchs, *Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes*, Comment. Math. Helv. **33** (1959), 258–295.
- [EW68] A. Edrei and A. Weitsman, *Asymptotic behavior of meromorphic functions with extremal deficiencies*, Bull. Amer. Math. Soc. **74** (1968), 140–144.
- [El34] G. Elfving, *Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung*, Acta Soc. Sci. Fenn. (N.S.) **3** (2) (1934), 1–60.
- [Er86] A. Eremenko, *Inverse problem of the theory of distribution of values for finite-order meromorphic functions*, Sibirsk. Mat. Zh. **27** (1986), 87–102 (in Russian).
- [Er93] A. Eremenko, *Meromorphic functions with small ramification*, Indiana Univ. Math. J. **42** (1993), 1193–1218.
- [EL92] A. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier **42** (1992), 1–32.
- [EM] A. Eremenko and S. Merenkov, *Nevanlinna functions with real zeros*, Manuscript.
- [Go54] A.A. Gol'dberg, *On the inverse problem of value distribution of meromorphic functions*, Ukrain. Math. Zh. **6** (1954), 385–397 (in Russian).

- [Go64a] A.A. Gol'dberg, *On the lower order of entire functions with a finite defective value*, Sibirian Math. J. **5** (1964), 54–76 (in Russian).
- [Go64b] A.A. Gol'dberg, *On the problem of conformal welding of strips*, Ukrain. Mat. Zh. **16** (1964), 586–592 (in Russian).
- [GO74] A.A. Gol'dberg and I.V. Ostrovskii, *Distribution of Values of Meromorphic Functions*, Nauka, Moscow (1974) (in Russian).
- [Gr28a] H. Grötzsch, *Über einige Extremalprobleme der konformen Abbildung*, Ber. Sächs. Akad. Leipzig **80** (1928), 367–376.
- [Gr28b] H. Grötzsch, *Über die Verzerrung bei schlichten nichtkonformen Abbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes*, Ber. Sächs. Akad. Leipzig **80** (1928), 503–507.
- [Hab52] H. Habsch, *Die Theorie der Grundkurven und das Äquivalenzproblem bei der Darstellung riemannscher Flächen*, Mitt. Math. Seminar Giessen **42** (1952), 1–51.
- [Ham02] D.H. Hamilton, *Quasiconformal welding*, Handbook of Complex Analysis: Geometric Function Theory, Vol. 1, North-Holland, Amsterdam (2002), 137–146.
- [Hay64] W.K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford (1964).
- [He56] J. Hersch, *Contribution à la théorie des fonctions pseudo-analytiques*, Comment. Math. Helv. **30** (1956), 1–19.
- [Hu56a] F. Huckemann, *Zur Darstellung von Riemannschen Flächen durch Streckenkomplexe*, Math. Z. **65** (1956), 215–239.
- [Hu56b] F. Huckemann, *Über den Einfluss von Randstellen Riemannscher Flächen auf die Wertverteilung*, Math. Z. **65** (1956), 240–282.
- [Hu61] F. Huckemann, *Bericht über die Theorie der Wertverteilung*, Jahresber. Deutsch. Math.-Verein. **64** (1961), 135–188.
- [J58] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1958).
- [Ka37] S. Kakutani, *Applications of the theory of pseudo-regular functions to the type-problem of Riemann surfaces*, Japan. J. Math. **13** (1937), 375–392.
- [Ka49] S. Kakutani, *Two-dimensional Brownian motion and the type problem of Riemann surfaces*, Proc. Japan. Acad. **21** (1949), 138–140.
- [Ka53] S. Kakutani, *Random Walk and the Type Problem of Riemann Surfaces*, Princeton Univ. Press, Princeton, NJ (1953), 95–101.
- [Ko35] Z. Kobayashi, *Theorems on the conformal representation of Riemann surfaces*, Sci. Rep. Tokyō Bunrika Daigaku Sect. A **39** (1935), 125–166.
- [Ku55] H. Künzi, *Konstruktion Riemannscher Flächen mit vorgegebener Ordnung der erzeugenden Funktionen*, Math. Ann. **128** (1955), 471–474.
- [Ku56] H. Künzi, *Zur Theorie der Viertelsenden Riemannscher Flächen*, Comment. Math. Helv. **30** (1956), 107–112.
- [KW59] H. Künzi and H. Wittich, *The distribution of the a -points of certain meromorphic functions*, Michigan Math. J. **6** (1959), 105–121.
- [Lan92] J. Langley, *On the deficiencies of composite entire functions*, Proc. Edinburgh Math. Soc. **36** (1992), 151–164.
- [Lan98] J. Langley, *Quasiconformal modifications and Bank–Laine functions*, Arch. Math. **71** (1998), 233–239.
- [Lav35] M. Lavrentiev, *Sur une classe de représentations continues*, Math. Sb. **42** (1935), 407–423.
- [LV73] O. Lehto and K. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd edn, Springer-Verlag, New York (1973) (translated from German).
- [L-V49] T. Le-Van, *Über das Umkehrproblem der Wertverteilungslehre*, Comment. Math. Helv. **23** (1949), 26–49.
- [L-V47] T. Le-Van, *Beitrag zum Typenproblem der Riemannschen Flächen*, Comment. Math. Helv. **20** (1947), 270–287.
- [Le80] B.Ja. Levin, *Distribution of Zeros of Entire Functions* (transl. from Russian), Transl. Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, RI (1980).
- [Me03] S. Merenkov, *Determining biholomorphic type of a manifold using combinatorial and algebraic structures*, Thesis, Dept. Math., Purdue University (2003).

- [My35] P.J. Myrberg, *Über die Bestimmung des Typus einer Riemannschen Fläche*, Ann. Acad. Sci. Fenn. A I **45** (1935).
- [fN29] F. Nevanlinna, *Über eine Klasse meromorpher Funktionen*, Congr. Math. Scand., Oslo (1929), 81–83.
- [N29] R. Nevanlinna, *Le Théorème de Picard–Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris (1929).
- [N32a] R. Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten*, Acta Math. **58** (1932), 295–373.
- [N32b] R. Nevanlinna, *Ein Satz über die konforme Abbildung von Riemannschen Flächen*, Comment. Math. Helv. **5** (1932), 95–107.
- [N32c] R. Nevanlinna, *Über die Riemannsche Fläche einer analytischen Funktion*, Verh. Internat. Math-Kongr., Zürich (1932), 221–239.
- [N52] R. Nevanlinna, *Über die Polygondarstellung einer Riemannschen Fläche*, Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys. **122** (1952), 1–9.
- [N66] R. Nevanlinna, *Über die Konstruktion von meromorphen Funktionen mit gegebenen Wertzuordnungen*, Festschrift Gedächtnisfeier K. Weierstrass, Westdeutschen Verlag, Köln (1966), 579–582.
- [N70] R. Nevanlinna, *Analytic Functions* (transl. from German by Ph. Emig), Springer-Verlag, Berlin (1970).
- [Oh04] B.-G. Oh, *Aleksandrov surfaces and hyperbolicity*, Thesis, Purdue University (2003).
- [Osg85] C.F. Osgood, *Sometimes effective Thue–Siegel–Roth–Schmidt–Nevanlinna bounds or better*, J. Number Theory **21** (1985), 347–389.
- [Oss53] R. Osserman, *Riemann surfaces of class A*, Trans. Amer. Math. Soc. **83** (1953), 217–245.
- [P53] D. Potyagailo, *Condition of hyperbolicity of a class of Riemann surfaces*, Ukrain. Mat. Zh. **5** (1953), 459–463 (in Russian).
- [R93] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, New York (1993).
- [Se54] P. Seifert, *Über die bei Deformationen Riemannscher Flächen mit endlich vielen Windungssorten entstehenden Randstellen*, Arch. Math. **5** (1954), 389–400.
- [Sp29] A. Speiser, *Probleme aus dem Gebiet der ganzen transzendenten Funktionen*, Comment. Math. Helv. **1** (1929), 289–312.
- [Sp30] A. Speiser, *Über Riemannsche Flächen*, Comment. Math. Helv. **2** (1930), 284–293.
- [Ste86] N. Steinmetz, *Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes*, J. Reine Angew. Math. **368** (1986), 134–141.
- [Sto38] S. Stoilow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Gauthier-Villars, Paris (1938).
- [Ta62] V.G. Tairova, *On line complexes of closed Riemann surfaces of genus zero*, Izv. Vysh. Učebn. Zaved. Matematika **4** (29) (1962), 155–160 (in Russian).
- [Ta64] V.G. Tairova, *On line complexes of some classes of closed Riemann surfaces*, Sibirsk. Math. Zh. **5** (1964), 929–951 (in Russian).
- [Te37] O. Teichmüller, *Eine Anwendung quasikonformer Abbildungen auf das Typenproblem*, Deutsche Math. **2** (1937), 321–327.
- [Te38] O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. **3** (1938), 621–678.
- [Te39] O. Teichmüller, *Vermutungen und Sätze über die Wertverteilung gebrochener Funktionen endlicher Ordnung*, Deutsche Math. **4** (1939), 161–190.
- [Te44] O. Teichmüller, *Einfache Beispiele zur Wertverteilungslehre*, Deutsche Math. **7** (1944), 360–368.
- [U36a] E. Ullrich, *Zum Umkehrproblem der Wertverteilungslehre*, Nachr. Ges. Wiss. Göttingen N.F. **1** (1936), 135–150.
- [U36b] E. Ullrich, *Flächenbau und Wachstumsordnung bei gebrochenen Funktionen*, Jahresber. Deutsch. Math.-Verein. **46** (1936), 232–274.
- [V50] L. Volkovskii, *Investigation of the type problem for a simply-connected surface*, Trudy Mat. Inst. Steklov **171** (34) (1950), 1–171 (in Russian).
- [V54] L. Volkovskii, *Quasiconformal Mappings*, Lvov University (1954) (in Russian).
- [We72] A. Weitsman, *A theorem on Nevanlinna deficiencies*, Acta Math. **128** (1972), 41–52.
- [Wi39] H. Wittich, *Über die konforme Abbildung einer Klasse Riemannscher Flächen*, Math. Z. **45** (1939), 642–668.
- [Wi48] H. Wittich, *Zum Beweis eines Satzes über quasikonforme Abbildungen*, Math. Z. **51** (1948), 278–288.

- [Wi55] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer-Verlag, Berlin (1955).
- [Y1] K. Yamanoi, *Defect relation for rational functions as targets*, Forum Math. **17** (2005), 169–189.
- [Y2] K. Yamanoi, *The second main theorem for small functions and related topics*, Acta Math. **195** (2004).
- [Z98] L. Zalcman, *Normal families: New perspectives*, Bull. Amer. Math. Soc. **35** (1998), 215–230.

CHAPTER 19

Bibliography of Geometric Function Theory

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HANDBOOK OF COMPLEX ANALYSIS: GEOMETRIC FUNCTION THEORY,
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In the following books about Geometric Function Theory (GFT) and books with some parts about GFT are listed. Also books about applications of GFT (mainly conformal mappings) and also some collections of articles or survey articles are included. Generally usual textbooks are not listed. Because we are here mainly interested in schlicht mappings also the theory of Riemann surfaces itself is not included. In the small comments only those parts of the books which concern GFT are mentioned.

- [1] W. Abikoff, *The Real Analytic Theory of Teichmüller Space*, Lecture Notes Math., Vol. 820, Springer-Verlag, Berlin–Heidelberg–New York (1980); Russian transl.: Mir, Moscow (1985).
Something about extremal quasiconformal mappings, extremal length.
- [2] M.J. Ablowitz and A.S. Fokas, *Complex Variables: Introduction and Applications*, 2nd edn, Cambridge Univ. Press, Cambridge (2003).
Also something about conformal mappings and applications: Schwarz–Christoffel transformations; mappings of curvilinear polygons involving circular arcs; modulus of a quadrilateral; Theodorsen integral equation; applications in electrostatics and hydrodynamics.
- [3] Academy of Sciences of the USSR, Siberian Branch, *Some Problems of Mathematics and Mechanics* (60th birthday of M.A. Lavrent'ev), Izd. Sibirsk. Otd. Acad. Nauk SSSR, Novosibirsk (1961) (in Russian).
Also some papers about GFT; work of M.A. Lavrent'ev, with list of his papers.
- [4] Academy of Sciences of the USSR, Siberian Branch, *Some Problems of Mathematics and Mechanics* (70th birthday of M.A. Lavrent'ev), Nauka Leningr. Otd., Leningrad (1970) (in Russian).
Also some papers about GFT; work of M.A. Lavrent'ev, with list of his papers.
- [5] *Aerodynamic Theory*, Vol. I, W.F. Durand, ed., Springer-Verlag, Berlin (1934).
Some simple applications of conformal maps.
- [6] D. Aharonov, *Special Topics in Univalent Functions*, Lecture Notes, Univ. Maryland (1971).
- [7] L.V. Ahlfors, *Variational Methods in Function Theory*, Lecture Notes by E.C. Schlesinger, Harvard Univ. (1953).
- [8] L.V. Ahlfors, *Conformal Invariants – Topics in Geometric Function Theory*, McGraw-Hill, New York (1973).
Schwarz lemma, Poincaré metric; capacity, transfinite diameter, symmetrization; harmonic measure; extremal length with many examples; univalent functions (area theorem, Grunsky and Golusin inequalities); Löwner's method; Schiffer variation; quadratic differentials; Riemann surfaces, uniformization.
- [9] L.V. Ahlfors, *Complex Analysis*, 3rd edn, McGraw-Hill, New York (1979).
Riemann mapping theorem, reflection principle, Schwarz–Christoffel formula, some simple canonical mappings of multiply-connected regions.
- [10] L.V. Ahlfors, *Möbius Transformations in Several Dimensions*, School of Math., Univ. Minnesota (1981); Russian transl.: L. Al'fors, ..., Mir, Moscow (1986).
In several dimensions: Möbius transformations, Schwarz derivative, hyperbolic geometry, groups of Möbius transformations, quasiconformal deformations.
- [11] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, NJ (1966); reprinted (1987); Russian transl.: L. Al'fors, ..., Mir, Moscow (1969).
Standard book about the theory.
- [12] L. Al'fors (Ahlfors) and L. Bers, *Spaces of Riemann Surfaces and Quasiconformal Mappings*, Izd. Inostr. Lit., Moscow (1961) (in Russian).
Collection of translated original papers in form of a booklet.
- [13] L.V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton, NJ (1960).
Also something about the corresponding principal functions and canonical conformal mappings.
- [14] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI (1990); Russian original in the 2nd edn: Nauka, Moscow (1970).
Applications: conformal mapping of a rectangle onto a half-plane, conformal mapping of a doubly-connected polygonal domain onto an annulus, two examples.

- [15] I.A. Aleksandrov, *Conformal Mapping of Simply and Multiply Connected Domains*, Izd. Tomsk. Univ., Tomsk (1976) (in Russian).
Booklet; Schwarz lemma, Riemann mapping theorem; boundary behavior; classes S and Σ , etc., area principle; mappings with quasiconformal extension; some about multiply-connected domains.
- [16] I.A. Aleksandrov, *Parametrical Extensions in the Theory of Univalent Functions*, Nauka, Moscow (1976) (in Russian).
Löwner–Kufarev differential equation for univalent mappings, also onto nonoverlapping domains; also for doubly- and multiply-connected domains; many examples in great detail; a translation would be desirable.
- [17] Yu.E. Alenitsyn and G.V. Kuz'mina, see *Extremal Problems*...
- [18] G.D. Anderson, M.K. Vamanamurthy and M.K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York (1997).
Something about quasiconformal mappings, also in n -space; Grötzsch and Teichmüller ring, the corresponding special functions (like, e.g., elliptic integrals) and distortion theorems.
- [19] C. Andreian Cazacu, C. Constantinescu and M. Jurchescu, *Probleme Moderne de Teoria Funcțiilor*, Ed. Acad. R. P. R., București (1965) (in Rumanian).
Introduction into the theory of quasiconformal mappings; boundary behavior; change of conformal modules; existence theorem for the Beltrami equation.
- [20] C. Andreian-Cazacu, A. Deleanu and M. Jurchescu, *Topologie – Categorii – Suprafețe Riemanniene*, Ed. Acad. R. S. R., București (1966) (in Rumanian).
Something about quasiconformal mappings and applications in the theory of Riemann surfaces.
- [21] V.V. Andrievskii, V.I. Belyi and V.K. Dzjadyk, *Conformal Invariants in Constructive Theory of Functions of Complex Variable*, World Federation Publisher, Atlanta, GA (1995); Russian original: Naukova Dumka, Kiev (1998).
Module of a curve family, quasiconformal mappings, Riemann mapping theorem, boundary behavior, Faber polynomials, quasidisks; applications in approximation theory.
- [22] V.V. Andrievskii and H.-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer-Verlag, New York (2002).
Conformal and quasiconformal mappings, Faber polynomials and Grunsky coefficients, quasiconformal curves, Fekete points; applications.
- [23] B.N. Apanasov, *Discrete Groups in Space and Uniformization Problems*, Math. Appl. Soviet Ser., Vol. 40, Kluwer, Dordrecht (1991); Transl. from the Russian, revised and enlarged; Russian original: *Discrete Transformation Groups and Manifold Structures*, Nauka Sibirsk. Otd., Novosibirsk (1983).
The role of quasiconformal mappings in this theory.
- [24] B.N. Apanasov, *Geometry of Discrete Groups and Manifolds*, Nauka, Moscow (1991) (in Russian).
Extension of the previous book.
- [25] *Aspects of Contemporary Complex Analysis*, D.A. Brannan and J.G. Clunie, eds, Academic Press, London (1979).
Survey articles about some topics in GFT, e.g.: Conformal mappings with quasiconformal extensions; Löwner differential equation; extreme points; extremal problems for univalent functions; boundary behavior of conformal mappings; quadratic inequalities and univalent functions; nonvanishing univalent functions; conformal slit mappings; coefficient estimates for inverses of schlicht functions.
- [26] F.G. Avkhadiev, *Conformal Mappings and Boundary Value Problems*, Kazanskiĭ fond "Matematika", Kazan (1996) (in Russian).
Many sufficient conditions for univalence of analytic functions; some applications in the theory of boundary value problems.
- [27] K.I. Babenko, *The Theory of Extremal Problems for Univalent Functions of Class S* , Proc. Steklov Inst. Math., Vol. 101, Transl. Amer. Math. Soc., Providence, RI (1975); Russian original: Nauka, Moscow (1972).
Theory of first and second variation, coefficient bodies for coefficients; contents till now not completely known and accessible.
- [28] I. Babuška, K. Rektorys and F. Vyčichlo, *Mathematische Elastizitätstheorie der ebenen Probleme*, Akademie-Verlag, Berlin (1960); Czech original: Nakl. Českoslov. Akad. Věd, Praha (1955).
Applications of conformal mappings in the theory of elasticity.
- [29] A. Baernstein II, D. Drasin, P. Duren and A. Marden, see *The Bieberbach Conjecture*.

- [30] A.F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York (1983).
Also some material about the Möbius transformations and hyperbolic geometry.
- [31] E.F. Beckenbach, see *Constructions*...
- [32] H.G.W. Begehr, *Complex Analytic Methods for Partial Differential Equations*, World Scientific, Singapore (1994).
Also Riemann mapping theorem, Green's function, Bergman(n) kernel function, Beltrami equation.
- [33] H. Begehr and R.P. Gilbert, *Transformations, Transmutations, and Kernel Functions*, Vol. 2, Longman, New York (1993).
Orthonormal expansions for some canonical quasiconformal mappings and a corresponding kernel function.
- [34] H. Behnke und F. Sommer, *Theorie der analytischen Funktionen einer komplexen Veränderlichen*, Nachdruck der 3. Auflage, Springer-Verlag, Berlin–Heidelberg–New York (1972); also “Studienausgabe” (1976).
Schwarz lemma; invariant metric; Riemann mapping theorem; boundary behavior; reflection principle; Schwarz–Christoffel integral; classical distortion theory with the area theorem; theorem of Bloch.
- [35] P.P. Belinskiĭ, *General Properties of Quasiconformal Mappings*, Nauka Sibirsk. Otd., Novosibirsk (1974) (in Russian).
Distortion theorems, behavior in an isolated point, variational method.
- [36] S. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mappings*, CRC Press, Boca Raton, FL (1992).
Something about Bergman(n) kernel, Szegő kernel, and conformal mapping.
- [37] S. Bergman, *The Kernel Function and Conformal Mapping*, Amer. Math. Soc., Providence, RI (1950); 2nd edn (1970).
The invariant metric; representation of canonical conformal mappings with the kernel function; coefficient conditions of Grunsky type; variational methods.
- [38] S.D. Bernardi, *Bibliography of Schlicht Functions*, Courant Inst. Math. Sci., New York Univ. (1966); Part II (1977); Reprinted by Mariner, Tampa, Florida (1983) (Part III added).
Bibliography; also topics in the theory of schlicht functions.
- [39] L. Bers, *Theory of Pseudo-Analytic Functions*, Lecture Notes, Inst. Math. Mech., New York Univ., New York (1953).
With relations to quasiconformal mappings.
- [40] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Wiley, New York/Chapman & Hall, London (1958); Russian transl.: Izd. Inostr. Lit., Moscow (1961).
Some concise remarks about quasiconformal mappings and applications in gas dynamics.
- [41] A. Betz, *Konforme Abbildung*, 2. Auflage, Springer-Verlag, Berlin–Göttingen–Heidelberg (1964).
Elementary introduction into the theory of conformal mappings with many examples.
- [42] L. Bieberbach, *Einführung in die konforme Abbildung*, 6. Auflage, de Gruyter, Berlin (1967); English transl. of the 4th edn: *Conformal Mapping*, Chelsea, New York (1953).
Booklet; some elementary mappings, Riemann mapping theorem, boundary behavior, Schwarz reflection principle, some simple distortion theorems for classes S and Σ , Löwner theory, something about non-schlicht mappings, uniformization, canonical mappings of multiply-connected domains.
- [43] L. Bieberbach, *Lehrbuch der Funktionentheorie*, Bd. II: *Moderne Funktionentheorie*, 2. Auflage, Teubner, Leipzig (1931); Reprinted by Johnson Reprint Corp., New York (1968).
Riemann mapping theorem; boundary behavior; Schwarz–Christoffel formula; elliptic modular function; hyperbolic metric in the disk; Schwarz lemma; Löwner lemma; elementary distortion theorems for schlicht functions; uniformization with conformal welding.
- [44] *The Bieberbach Conjecture*, Proc. Symp. Occasion of the Proof, A. Baernstein II, D. Drasin, P. Duren and A. Marden, eds, Amer. Math. Soc., Providence, RI (1986).
A collection of survey articles about GFT.
- [45] M. Biernacki, *Les fonctions multivalentes*, Hermann, Paris (1938).
Some criteria and coefficient inequalities for multivalent functions.
- [46] D.E. Blair, *Inversion Theory and Conformal Mapping*, Amer. Math. Soc., Providence, RI (2000).
The only conformal maps in $\mathbb{R}^d$, $d \geq 3$, are Möbius transformations (Liouville). Also about related topics, hyperbolic geometry, Möbius transformations.

- [47] F. Bowman, *Introduction to Elliptic Functions, with Applications*, English Univ. Press, London (1953); corr.: Dover, New York (1961).
With applications to some conformal mappings (e.g., condenser problems).
- [48] D.A. Brannan and J.G. Clunie, see *Aspects...*
- [49] J.W. Brown and R.V. Churchill, *Complex Variables and Applications*, 7th edn, McGraw-Hill, Boston (2004).
Applications of conformal mappings and Schwarz–Christoffel integrals in electrostatics, heat conduction, potential flow (in the plane); at the end a table of conformal transformations.
- [50] R.B. Burckel, *An Introduction to Classical Complex Analysis*, Vol. I, Birkhäuser, Basel–Stuttgart (1979).
Also Schwarz lemma and Riemann mapping theorem with many related topics, references and historical remarks.
- [51] P. Caraman, *n-Dimensional Quasiconformal (QCf) Mappings*, Ed. Acad. Române, București/Abacus Press, Tunbridge Wells, Kent (1974); Romanian original: Ed. Acad. R. S. R., București (1968).
About the definitions of n -dimensional quasiconformal mappings; a great bibliography, also for quasiconformal mappings in the plane.
- [52] C. Carathéodory, *Conformal Representation*, 2nd edn, Cambridge Univ. Press (1952); Reprint: Cambridge Univ. Press, London (1998); Russian transl.: K. Karateodori, *Conformal Mapping*, Gostekhizdat, Moscow–Leningrad (1934).
Charming classical booklet from 1932.
- [53] C. Carathéodory, *Theory of Functions of a Complex Variable*, Vols I and II, 2nd edn, Chelsea, New York (1958, 1960); German original in the 2nd edn: *Funktionentheorie*, Vols I and II, Birkhäuser, Basel–Stuttgart (1960–1961).
I: Euclidean, spherical and hyperbolic geometry in the context of Möbius transformations. II: Riemann mapping theorem, some boundary behavior, reflection principle, simple distortion theorems, mapping of circular triangles.
- [54] L. Carleson and T.W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York (1993).
Something about quasiconformal mappings and the role in complex dynamics.
- [55] G.F. Carrier, M. Krok and C.E. Pearson, *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill Book Co., New York–Toronto–London (1966).
Emphasis on applications, also of conformal mappings, Schwarz–Christoffel transformations.
- [56] I. Cerný, *Foundations of Analysis in the Complex Domain*, Ellis Horwood, New York/Academia, Publishing House of the Czechoslovak Acad. Sci., Prague (1992).
Also Riemann mapping theorem, some boundary behavior; parallel slit mappings; applications in fluid dynamics.
- [57] R.V. Churchill, J.W. Brown and R.F. Verhey, *Complex Variables and Applications*, 4th edn, McGraw-Hill Book Company, New York (1984).
Elementary conformal mappings, Schwarz–Christoffel transformation, small list of conformal transformations; some applications of conformal mappings: heat conduction, electrostatic problems, potential flow.
- [58] H. Cohn, *Conformal Mapping on Riemann Surfaces*, McGraw-Hill, New York (1967).
- [59] *Complex Analysis*, Articles dedicated to Albert Pfluger on the Occasion of his 80th birthday, J. Hersch and A. Huber, eds, Birkhäuser, Basel–Boston–Berlin (1988).
With a list of publications of A. Pfluger.
- [60] *Constructions and Applications of Conformal Maps*, E.F. Beckenbach, ed., Proc. Symp., National Bureau of Standards, Appl. Math. Ser., Vol. 18 (1952).
Collection of small survey articles.
- [61] *Contemporary Problems in Theory Anal. Functions (Internat. Conf., Erevan 1965)*, Nauka, Moscow (1965) (in Russian).
With several articles in GFT, also some in English.
- [62] J.B. Conway, *Functions of One Complex Variable*, Vol. II, Springer-Verlag, New York (1995).
Also something about conformal mappings of simply-connected and finitely-connected regions, prime ends, de Branges proof of the Bieberbach conjecture, harmonic measure, logarithmic capacity.
- [63] E.T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Clarendon Press, Oxford (1935).
Also Riemann mapping theorem, Schwarz–Christoffel formula.

- [64] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, with an Appendix by M. Schiffer, *Some Recent Developments in the Theory of Conformal Mapping*, Interscience, New York–London (1950); Russian transl.: Izd. Inostr. Lit., Moscow (1953).
Many about conformal slit mappings (also non-schlicht), their role in the solution of Plateau's problem; Appendix: Canonical conformal mappings, kernel functions, extremal mappings (variational method, extremal length).
- [65] P.J. Davis, *The Schwarz Function and Its Applications*, Carus Math. Monogr., Math. Assoc. Amer., Washington, DC (1974).
In an elementary manner something about the reflection at an analytic curve; many examples, relations to Möbius transformations and geometric interpretation, iteration theory etc.
- [66] W.R. Derrick, *Introductory Complex Analysis and Applications*, 2nd edn, Academic Press, New York–London (1973).
Applications of conformal mappings and Schwarz–Christoffel integrals in electrostatics, heat conduction, potential flow (in the plane).
- [67] S. Dineen, *The Schwarz Lemma*, Oxford Math. Monographs, Oxford Univ. Press, New York (1989).
With generalizations on complex manifolds.
- [68] T.A. Driscoll and L.N. Trefethen, *Schwarz–Christoffel Mapping*, Cambridge Univ. Press, Cambridge (2002).
With many practical and numerical aspects.
- [69] V.N. Dubinin, *Symmetrization in the Geometric Theory of Functions of a Complex Variable*, Russian Math. Surveys **49** (1) (1994), 1–79; Russian original: Uspehi Mat. Nauk **49** (1) (1994), 3–76.
Great survey.
- [70] É. Durand, *Électrostatique et Magnétostatique*, Masson, Paris (1953).
Something about electrostatic problems in the plane with function theory and conformal mapping; many examples.
- [71] W.F. Durand, see *Aerodynamic Theory*.
- [72] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York–Berlin–Heidelberg–Tokyo (1983).
Standard book about GFT, preferable in the case of simply-connected domains.
- [73] P.L. Duren, *Harmonic Mappings in the Plane*, Cambridge Univ. Press, Cambridge (2003).
- [74] P. Duren, J. Heinonen, B. Osgood and B. Palka, see *Quasiconformal Mappings and Analysis*.
- [75] Dzh. Dzhenkins, see J.A. Jenkins.
- [76] B. Epstein, *Orthogonal Families of Analytic Functions*, Macmillan, New York/Collier-Macmillan, London (1965).
Also the connection between the Bergman(n) kernel function and conformal mapping.
- [77] *Extremal Problems in Geometric Function Theory*, Yu.E. Alenitsyn and G.V. Kuz'mina, eds, Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. Steklov. Akad. Nauk SSSR, Tom 24, Nauka Leningr. Otd., Leningrad (1972) (in Russian).
A collection of papers in GFT.
- [78] P.F. Fil'chakov, *Theory of Filtration under Hydrotechnical Constructions*, Vols I and II, Izd. Akad. Nauk USSR, Kiev (1959, 1960) (in Russian).
With conformal mappings.
- [79] P.F. Fil'chakov, *Approximation Methods of Conformal Mappings*, Naukova Dumka, Kiev (1964) (in Russian).
Some elementary mappings; Schwarz–Christoffel integral; elliptic integrals and functions; some approximation methods; Lavrent'ev's variational formulas; hydrotechnical applications (filtration problems); many numerical examples.
- [80] V.P. Fil'chakova, *Conformal Mappings of Domains of Special Type*, Naukova Dumka, Kiev (1972) (in Russian).
Mainly conformal mapping of lattice domains (the boundary components are in a periodical order), the corresponding procedures in great detail, including a great numerical material.
- [81] W. Fischer and I. Lieb, *Funktionentheorie*, 8. Auflage, Friedrich Vieweg & Sohn, Braunschweig–Wiesbaden (2003).
Also Riemann mapping theorem, hyperbolic geometry.

- [82] W. Fischer und I. Lieb, *Ausgewählte Kapitel aus der Funktionentheorie*, Friedrich Vieweg & Sohn, Braunschweig–Wiesbaden (1988).
Bloch and Landau constant, some boundary behavior of conformal maps, mappings of circular polygons, Bergman(n) and Szegő kernel, Riemann mapping.
- [83] S.D. Fisher, *Complex Variables*, Wadsworth & Brooks/Cole, Belmont–Monterey, CA (1986).
Also Riemann mapping theorem, Schwarz–Christoffel transformations; applications in fluid dynamics.
- [84] W. Forst und D. Hoffmann, *Funktionentheorie erkunden mit Maple*, Springer-Verlag, Berlin (2002).
Also something about Möbius transformations and the Joukowski transformation with Maple.
- [85] P. Frank und R. von Mises (Herausgeber), *Die Differential- und Integralgleichungen der Mechanik und Physik*, 2. Auflage, Friedrich Vieweg & Sohn, Braunschweig, 1. Teil (1930), 2. Teil (1935).
1. Teil: Also Schwarz–Christoffel formulas, Schwarz lemma, Riemann mapping theorem (proof with the “Koebe Schmiegungsverfahren”), minimal area property (K. Löwner). 2. Teil: Also some applications of conformal mappings in electrostatics (F. Noether).
- [86] W.H.J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Van Nostrand, Princeton (1967).
Extremal length, capacity, transfinite diameter.
- [87] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1964).
Standard book about the numerical procedures for canonical conformal mappings (simply- and multiply-connected domains); great bibliography up to 1964.
- [88] D. Gaier, *Lectures on Complex Approximation*, Birkhäuser, Boston–Basel–Stuttgart (1987); German original: Birkhäuser, Boston–Basel–Stuttgart (1980); Russian transl.: Mir, Moscow (1986); Chinese transl.: Hunan Educational Publ. House (1985).
Orthonormal series and Bergman(n) kernel function for calculation of the Riemann mapping function.
- [89] T.W. Gamelin, *Complex Analysis*, Springer-Verlag, New York (2001).
Schwarz lemma, Schwarz reflection principle, Schwarz–Christoffel formula, hyperbolic metric, Riemann mapping theorem, uniformization.
- [90] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York (1987).
Also something about Riemann surfaces, quasiconformal mappings, quadratic differentials, extremal quasiconformal mappings.
- [91] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Amer. Math. Soc., Providence, RI (2000).
Also something about quasiconformal mappings, quadratic differentials, extremal quasiconformal mappings.
- [92] J.B. Garnett, *Applications of Harmonic Measure*, Wiley, New York (1986).
Also with applications on conformal maps; Hayman–Wu theorem; results of Makarov.
- [93] C. Gattegno et A. Ostrowski, *Représentation conforme à la frontière: Domaines généraux*, Mém. Sci. Math., Vol. 109, Paris (1949).
General theory of boundary behavior; Schwarz reflection principle; Löwner–Montel lemma; also some more special questions.
- [94] C. Gattegno et A. Ostrowski, *Représentation conforme à la frontière: Domaines particuliers*, Mém. Sci. Math. 110, Paris (1949).
- [95] F.W. Gehring, *Characteristic Properties of Quasidisks*, Les Presses de L’Université de Montréal, Montréal (1982).
Plane quasiconformal mappings, quasidisks and their characteristic properties.
- [96] F.W. Gehring and O. Lehto, *Lectures on Quasiconformal Mappings*, Lecture Note 14, Dept. of Math., Univ. Maryland, College Park, MD (1975).
Introduction for the plane by O. Lehto, for $\mathbb{R}^n$ by F.W. Gehring.
- [97] *Geometric Analysis and Applications*, Izd. Volgogradskogo Gosud. Univ., Volgograd (1999) (in Russian).
Collection of papers on minimal surfaces and connections with GFT (also extremal length and quasiconformal mappings).
- [98] W.J. Gibbs, *Conformal Transformations in Electrical Engineering*, Chapman & Hall, London (1958).

- [99] A.A. Gol'dberg and I.V. Ostrovskii, *Value Distribution of Meromorphic Functions*, Nauka, Moscow (1970).
Also something about quasiconformal mappings as a tool.
- [100] V.M. Gol'dstein and Yu.G. Reshetnyak, *Introduction to the Theory of Functions with Generalized Derivatives, and Quasiconformal Mappings*, Nauka, Moscow (1983) (in Russian); Revised transl.: *Quasiconformal Mappings and Sobolev Spaces*, Math. Appl., Soviet Ser., Vol. 54, Kluwer, Dordrecht (1990).
Relates also Sobolev spaces and quasiconformal mappings to each other.
- [101] G.M. Goluzin, *Some Questions in the Theory of Schlicht Functions*, Trudy Matem. Inst. Steklov. **27**, Moscow–Leningrad (1949) (in Russian).
Booklet before the following book.
- [102] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Transl. Math. Monogr., Vol. 29, Amer. Math. Soc., Providence, RI (1969); German transl. from the Russian: G.M. Goluzin, *Geometrische Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1957); Russian original, 2nd edn: Nauka, Moscow (1966).
Was the first standard book about GFT. In the German translation some additional remarks and references. In the 2nd edn some additions and a list of Goluzin's publications.
- [103] G.M. Goluzin, L.V. Kantorovich, V. Krylov, P. Melent'ev, M. Muratov and N. Stenin, *Conformal Mapping of Simply- and Multiply-Connected Domains*, ONTI, Moscow–Leningrad (1937) (in Russian).
- [104] Sheng Gong, *The Bieberbach Conjecture*, Amer. Math. Soc., Providence, RI/Intern. Press, Cambridge, MA (1999). Transl. from the 1989 Chinese original and revised by the author, with preface by C.H. FitzGerald.
A monograph about the Bieberbach conjecture.
- [105] Sheng Gong, *Concise Complex Analysis*, World Scientific, Singapore (2001).
Also Gaussian curvature of conformal metrics, the Schwarz–Ahlfors lemma.
- [106] M.O. González, *Complex Analysis – Selected Topics*, Dekker, New York–Basel–Hong Kong (1992).
Riemann mapping theorem, boundary behavior, Schwarz–Christoffel formula, univalent functions in the disk, area theorem with classical distortion theorems, introduction to quasiconformal mappings.
- [107] A.W. Goodman, *Univalent Functions*, Vols 1 and 2, Mariner, Tampa, FL (1983).
The classical theory of univalent functions with some special topics: functions of positive real part, convex and starlike functions, radius problems (Koebe domains), convolution.
- [108] E. Graeser, *Einführung in die Theorie der elliptischen Funktionen und deren Anwendungen*, Oldenbourg, München (1950).
Conformal mappings of some special doubly-connected domains onto an annulus.
- [109] I. Graham and G. Kohr, *GFT in One and Higher Dimensions*, Marcel Dekker, New York, NY (2003).
Some special classes of schlicht functions, Loewner chains, linear invariant families univalence criteria.
- [110] Ch. Gram, see *Selected Numerical Methods*.
- [111] R.E. Greene and S.G. Krantz, *Function Theory of One Complex Variable*, 2nd edn, Amer. Math. Soc., Providence, RI (2002).
Schwarz lemma, Riemann mapping theorem, Schwarz reflection principle, boundary behavior, Bergman(n) kernel.
- [112] H. Grunsky, *Lectures on Theory of Functions in Multiply Connected Domains*, Vandenhoeck & Ruprecht, Göttingen (1978).
Canonical conformal mappings of multiply-connected domains with corresponding extremal properties; many references.
- [113] W.K. Hayman, *Multivalent Functions*, 2nd edn, Cambridge Univ. Press, Cambridge (1994). Russian transl. of the 1st edn: Izd. Inostr. Lit., Moscow (1960).
Standard book also about the classical theory of univalent and multivalent functions, including symmetrization and Löwner theory; in the 2nd edn also de Branges' theorem.
- [114] W.K. Hayman, *Research Problems in Function Theory*, Athlone Press, London (1967).
Also something about schlicht and multivalent functions.
- [115] W.K. Hayman, *Subharmonic Functions*, Vol. 2, Academic Press, London (1989).
Also something about capacity, weakly univalent functions, p -valent functions, extremal length, Baernstein's star function, means and symmetrization.
- [116] M. Heins, *Complex Function Theory*, Academic Press, New York–London (1968).
Also some questions concerning univalent functions.

- [117] P. Henrici, *Applied and Computational Complex Analysis*, Vol. III, Wiley, New York (1986); reprint (1993). Several methods for construction of conformal maps (simply- and multiply-connected regions) with many references. Something about univalent functions, first proof of the Bieberbach conjecture in a book.
- [118] J. Hersch and A. Huber, see *Complex Analysis*.
- [119] E. Hille, *Analytic Function Theory*, Vol. II, 2nd edn, Chelsea, New York, NY (1973).
Also Riemann mapping theorem, Bergman(n) kernel function, schlicht functions, harmonic measure.
- [120] G. Holzmüller, *Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildungen, verbunden mit Anwendungen auf mathematische Physik*, Teubner, Leipzig (1882).
In great detail elementary conformal mappings (including elliptic integrals of the first kind); with many figures of a surprisingly high quality; still nicely to read.
- [121] J.A. Hummel, *Lectures on Variational Methods in the Theory of Univalent Functions*, Lecture Notes, Univ. Maryland (1972).
- [122] A. Hurwitz, *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen*, Herausgegeben und ergänzt durch einen Abschnitt über Geometrische Funktionentheorie von R. Courant, Springer-Verlag, Berlin (1922); 4. Auflage mit einem Anhang von H. Röhl, Springer-Verlag, Berlin-Göttingen-Heidelberg-New York (1964); Russian transl. of the 4th edn without the Appendix by H. Röhl: Nauka, Moscow (1968); the 5th German edn without the Appendixes by R. Courant and H. Röhl; Russian transl. of Courant's part: R. Kuran, *Geometric Function Theory*, Gostechizdat, Moscow-Leningrad (1934).
In the Courant Appendix: Riemann mapping theorem, boundary behavior, Schwarz-Christoffel mapping, circular polygons, simple distortion theorems, some canonical conformal mappings of multiply-connected domains; in the Röhl Appendix additionally some elementary things about quasiconformal mappings, conformal modulus of quadrilaterals.
- [123] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo (1992).
Including something about quasiconformal mappings without mentioning H. Grötzsch.
- [124] *Investigations in Modern Problems of Complex Function Theory*, A.I. Markushevich, ed., Gosud. Izd. Fiz.-Mat. Lit., Moscow (1960, 1961) (in Russian).
Also some papers about GFT.
- [125] O.V. Ivanov and G.D. Suvorov, *Complete Lattices of Conformally Invariant Compactifications of a Domain*, Naukova Dumka, Kiev (1982) (in Russian).
Theory of prime ends in a general context.
- [126] V. Ivanov and M.K. Trubetskov, *Handbook of Conformal Mapping with Computer-Aided Visualization*, CRC Press, Boca Raton, FL (1995).
A great number of explicit conformal mappings with pictures, including Schwarz-Christoffel integrals; applications to physical problems: Hydrodynamics, filtration, heat conduction, elasticity; also a catalog of conformal mappings with graphs; including a disk containing a program CONFORM.
- [127] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-Linear Analysis*, Oxford Math. Monogr., Clarendon Press/Oxford University Press, New York (2001).
Mainly the n -dimensional theory.
- [128] M.A. Jaswon and G.T. Symm, *Integral Equation Methods in Potential Theory and Electrostatics*, Acad. Press, London-New York-San Francisco (1977).
Also something about integral equation methods in conformal mapping theory.
- [129] J.A. Jenkins, *Univalent Functions and Conformal Mapping*, Springer-Verlag, Berlin-Göttingen-Heidelberg (1958); corr. edn: Springer-Verlag, Berlin-Heidelberg-New York (1965); Russian transl.: Dzh. Dzhenkins, . . . , Izd. Inostr. Lit., Moscow (1962).
Comprehensive application of the method of extremal metric in form of the General Coefficient Theorem; quadratic differentials; canonical conformal mappings; symmetrization; multivalent functions.
- [130] G. Julia, *Leçons sur la représentation conforme des aires simplement connexes*, 2nd edn, Gauthier-Villars, Paris (1950).
Möbius transformations, Schwarz reflection principle, Riemann mapping theorem, examples, Schwarz-Christoffel formula, area theorem and related classical distortion theorems.
- [131] G. Julia, *Leçons sur la représentation conforme des aires multiplement connexes*, 2nd edn, Gauthier-Villars, Paris (1955).
Canonical conformal mappings (also multivalent); including mappings after Julia and de la Vallée Poussin.

- [132] L.V. Kantorovich and V.I. Krylov, *Approximate Methods of Higher Analysis*, Noordhoff, Groningen (1958); German transl.: L.W. Kantorowitsch und W.I. Krylow, *Näherungsmethoden der höheren Analysis*, VEB Deutscher Verlag Wiss., Berlin (1956); Russian original in the 4th edn: Gos. Izd. Tekhn.-Teor. Lit., Moscow–Leningrad (1952).
Without proof some canonical conformal mappings (onto a disk, an annulus, a parallel-slit domain); use of minimum properties (area, length) for the construction; orthogonal polynomials; several other special methods; integral equation method; Schwarz–Christoffel formula with the parameter problem; application of conformal mappings to the solution of boundary value problems, also for the biharmonic equation; always many numerical examples.
- [133] K. Karateodori, see C. Carathéodory.
- [134] M.V. Keldysh and L.I. Sedov, *Applications of Complex Function Theory in Hydrodynamics and Aerodynamics*, Nauka, Moscow (1964) (in Russian).
- [135] O.D. Kellogg, *Foundations of Potential Theory*, Springer-Verlag, Berlin–Heidelberg–New York (1967) (Reprint).
Also a crash course on conformal mappings: Riemann mapping function, Green’s function, Schwarz–Christoffel formula.
- [136] V.K. Kheĭman, see W.K. Hayman.
- [137] H. Kober, *Dictionary of Conformal Representations*, Dover, New York (1952).
Many examples, including applications of elliptic integrals and Schwarz–Christoffel integrals.
- [138] N.E. Kochin, I.A. Kibel’ and N.V. Roze, *Theoretical Hydrodynamics*, Interscience Publ. Wiley, New York–London–Sydney (1964). Russian original in the 4th edn: Gostekhizdat, Moscow (1948); German transl.: N.J. Kotschin, I.A. Kibel und N.W. Rose, *Theoretische Hydrodynamik*, Bd. I, Akademie-Verlag, Berlin (1954).
Applications of conformal mappings in hydrodynamics (including Schwarz–Christoffel formula).
- [139] P. Koebe, *Riemannsche Mannigfaltigkeiten und nichteuklidische Raumformen*, 1.–8. Mitteilung, Sitzungsberichte der Preuß. Akad. Wiss. Phys.-Math. Kl., Berlin (1927), 164–196; (1928), 345–384, 385–442; (1929), 414–457; (1930), 304–364, 505–541; (1931), 506–534; (1932), 249–284.
This is in form not a book, but also unusual as a series of “papers”; nowadays it would be probably a book; therefore this is exceptionally also listed here, especially because this text is almost unknown up to now.
- [140] Y. Komatu and K. Noshiro, *List of Books and Papers on the Theory of Functions*, ed. with the cooperation of K. Oikawa, M. Ozawa, J. Tamura, The Research Division of Function Theory, Math. Soc. Japan (1960); Suppl. 1 (1964); Suppl. 2 (1968).
Contains books and papers published by the actual members of the Research Division of Function Theory of the Mathematical Society of Japan.
- [141] R. König and K.H. Weise, *Mathematische Grundlagen der höheren Geodäsie und Kartographie*, 1 Bd: *Das Erdsphäroid und seine konformen Abbildungen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1951).
Including many interesting material about conformal maps.
- [142] W. von Koppenfels und F. Stallmann, *Praxis der konformen Abbildung*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1959); Russian transl.: Fizmatgiz., Moscow (1963).
Some elementary mappings; Schwarz lemma; Riemann mapping theorem; Schwarz reflection principle; mainly: mapping of polygonal domains, also with circular arcs or arcs in more general nets; also doubly-connected polygonal domains; many examples in great detail; some integral equations; approximation procedures.
- [143] A.P. Kopylov, *Stability in the C-Norm of Classes of Mappings*, Nauka Sibirsk. Otd., Novosibirsk (1990) (in Russian).
About mappings in n -space whose dilatation is close to 1.
- [144] S.G. Krantz, *Complex Analysis: The Geometric Viewpoint*, Carus Math. Monogr., Math. Assoc. Amer., Washington, DC (1990).
Schwarz lemma and several related aspects, Carathéodory and Kobayashi metric.
- [145] S.G. Krantz, *Handbook of Complex Variables*, Birkhäuser, Boston, MA (1999).
Some simple conformal mappings, applications of conformal mappings, a pictorial catalog of conformal mappings, numerical problems.

- [146] S.L. Krushkal and R. Kühnau, *Quasikonforme Abbildungen – neue Methoden und Anwendungen*, Teubner, Leipzig (1983); Russian edn: Nauka Sibirsk. Otd., Novosibirsk (1984).
Teichmüller spaces, Kobayashi and Carathéodory metric; conformal mappings with quasiconformal extension; extremal problems; mappings, quasiconformal in the mean; applications in electrostatics.
- [147] S.L. Krushkal', *Variational Methods in the Theory of Quasiconformal Mappings*, Novosibirsk. Gos. Univ., Novosibirsk (1974) (in Russian).
Booklet: Quasiconformal mappings (also of Riemann surfaces), quadratic differentials, variational formulas, Teichmüller problems, extremal problems, also for schlicht functions with a quasiconformal extension.
- [148] S.L. Krushkal', *Quasiconformal Mappings and Riemann Surfaces*, Winston, Washington, DC/Wiley, New York (1979); Russian original: Nauka Sibirsk. Otd., Novosibirsk (1975).
Comprehensive monography about quasiconformal mappings in the plane: Riemann surfaces and Teichmüller spaces, differentials, extremal problems with variational formulas, quasiconformal mappings with a given boundary correspondence, Kleinian groups.
- [149] S.L. Krushkal', B.N. Apanasov and N.A. Gusevskii', *Kleinian Groups in Examples and Problems*, Novosibirsk. Gos. Univ., Novosibirsk (1978) (in Russian).
- [150] S.L. Krushkal', B.N. Apanasov and N.A. Gusevskii', *Uniformization and Kleinian Groups*, Novosibirsk. Gos. Univ., Novosibirsk (1979) (in Russian).
This and the foregoing booklet in preparation of the following book.
- [151] S.L. Krushkal', B.N. Apanasov and N.A. Gusevskii', *Kleinian Groups and Uniformization in Examples and Problems*, Amer. Math. Soc., Providence, RI (1986); Russian original: Nauka Sibirsk. Otd., Novosibirsk (1981).
The role of quasiconformal mappings in the theory of Riemann surfaces and Kleinian groups.
- [152] J.G. Krzyż, *Problems in Complex Variable Theory*, Elsevier, New York/PWN-Polish Sci. Publ., Warszawa (1971).
Möbius transformations, hyperbolic geometry, elementary conformal mappings, Schwarz–Christoffel integrals, conformal mappings associated with elliptic functions; Riemann mapping theorem, harmonic measure, Bergman(n) kernel; stationary two-dimensional flow, electrostatic fields; univalent functions: functions of positive real part, starshaped and convex functions, general univalent functions, inner radius; symmetrization.
- [153] J. Krzyż and J. Ławrynowicz, *Elementy analizy zespolonej*, Wydawnictwa Naukowo–Techniczne, Warszawa (1981) (in Polish).
Something about Beltrami equation, quasiconformal mappings, Schwarz reflection principle, Schwarz–Christoffel formula.
- [154] R. Kühnau, *Geometrie der konformen Abbildungen auf der hyperbolischen und der elliptischen Ebene*, VEB Deutscher Verlag Wiss., Berlin (1974).
Conformal mappings on the hyperbolic plane (unit disk) and the elliptic plane (Riemann sphere with identification of antipodal points); extremal problems; hyperbolic and elliptic transfinite diameter; the finished English translation was not desirable; at p. 15, lines 14 and 15, a great mistake.
- [155] H.P. Künzi, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
Survey about the theory up to 1960.
- [156] R. Kurant, see A. Hurwitz. . .
- [157] G.V. Kuz'mina, *Methods of Geometric Function Theory*, Vols I and II, St. Petersburg Math. J. **9** (3) (1998), 455–507; (5), 889–930; corr: **10** (3) (1999), 577; Russian original: Algebra i Analiz **9** (3) (1997), 41–103; (5), 1–50; corr.: **10** (3) (1998), 223.
Great survey.
- [158] G.V. Kuz'mina, *Moduli of Families of Curves and Quadratic Differentials*, Proc. Steklov Inst. Math. (1982); Russian original: Tom **139** (1980).
Method of extremal metric; continua of minimal capacity (transfinite diameter), also in the hyperbolic sense; characterization with quadratic differentials; capacity and symmetrization; applications.
- [159] Yue Kuen Kwok, *Applied Complex Variables for Scientists and Engineers*, Cambridge Univ. Press, Cambridge (2002).
Also applications of conformal mappings and Schwarz–Christoffel integrals in electrostatics, heat conduction, potential flow (in the plane).

- [160] P.K. Kythe, *Computational Conformal Mappings*, Birkhäuser, Boston–Basel–Berlin (1998). Schwarz–Christoffel formula, parameter problem; polynomial approximation; nearly circular domains; integral equation method; airfoils; doubly- and multiply-connected domains; grids; Riemann–Hilbert problem.
- [161] H. Lamb, *Hydrodynamics*, 6th edn, paperb. edn, Cambridge Univ. Press, Cambridge (1993); German transl.: *Lehrbuch der Hydrodynamik*, 2. Auflage, Teubner, Leipzig–Berlin (1931). Schwarz–Christoffel formula with applications to the theory of wake.
- [162] E. Landau und D. Gaier, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 3. erweiterte Auflage, Springer-Verlag, Berlin (1986). Also several classical things about conformal maps; D. Gaier has brought the book up to date with a great bibliography.
- [163] M.A. Lavrent'ev, *Conformal Mappings with Applications in Several Questions of Mechanics*, Gostekhzdat, Moscow–Leningrad (1946) (in Russian).
- [164] M.A. Lavrent'ev, *Variational Methods for Boundary Value Problems for Systems of Elliptic Equations*, Noordhoff, Groningen (1963); Russian original: Izd. Akad. Nauk SSSR, Moscow (1962). Very concise booklet about variational formulas (for the mapping functions if a domain lies in a neighborhood of another); applications in hydrodynamics (e.g., water waves); quasiconformal mappings (some distortion theorems, linear and nonlinear systems).
- [165] M.A. Lavrent'ev and L.A. Ljusternik, *Foundations of Variational Calculus* (Appendix II: *About Some Extremal Problems in the Theory of Conformal Mappings*), ONTI, Moscow–Leningrad (1935) (in Russian).
- [166] M.A. Lavrent'ev and B.V. Shabat, *Problems of Hydrodynamics and Their Mathematical Models*, Nauka, Moscow (1973) (in Russian). From the point of view of applications in hydrodynamics very concise something about conformal and quasiconformal mappings and the corresponding variational principles.
- [167] V.I. Lavrik, V.P. Fil'chakova and A.A. Yashin, *Conformal Mappings of Physical–Topological Models*, Naukova Dumka, Kiev (1990) (in Russian). Application in underground water hydrodynamics, hydrodynamic theory of lattices, microelectronics.
- [168] V.I. Lavrik and V.N. Savenkov, *A Handbook on Conformal Mappings*, Naukova Dumka, Kiev (1970) (in Russian). Conformal mappings by elementary functions and Schwarz–Christoffel integrals; finally a catalog of 115 mappings.
- [169] M.A. Lawrentjew und B.W. Schabat, *Methoden der komplexen Funktionentheorie*, VEB Deutscher Verlag Wiss., Berlin (1967); Russian original in the 5th edn: M.A. Lavrent'ev and B.V. Shabat, ..., Nauka, Moscow (1987); French transl.: M. Lavrentiev and B. Chabat: ..., Nauka, Moscow (1972). Many special conformal mappings including Schwarz–Christoffel integrals; many applications in hydrodynamics etc; mappings of neighborly domains.
- [170] J. Ławrynowicz, in cooperation with J. Krzyż, *Quasiconformal Mappings in the Plane: Parametrical Methods*, Lecture Notes in Math., Vol. 978, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo (1983). Basic concepts, extremal problems.
- [171] N.A. Lebedev, *The Area Principle in the Theory of Univalent Functions*, Nauka, Moscow (1975) (in Russian). Solution of extremal problems with the area method (coefficient estimates and distortion theorems of a very general character) in the classes S and Σ , also in classes of mappings onto nonoverlapping domains; very rich material; a translation would be very desirable.
- [172] O. Lehto, *Conformal Mappings and Teichmüller Spaces*, Technion Lecture Notes, Israel Inst. Tech., Haifa (1973). Preparation of the following book.
- [173] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York (1987). Quasiconformal mappings, Beltrami equation, quasidisks; something about univalent functions, also with quasiconformal extension; universal Teichmüller space, Teichmüller metric, inner radius of univalence; Riemann surfaces, uniformization, groups of Möbius transformations, quadratic differentials; Teichmüller spaces of Riemann surfaces.

- [174] O. Lehto und K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin–Heidelberg–New York (1965); 2nd edn: *Quasiconformal Mappings in the Plane*, Springer-Verlag, Berlin–Heidelberg–New York (1973).
Standard book for the foundation of the theory.
- [175] J. Lelong-Ferrand, *Représentation conforme et transformations à intégrale de Dirichlet bornée*, Gauthier-Villars, Paris (1955).
Mappings with a bounded Dirichlet integral (more general than quasiconformal mappings).
- [176] T.A. Leont'eva, V.S. Panferov and V.S. Serov, *Problems in the Function Theory of one Complex Variable*, Izd. MGU, Moscow (1992) (in Russian).
With applications of conformal mappings in mechanics and physics.
- [177] L. Lewent (hrsg. E. Jahnke, mit einem Beitrag von W. Blaschke), *Konforme Abbildung*, Teubner, Leipzig–Berlin (1912).
A small charming book of elementary character with a tragical history.
- [178] L. Lichtenstein, *Neuere Entwicklung der Potentialtheorie. Konforme Abbildung*, Encyklopädie der Math. Wiss. Bd. II, 3. Teil, 1. Hälfte C 3, Teubner, Leipzig (1919), 177–377.
Great survey, also about GFT, up to 1919; still very interesting not only from the historical point of view.
- [179] J.E. Littlewood, *Lectures on the Theory of Functions*, Oxford Univ. Press, Oxford (1944).
Riemann mapping theorem, Schwarz lemma, Möbius transformations, boundary behavior, functions schlicht in the unit disk.
- [180] A.I. Markushevich, *Complex Numbers and Conformal Mappings*, Pergamon Press, New York (1963); Russian original: Fizmatgiz, Moscow (1960); German transl.: A.I. Markuschewitsch, ..., 4. Auflage, VEB Deutscher Verlag Wiss., Berlin (1973).
Elementary account.
- [181] A.I. Markushevich, see *Investigations...*
- [182] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin (1988).
Also something about quasiconformal mappings.
- [183] H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1962).
Representation of canonical conformal mappings with the kernel function.
- [184] I.M. Milin, *Univalent Functions and Orthonormal Systems*, Amer. Math. Soc., Providence, RI (1977); Russian original: Nauka, Moscow (1971).
In the simply-connected case: Faber polynomials, area theorem, coefficients, Grunsky conditions, logarithmic coefficients, asymptotic estimates; the second part contains many results about the corresponding theory for multiply-connected domains, not well known.
- [185] L.M. Milne-Thomson, *Theoretical Hydrodynamics*, 4th edn, Macmillan, London/St. Martin's Press, New York (1960).
Also application of conformal maps, including Schwarz–Christoffel transformation.
- [186] L.M. Milne-Thomson, *Plane Elastic Systems*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
Also applications of conformal maps.
- [187] L.M. Milne-Thomson, *Antiplane Elastic Systems*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1962).
Some concrete applications of conformal maps in the theory of elasticity.
- [188] I.P. Mityuk, *Application of Symmetrization Methods to the Geometric Function Theory*, Krasnodar (1985) (in Russian).
- [189] V.N. Monakhov, *Boundary-Value Problems with Free Boundaries for Elliptic Systems of Equations*, Amer. Math. Soc., Providence, RI (1983); Russian original: Nauka Sibirsk. Otd., Novosibirsk (1977).
Also Schwarz–Christoffel formula, some boundary behavior of conformal maps, quasiconformal maps in connection with elliptic (also nonlinear) systems.
- [190] P. Montel, *Leçons sur les fonctions univalentes ou multivalentes*, Gauthier-Villars, Paris (1933).
Special classes of conformal mappings (starlike mappings, polynomials); classical distortion theorems; parallel slit mappings; Bloch theorem.
- [191] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, corr. 4th edn, Noordhoff, Leyden (1975); Russian original in several editions; German transl.: N.I. Mušchelischwili, *Einige Grundaufgaben der mathematischen Elastizitätstheorie*, Carl Hanser Verlag, München (1971).
Also applications of conformal mappings.

- [192] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley, New York (1988).
Something about quasiconformal mappings and the role in the theory of Teichmüller spaces.
- [193] T. Needham, *Visual Complex Analysis*, Clarendon Press, Oxford (1997); German transl.: *Anschauliche Funktionentheorie*, Oldenbourg, München–Wien (2001).
An unusual book about the elementary theory, but with modern computer graphics and many history.
- [194] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York–Toronto–London (1952); Reprint: Dover, New York (1975).
At the beginning of introductory character; then Riemann mapping theorem, Schwarz lemma, Schwarz–Christoffel formula and other special mappings; univalent functions (classes S and Σ), kernel function; mappings of multiply-connected domains onto some canonical domains; interesting exercises.
- [195] R. Nevanlinna, *Uniformisierung*, 2. Auflage, Springer-Verlag, Berlin–Göttingen–Heidelberg (1967).
Riemann mapping theorem; some slit mappings; span; capacity; harmonic measure.
- [196] R. Nevanlinna, *Eindeutige analytische Funktionen*, 2. Auflage, Reprint, Springer-Verlag, Berlin–Göttingen–Heidelberg (1954); English transl.: *Analytic Functions*, Springer-Verlag, Berlin (1970); Russian transl.: Gostekhizdat, Moscow (1951).
Something about harmonic and hyperbolic measure, Carleman's principle.
- [197] R. Nevanlinna and V. Paatero, *Introduction to Complex Analysis*, 2nd edn, Chelsea, New York (1982); German original: *Einführung in die Funktionentheorie*, Birkhäuser, Basel–Stuttgart (1965).
Riemann mapping theorem, boundary behavior, Schwarz–Christoffel formula, reflection principle.
- [198] *Numerical Conformal Mapping*, L.N. Trefethen, ed., North-Holland, Amsterdam–New York–Oxford (1986).
A collection of survey articles.
- [199] M. Ohtsuka, *Dirichlet Problem, Extremal Length, and Prime Ends*, Van Nostrand, New York (1970).
Extremal length of a curve family with many examples; something about symmetrization; connection with the Dirichlet problem; prime end theory (boundary correspondence under conformal mapping).
- [200] B.P. Palka, *An Introduction to Complex Function Theory*, Springer-Verlag, New York (1991).
Also Möbius transformations, Riemann mapping theorem, Schwarz–Christoffel mappings.
- [201] D. Partyka, *The Generalized Neumann–Poincaré Operator and Its Spectrum*, Dissertationes Math., Vol. 366, Inst. Mat., Polska Akad. Nauk, Warszawa (1997).
Connection between extremal quasiconformal mappings with given boundary values, corresponding Fredholm eigenvalues, Grunsky inequalities.
- [202] A. Pfluger, *Theorie der Riemannschen Flächen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1957).
With some relations to GFT, e.g., Riemann mapping theorem, parallel slit mappings, span, etc.
- [203] A. Pfluger, *Lectures on Conformal Mapping*, Lecture Notes, Dept. Math., Indiana Univ. (1969).
- [204] E.G. Phillips, *Some Topics in Complex Analysis*, Pergamon Press, Oxford (1966).
Schwarz–Christoffel transformation; some elementary conformal transformations; Schwarz lemma; elementary distortion theory in the class S .
- [205] S.I. Pinchuk, *Methods of the Geometric Theory of Functions of a Complex Variable*, Bashkir. Gos. Univ. im. 40-let. Okt., Ufa (1984) (in Russian).
Small booklet: Schwarz lemma, Riemann mapping theorem, something about the boundary behavior, hyperbolic and Carathéodory metric, subharmonic functions, Dirichlet problem.
- [206] J. Plemelj, *Theory of Analytic Functions*, Acad. Scient. et Artium Slovenica, Ljubljana (1953) (in Slovenian).
Also some GFT.
- [207] P.Ya. Polubarinova-Kochina, *Theory of Ground Water Movement*, Princeton Univ. Press, Princeton, NJ (1962); Russian original: Gos. Izd. Tekhn.-Teor. Lit., Moscow (1952).
Also with the use of conformal mappings.
- [208] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton Univ. Press, Princeton (1951); Russian transl.: Gos. Izd. Fiz.-Mat. Lit., Moscow (1962).
Solution of some isoperimetric inequalities with conformal mappings; something about the conformal radius.

- [209] G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, Vols I and II, Springer-Verlag, Berlin (1998); Reprint; German original in the 4th edn: Springer-Verlag, Berlin–Heidelberg–New York (1970, 1971); Hungarian transl.: Tankönyvkiadó Vállalat, Budapest (1980, 1981).
Examples of conformal maps; simple geometric properties (starlikeness, convexity); simple classical distortion properties; area theorem; Riemann mapping theorem; proof with the Koebe “Schmiegunungsverfahren”; conformal radius with examples.
- [210] Ch. Pommerenke, *Univalent Functions, with a Chapter on Quadratic Differentials by Gerd Jensen*, Vandenhoeck & Ruprecht, Göttingen (1975).
Standard book about GFT, preferable in the case of simply-connected domains.
- [211] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin (1991).
Standard book about the boundary behavior of a conformal mapping of the unit disk onto an arbitrary simply-connected domain.
- [212] I.I. Priwalow, *Randeigenschaften analytischer Funktionen*, 2. Auflage, VEB Deutscher Verlag Wiss., Berlin (1956); Russian original: 2nd edn, Gos. Izd. Tekhn.-Teor. Lit., Moscow–Leningrad (1950).
Something about boundary behavior of conformal maps, some exotic examples.
- [213] D.V. Prokhorov, *Reachable Set Methods in Extremal Problems for Univalent Functions*, Saratov Univ. Publishing House (1993).
About the Löwner method in the classes S and Σ .
- [214] *Quasiconformal Mappings and Analysis*, Collection of Papers Honoring Frederick W. Gehring to His 70th Birthday, Proc. Intern. Symp., Ann Arbor, MI, August, 1995, P. Duren, J. Heinonen, B. Osgood and B. Palka, eds, Springer-Verlag, New York, NY (1998).
With some articles about GFT.
- [215] *Quasiconformal Space Mappings, a Collection of Surveys, 1960–1990*, ed. M. Vuorinen, Lecture Notes in Math., Vol. 1508, Springer-Verlag, Berlin (1992).
- [216] R. Remmert, *Classical Topics in Complex Function Theory*, Springer-Verlag, New York (1998). German original in the 2nd edn: *Funktionentheorie II*, Springer-Verlag, Berlin (1995).
Riemann mapping theorem with historical remarks, also the constructive proof of P. Koebe (“Schmiegunungsverfahren”).
- [217] H. Renelt, *Elliptic Systems and Quasiconformal Mappings*, Wiley, Chichester (1988); German original: B.G. Teubner Verlagsgesellschaft, Leipzig (1982).
Quasiconformal mappings as solutions of linear systems of elliptic differential equations (not necessary of Beltrami type); variational method to solve extremal problems, also in classes of mappings which satisfy a fixed elliptic system.
- [218] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, Amer. Math. Soc., Providence, RI (1989). Russian original: Nauka Sibirsk. Otd., Novosibirsk (1982).
Quasiconformal and more general mappings in n -space.
- [219] Yu.G. Reshetnyak, *Stability Theorems in Geometry and Analysis*, Nauka Sibirsk. Otd., Novosibirsk (1982) (in Russian).
Quasiconformal mappings in $\mathbb{R}^n$ with a small dilatation.
- [220] S. Rickman, *Quasiregular Mappings*, Springer-Verlag, Berlin (1993).
Something about the method of modulus of path families and about conformal capacity.
- [221] B. Rodin, L. Sario and M. Nakai, *Principal Functions*, Van Nostrand, Princeton, NJ (1968).
Something about canonical conformal mappings, construction of reproducing differentials, stability of boundary components, extremal length.
- [222] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser, Basel (1994).
Also proof of the Bieberbach conjecture (de Branges) with all aspects.
- [223] R. Rothe, F. Ollendorf and K. Pohlhausen, see *Theory of Functions*...
- [224] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Presses de l' Université de Montréal, Montréal (1982).
About the connection of convolution, schlichtness, convexity etc.
- [225] E. Saff and A.D. Snider, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 2nd edn (with an Appendix by L.N. Trefethen), Prentice-Hall, Englewood Cliffs, NJ (1993).

- Appendix on applications of conformal mappings and Schwarz–Christoffel integrals in electrostatics, heat conduction, potential flow (in the plane).
- [226] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, Berlin (1997).
The usual material about extremal point methods, transfinite diameter, orthogonal polynomials, related basic results of potential theory.
 - [227] S. Saks and A. Zygmund, *Analytic Functions*, 2nd edn enlarged, Polish Sci. Publishers, Warsaw (1965).
Also Schwarz lemma, Riemann mapping theorem, Schwarz–Christoffel formula.
 - [228] G. Sansone and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable, Vol. II: Geometric Theory*, Wolters-Noordhoff, Groningen (1969).
Riemann mapping with boundary behavior, Schwarz–Pick lemma, Schwarz reflection principle, Schwarz–Christoffel formula, length–area principle, area theorem with classical distortion theorems, Carathéodory kernel convergence, Löwner theory, conformal mapping of circular polygons.
 - [229] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer-Verlag, Berlin–Heidelberg–New York (1970).
Something about the modulus method, extremal length, mapping onto a radial or circular slit disk, span, Koebe circular mapping.
 - [230] L. Sario and K. Oikawa, *Capacity Functions*, Springer-Verlag, Berlin–Heidelberg–New York (1969).
Canonical conformal mappings of domains of arbitrary connectivity (also on Riemann surfaces), related capacities etc.
 - [231] G.N. Savin, *Stress Concentration Around Holes*, Pergamon Press, New York–Oxford–London–Paris (1961); Russian original: Gos. Izd. Tekhn.-Teor. Lit., Moscow–Leningrad (1951); German translation: G.N. Sawin, *Spannungserhöhung am Rande von Löchern*, Verlag Technik, Berlin (1956).
Applications of conformal mappings in the theory of elasticity.
 - [232] A.C. Schaeffer and D.C. Spencer, *Coefficient Regions for Schlicht Functions* (with a chapter by A. Grad: *The Region of Values of the Derivative of a Schlicht Function*), Amer. Math. Soc. Colloq. Publ., Vol. 35, Amer. Math. Soc., New York (1950).
Determination of the coefficient region for schlicht functions $f(z) = z + a_2z^2 + a_3z^3 + \dots$, mainly with the variational method; very concrete result for the region of (a_2, a_3) .
 - [233] J.L. Schiff, *Normal Families*, Springer-Verlag, New York (1993).
Application: Proof of the Riemann mapping theorem.
 - [234] M. Schiffer and D.C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton, NJ (1954); Russian transl.: Izd. Inostr. Lit., Moscow (1957).
Variational theory etc. of very general character on Riemann surfaces; but also something about univalent mappings.
 - [235] R. Schinzinger and P.A.A. Laura, *Conformal Mapping: Methods and Application*, Elsevier, Amsterdam–Oxford–New York–Tokyo (1991).
Some applications of conformal maps and of Schwarz–Christoffel formulas in calculation of fields, also in inhomogeneous or anisotropic situations; some applications of quasiconformal mappings.
 - [236] F.J. Schnitzer, *Schlichte Funktionen*, Ber. Math.-Stat. Sect. Forschungszentr. Graz, Nr. 209 (1983).
A survey on the theory up to de Branges; almost 200 references.
 - [237] G. Schober, *Univalent Functions—Selected Topics*, Lecture Notes in Math., Vol. 478, Springer-Verlag, Berlin–Heidelberg–New York (1975).
Mapping classes S and Σ , also special classes (convex, starlike, etc.), distortion, coefficients, extreme points, variation; Pólya–Schoenberg conjecture; extremal length; quasiconformal mappings, schlicht functions with quasiconformal extension.
 - [238] L.I. Sedov, *Two-Dimensional Problems in Hydrodynamics and Aerodynamics*, Wiley, New York–London–Sydney (1965); Russian original in the 3rd edn: Nauka, Moscow (1980).
Applications of conformal mappings in hydrodynamics; many special mappings; e.g., also of doubly-connected domains (with elliptic functions); domains bounded by slits on conics or in form of lattices.
 - [239] *Selected Numerical Methods for Linear Equations, Polynomial Equations, Partial Differential Equations, Conformal Mapping*, Ch. Gram, ed., Regnecentralen, Copenhagen (1962); Part III: *Conformal Mapping*, by Chr. Anderson, S.E. Christiansen, O. Møller, H. Tornehave.
Some explicit mappings; orthogonal polynomials; some integral equation methods, also from the point of view of practice, with ALGOL code for the Gershgorin–Lichtenstein method; mapping of circular regions.

- [240] M. Seppälä and T. Sorvali, *Geometry of Riemann Surfaces and Teichmüller Spaces*, North-Holland, Amsterdam (1992).
Also Möbius transformations, hyperbolic metric, quasiconformal mappings.
- [241] H.S. Shapiro, *The Schwarz Function and Its Generalization to Higher Dimensions*, Univ. Arkansas, Lecture Notes in Math., Spec. Vol. 9, Wiley, New York (1992).
Schwarz function; analytic reflection at an analytic Jordan curve.
- [242] V.G. Sheretov, *Analytic Functions with a Quasiconformal Extension*, Tverskoj. Gosud. Univ., Tver' (1991).
Quasiconformal mappings, Beltrami equation, extremal mappings and quadratic differentials; area method, distortion theorems, Grunsky conditions for univalent functions with a quasiconformal extension.
- [243] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht (2001).
Iterations of the holomorphic maps of the unit disk into itself are used to study, e.g., the Löwner–Kufarev representation of univalent functions.
- [244] R.A. Silverman, *Complex Analysis with Applications*, Prentice-Hall, Englewood Cliffs, NJ (1974).
With some applications in physics.
- [245] I.S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd edn, McGraw-Hill, New York–Toronto–London (1956).
Also with the use of conformal mappings.
- [246] *Some Problems of Mathematics and Mechanics*, see Academy of Sciences of the USSR.
- [247] K. Strebel, *Quadratic Differentials*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo (1984).
Riemann surfaces; quadratic differentials; trajectories; associated metric; existence theorems; extremal properties concerning the conformal moduli.
- [248] E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Heft 2, Herausgegeben unter Mitwirkung von W. Blaschke: *Konforme Abbildung einfach–zusammenhängender Bereiche*, Teubner, Leipzig–Berlin (1913).
Although now very old, nevertheless something still nice and inspiring: Boundary behavior of conformal maps in examples, Schwarz–Christoffel formula, also the continuous limit case with the “Stützwinkel-funktion”, level curves, convex mappings.
- [249] P.K. Suetin, *Polynomials Orthogonal over a Region and Bieberbach Polynomials*, Proc. Steklov Inst. Math. **100** (1974); Russian original: Trudy Mat. Inst. Steklov **100** (1971).
Faber and Bieberbach polynomials etc., series and speed of convergence.
- [250] P.K. Suetin, *Series of Faber Polynomials*, Gordon and Breach, Reading, MA (1998); Russian original: Nauka, Moscow (1984).
Faber polynomials; applications in the theory of univalent functions (area method); some additions in the English translation.
- [251] G.D. Suvorov, *Families of Topological Mappings in the Plane*, Red.-Izd. Otd. Sibirsk. Otd. AN SSSR, Novosibirsk (1965) (in Russian).
Distortion of quasiconformal and more general mappings (with bounded Dirichlet integral); length–area method; boundary distortion.
- [252] G.D. Suvorov, *The Metric Theory of Prime Ends and Boundary Properties of Plane Mappings with Bounded Dirichlet Integral*, Naukova Dumka, Kiev (1981) (in Russian).
Theory of prime ends for mappings with bounded spherical Dirichlet integral (including quasiconformal mappings).
- [253] G.D. Suvorov, *The Generalized “Length and Area Principle” in Mapping Theory*, Naukova Dumka, Kiev (1985) (in Russian).
- [254] G.D. Suvorov, *Prime Ends and Sequences of Plane Mappings*, Naukova Dumka, Kiev (1986) (in Russian).
This theory in a more general context.
- [255] A.G. Sveshnikov and A.N. Tikhonov, *Theory of Functions of a Complex Variable*, Nauka, Moscow (1974) (in Russian).
Also Riemann mapping theorem, Schwarz–Christoffel transformations.
- [256] A.V. Sychev, *n-Dimensional Quasiconformal Mappings*, Novosibirsk. Gos. Univ., Novosibirsk (1975) (in Russian).
Booklet before the following book.

- [257] A.V. Sychev, *Moduli and n -Dimensional Quasiconformal Mappings*, Nauka Sibirsk. Otd., Novosibirsk (1983) (in Russian).
Extremal length, conformal capacity, coefficient of quasiconformality of domains, all in n -space.
- [258] O. Tammi, *Extremum Problems for Bounded Univalent Functions*, Lecture Notes in Math., Vol. 646, Springer-Verlag, Berlin–Heidelberg–New York (1978); Part II: Lecture Notes in Math., Vol. 913, Springer-Verlag, Berlin–Heidelberg–New York (1982).
Very intensive studies about the coefficients (estimates and coefficient bodies) of bounded univalent functions, with different methods.
- [259] *Theory of Functions as Applied to Engineering Problems*, R. Rothe, F. Ollendorf and K. Pohlhausen, eds, Dover, New York (1961); German original: *Funktionentheorie und ihre Anwendung in der Technik*, Springer-Verlag, Berlin (1931).
Schwarz–Christoffel formula; Schwarz reflection principle; several applications of conformal maps: two-dimensional electrostatic fields, hydrodynamics; also concrete technical applications.
- [260] O. Tietjens, *Hydro- und Aerodynamik* (nach Vorlesungen von L. Prandtl), I: *Gleichgewicht und reibungslose Bewegung*, Springer-Verlag, Berlin (1929); English transl.: Dover, New York (1957).
Also something about the use of conformal mappings.
- [261] O. Tietjens, *Strömungslehre, I: Hydro- und Aerostatik, Bewegung der idealen Flüssigkeit*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
Use of conformal maps in great detail; hodograph method.
- [262] L.N. Trefethen, see *Numerical Conformal Mapping*.
- [263] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo (1959); 2nd edn: Chelsea, New York (1975).
Capacity, transfinite diameter, mapping radius, also in the hyperbolic sense; Löwner differential equation; variable domains; something about conformal mapping of multiply-connected domains: e.g., circular slits-plane, radial slits-plane; circular domains.
- [264] J. Väisälä, *Lectures on n -Dimensional Quasiconformal Mappings*, Lecture Notes in Math., Vol. 229, Springer-Verlag, Berlin–Heidelberg–New York (1971).
Theory of quasiconformal mappings, preferable in the n -dimensional case with $n > 2$.
- [265] G. Valiron, *Fonctions Analytiques*, Presses Univ. de France, Paris (1954).
Polygonal mappings, schlicht functions, elementary distortion theory.
- [266] A. Vasil'ev, *Moduli of Families of Curves for Conformal and Quasiconformal Mappings*, Lecture Notes in Math., Vol. 1788, Springer-Verlag, Berlin (2002).
Applications of Grötzsch's strip method (in form of extremal length) to extremal problems; many references to papers in Russian.
- [267] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford–London–New York–Paris/Addison-Wesley, Reading, MA (1962).
Also some boundary behavior of conformal maps, Beltrami equation.
- [268] H. Villat, *Leçons sur la théorie des tourbillons*, Gauthier-Villars, Paris (1930).
Some applications of the theory of complex functions and conformal mappings.
- [269] H. Villat, *Mécanique des fluides*, 2nd edn, Gauthier-Villars, Paris (1938).
Some applications of the theory of complex functions and conformal mapping, including Schwarz–Christoffel formula.
- [270] L.I. Volkovskii, *Investigation of the Type Problem for a Simply Connected Riemann Surface*, Trudy Mat. Inst. Steklov **34**, Moscow–Leningrad (1950) (in Russian).
With quasiconformal mappings as essential tool.
- [271] L.I. Volkovskii, *Quasiconformal Mappings*, Izd. L'vovsk. Univ., L'vov (1954) (in Russian).
Quasiconformal mappings and the Beltrami equation.
- [272] L. Volkovskii, G. Lunts and I.G. Aramanovich, *Problems in the Theory of Functions of a Complex Variable*, 2nd edn, Mir, Moscow (1977); Russian original: Fizmatgiz, Moscow (1961); another English translation: Pergamon Press, Oxford, Addison-Wesley, Reading, MA (1965); Reprint: Dover, New York (1991); German translation from the English edn: L.I. Volkovskii, G.L. Lunts und I.G. Aramanovich, *Aufgaben und Lösungen zur Funktionentheorie*, B.I. Hochschultaschenbücher Bd. 195, Mannheim–Wien–Zürich (1973).

- Many special conformal mappings, also Schwarz–Christoffel integrals, elliptic functions; quasiconformal mappings: formulas for the correspondence between the coefficients of linear systems and the parameters of the transformed infinitesimal ellipses; applications in hydrodynamics, electrostatics etc.
- [273] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., Vol. 1319, Springer-Verlag, Berlin (1988).
Mainly in n -space: Möbius transformations, hyperbolic geometry, modulus, capacity, distortion theory, boundary behavior.
- [274] M. Vuorinen, see *Quasiconformal Space Mappings*.
- [275] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th edn, Amer. Math. Soc. Coll. Publ., Vol. 20, Providence, RI (1965).
With some relations to conformal mappings.
- [276] G. Walz, *Spline-Funktionen im Komplexen*, Bibl. Inst. & F.A. Brockhaus AG, Mannheim–Wien–Zürich (1991).
Including remarks about quasiconformal mappings, quasicircles, Beltrami equation.
- [277] C. Weber und W. Günther, *Torsionstheorie*, Vieweg & Sohn, Braunschweig/Akademie-Verlag, Berlin (1958).
Some concrete applications of conformal maps in the theory of elasticity.
- [278] F. Weinig, *Die Strömung um die Schaufeln von Turbomaschinen*, Johann Ambrosius Barth, Leipzig (1936).
Also some interesting applications of conformal mappings in the theory of the flow in a turbine.
- [279] Guo-Chun Wen, *Conformal Mappings and Boundary Value Problems*, Amer. Math. Soc., Providence, RI (1992) (transl. from Chinese).
Also some canonical conformal mappings, connections with boundary value problems and integral equations; univalent functions.
- [280] H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer-Verlag, Berlin–Göttingen–Heidelberg (1955); Russian transl.: Fizmatgiz, Moscow (1960), Suppl. by A.A. Gol'dberg, ed. by L.I. Volkovyskii.
Also something about conformal and quasiconformal mappings of rings, with applications in the theory of the type problem.
- [281] L.C. Woods, *The Theory of Subsonic Plane Flow*, Cambridge Univ. Press, Cambridge (1961).
Something about conformal mappings; Schwarz–Christoffel formula, also for doubly-connected domains; basic integral equations; mapping of nearly circular domains; Theodorsen–Garrick method; relation between conformal mappings and incompressible fluid motion; conformal transformations in general aerofoil theory.
- [282] J. Zajac, *Quasihomographies in the Theory of Teichmüller Spaces*, Dissertationes Math., Vol. 357, Inst. Mat., Polska Akademia Nauk, Warszawa (1996).
Quasiconformal mappings, conformal invariants, and special functions (e.g., elliptic integrals).
- [283] There exist some booklets and series of booklets, containing papers about GFT in Russian or Ukrainian:
Metric Problems of the Theory of Functions and Mappings, Naukova Dumka, Kiev.
Questions of the Metrical Theory of Mappings and Their Applications, Naukova Dumka, Kiev.
Questions of Geometric Function Theory, Izd. Tomsk. Univ., Tomsk.
Mathematical Analysis, Kubansk. Gos. Univ., Krasnodar.
Theory of Functions and Mappings, Naukova Dumka, Kiev.
Papers about the Theory of Functions of a Complex Variable and Their Application, Inst. Mat. AN USSR, Kiev.
Theory of Mappings and Approximation of Functions, Naukova Dumka, Kiev.
Theory of Mappings, Its Generalizations and Applications, Naukova Dumka, Kiev.
- [284] There exist *Gesammelte Abhandlungen (Collected Papers)* of B. Riemann, F. Klein and H.A. Schwarz, also of L.V. Ahlfors, L. Bers, A. Beurling, C. Carathéodory, H. Grunsky, G. Julia, M.A. Lavrent'ev (selected papers), J.E. Littlewood, K. Löwner (Ch. Loewner), G. Pólya, M. Schiffer (selected papers, in preparation), G. Szegő, O. Teichmüller, not, e.g., of L. Bieberbach, G.M. Goluzin, H. Grötzsch, P. Koebe.

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